ALGEBRA 10: normal subgroups and representations

10.1 Normal subgroups

Definition 10.1. Let G be a group and let x, y be its elements. Denote by x^y the element of the form yxy^{-1}. A subgroup $G_1 \subset G$ is called **normal**, if for any $x \in G_1$, $y \in G$ it holds that $x^y \in G_1$.

Exercise 10.1. The **centre** of the group G (denoted $Z(G)$) is the set of all elements $x \in G$ that commute with all elements of G. Prove that $Z(G) \subset G$ is a normal subgroup.

Exercise 10.2. Let $G_1 \subset G$ be a subgroup. **Left cosets** of the subgroup G_1 are subsets of G the form $G_1 \cdot x \subset G$, where x takes all values in G_1. **Right cosets** are subsets of G of the form $x \cdot G_1 \subset G$. Prove that right (left) cosets either intersect or coincide. Prove that right cosets are right (and vice versa) if and only if G_1 is a normal subgroup.

Exercise 10.3. Let $G_1 \subset G$ be a normal subgroup and let S_1, S_2 be its cosets. Take $x \in S_1$, $y \in S_2$. Prove that the coset of the product xy does not depend on the choice of x, y in S_1, S_2. Prove that the product thus defined makes the set G_2 of cosets of G_1 into a group.

Definition 10.2. In this case one says that G_2 is the **quotient group of G by G_1** (denoted $G_2 = G/G_1$), and G is an **extension of G_2 by G_1**. A group extension is denoted as follows:: $1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$.

Exercise 10.4. Let $G \xrightarrow{\varphi} G'$ be a homomorphism of groups. Prove that the kernel φ (i.e. the set of elements that are mapped to $1_{G'}$) is a normal group in G.

Exercise 10.5. Let $G \xrightarrow{\varphi} G'$ be a surjective homomorphism of groups. Prove that $G' \cong G/\ker\varphi$ where $\ker\varphi$ is the kernel of φ.

Exercise 10.6. Consider the set $\operatorname{Aut}(G)$ of automorphisms of a group G with the composition operation. Prove that it is a group. Prove that the correspondence $\varphi_y(x) \mapsto x^y$ defines a homomorphism $G \longrightarrow \operatorname{Aut}(G)$.

Definition 10.3. Let G, G' be groups and

$$G \longrightarrow \operatorname{Aut}(G')$$

be a homomorphism. In this case one says that G **acts on G' by automorphisms**. Automorphisms of the form $x \xrightarrow{\varphi_y} x^y$ are called **inner**.

Exercise 10.7. Find the group $\operatorname{Aut}(G)$ for $G = \mathbb{Z}/p\mathbb{Z}$ (p prime).

Exercise 10.8 (*). Find the group $\operatorname{Aut}(G)$ for $G = \mathbb{Z}/n\mathbb{Z}$ (n arbitrary).

Exercise 10.9. Consider a homomorphism $G_2 \xrightarrow{\varphi} \operatorname{Aut}(G_1)$. Define the following operation on the set of pairs (g_1, g_2): $(g_1, g_2) \cdot (h_1, h_2) = (g_1\varphi(g_2, h_1), g_2h_2)$. Prove that this defines a group.

Definition 10.4. This group is called a **semi-direct product of G_1 and G_2** and is denoted $G_1 \rtimes G_2$.

Exercise 10.10. In the previous problem setting prove that $(G_1, 1)$ defines a normal subgroup in G and that the quotient by this subgroup is isomorphic to G_2.

Exercise 10.11. Describe the group S_3 as a semi-direct product of two non-trivial Abelian groups.

Exercise 10.12 (!). Describe the dihedral group as a semi-direct product of two non-trivial Abelian groups.

Exercise 10.13 (*). The Klein group is the group of quaternions of the form $\pm 1, \pm I, \pm J, \pm K$, with the natural product. Is it possible to get the Klein group as a semi-direct product of two Abelian groups?

Exercise 10.14 (*). Consider a group extension $1 \longrightarrow G_1 \longrightarrow G \xrightarrow{\varphi} G_2 \longrightarrow 1$. Suppose that $G \xrightarrow{\psi} G_1$ is a homomorphism such that $\psi \circ \varphi$ is the identity automorphism of G_2 (in this case one says that φ **admits a section** or **splits**). Prove that G is not a semi-direct product $G_1 \rtimes G_2$.

