ALGEBRA 12: semisimple and nilponent operators

Artinian algebras over an algebraically closed field

Let R be an Artinian ring over a field k. Recall that in the exercise sheet 9 we have constructed a canonical decomposition $R \cong \oplus_i e_i R_i$ where e_i are indecomposable orthogonal idempotents and $e_i R$ is Artinian with no non-unit idemotents; moreover, this decomposition is unique.

Exercise 12.1 (!). Assume that R does not have non-unit idempotents and k is algebraically closed. Prove that if R is semisimple then $R = k$.

Exercise 12.2 (!). Assume that R does not have non-unit idempotents, and k is algebraically closed. Prove that $R = k \oplus \mathfrak{n}$ where $\mathfrak{n}$ is a nilradical.

Hint. Prove that $R/\mathfrak{n}$ is semisiple and apply the previous exercise.

Exercise 12.3 (!). Let R be an Artinian ring over an algebraically closed field k. Prove that $R = R_{ss} \oplus \mathfrak{n}$ where R_{ss} is a semisimple Artinian subring in R. Prove that $R_{ss} \subset R$ is uniquely defined.

Exercise 12.4 (*). Is this true if k is not algebraically closed?

We will further need the following statement.

Exercise 12.5 (!). Let R be a semi-simple Artinian ring over a field k, and $R \longrightarrow R'$ be a surjective homomorphism of k-algebras. Prove that R' is a semisimple Artinian ring too.

Hint. There is a similar problem in ALGEBRA 9.

Definition 12.1. Let R be an algebra over a field k. A **representation** of an algebra R is a homomorphim of algebras from R to $\mathrm{End}(V)$, where V is a vector space over k.

Exercise 12.6. Let R be an algebra over a field k. Consider a mapping $R \longrightarrow \mathrm{End}(R)$, defined by the formula $r \mapsto (v \longrightarrow rv)$. Prove that this is a representation.

Exercise 12.7. Let R be an algebra over k, isomorphic to a finite extension k, and let V be a finite dimensional representation of R. Prove that $V \cong R^n$, that is, V is isomorphic (as a representation of R) to a sum of several copies of R.

Exercise 12.8. Let V be a finite dimensional representation of the quaternion algebra $\mathbb{H}$ over $\mathbb{R}$. Prove that V is isomorphic to $\mathbb{H}^n$.

Exercise 12.9. Let G be a group, and k be a field. A **group algebra** G over k (denoted $k[G]$) is the vector space of linear combinations of the form $\sum \lambda_i g_i$ ($\lambda_i \in k$, $g_i \in G$) with multiplication defined by the formula

$$(\sum \lambda_i g_i)(\sum \lambda'_j g'_j) = \sum_{i,j} \lambda_i \lambda'_j g_i g'_j.$$

Prove that this is indeed an algebra. Prove that any representation of a group G can be uniquely extended to a representation of the group algebra.

Exercise 12.10 (!). Let G_1, G_2 be groups and $k[G_1 \times G_2]$ be the group algebra of their product. Prove that $k[G_1 \times G_2] \cong k[G_1] \otimes k[G_2]$.

Exercise 12.11 (!). Let $G = (\mathbb{Z}/2\mathbb{Z})^n$ be a product of n copies of $\mathbb{Z}/2\mathbb{Z}$. Prove that $k[G] \cong k^{\oplus 2^n}$ (direct