Exercise 10.15 (!). Consider a group G. Consider a subgroup $[G, G] \subset G$ generated by the elements of the form $xyx^{-1}y^{-1}$. Prove that this is a normal subgroup and the quotient by this subgroup is commutative.

Definition 10.5. $[G, G]$ is called the **commutant** of the group G.

Exercise 10.16 (*). Find the commutant of the symmetric group.

Exercise 10.17 (!). . Consider the group of even substitutions A_n, $n \geqslant 5$. Prove that it coincides with its commutant.

Hint. Compute $xyx^{-1}y^{-1}$ where x, y are cyclic permutations of order 3.

Solvable groups

Definition 10.6. A group G is called **solvable** if there exists a sequence $1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$ of normal subgroups such that all G_i/G_{i-1} are Abelian.

Exercise 10.18. Prove that a subgroup of a solvable group is solvable.

Exercise 10.19. Prove that the symmetric group S_3 is solvable.

Exercise 10.20. Prove that the symmetric group S_4 is solvable.

Exercise 10.21. Prove that the Klein group $\{\pm 1, \pm I, \pm J, \pm K\}$ is solvable.

Exercise 10.22 (!). Consider a group G_0 and its commutant G_1, then $G_2 = [G_1, G_1]$ – the commutant of the commutant and so on, $G_i = [G_{i-1}, G_{i-1}]$. Prove that G_0 is solvable if and only if at some stage we get $G_n = 1$.

Exercise 10.23 (!). Prove that the group of even permutations A_n, $n \geqslant 5$ is not solvable.

Exercise 10.24 (*). Prove that the group of motions of $\mathbb{R}^3$ is not solvable.

Hint. Construct an isomorphism between A_5 and the group of motions of an icosahedron and use the Problem 10.17.

Exercise 10.25. Consider a group G of order p^n. Prove that the centre of G contains more that one element.

Hint. Consider the action of G on itself by automorphisms. The order of G equals the sum of cardinalities of classes of the form x^G where x^G is the set of all elements of the form x^y, $y \in G$. First prove that if x is not in the centre then the order of x^G is divisible by p. We thus obtain that $|G| = 1 + \sum |y_i^G|$, and if G has no centre then all $|y_i^G|$ are divisible by p.

Exercise 10.26 (!). Let G be a group of order p^n. Prove that G is solvable.

Exercise 10.27 (*). Let G be a group of order p^2, where p is prime. Prove that G is Abelian.

Exercise 10.28 (*). Give an example of a non-Abelian group of order p^3 where p is any prime number.

Exercise 10.29 (*). Consider the set S of all upper-triangular matrices $n \times n$ with unity on the diagonal over the field k. Prove that these matrices form a subgroup in $GL(n, k)$. Prove that this group is solvable. Find its order for $k = \mathbb{Z}/p\mathbb{Z}$.

10.2 Representations and invariants

Definition 10.7. A **representation of a group** G **on a vector spac** V is a homomorphism $G \longrightarrow GL(V)$ from G into the group $GL(V)$ of invertible endomorphisms of V. If there is a representation of G on V one says that G **acts on** V. A **subrepresentation** V is a subspace that is preserved under the action of G.

Exercise 10.30. Let G act on vector spaces V, V'. Define the action G on $V \otimes V'$ by the formula $g(v \otimes v') = g(v) \otimes g(v')$. Prove that this definition is correct and defines a representation of G on $V \otimes V'$.

Definition 10.8. Let G be a group acting on a vector space V. A vector $v \in V$ is called **invariant under the action of** G or an **invariant of** G if $g(v) = v$ for any $g \in V$. The space of all G-invariant vectors is denoted V^G.

Exercise 10.31. Consider the action of the symmetric group S_n on $V = R^n$ defined by the permutations of coordinates. Find the space of invariants.

Exercise 10.32 (*). In the previous problem setting find the space of invariants of the action of S_n on $V \otimes V$.

Exercise 10.33. Consider the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on $V = R^n$ by the cyclic permutations of coordinates. Find the space of invariants.

Exercise 10.34 (*). In the previous problem setting find the space of invariants $(V \otimes V)^{\mathbb{Z}/n\mathbb{Z}}$ under the action of $\mathbb{Z}/n\mathbb{Z}$ on $V \otimes V$.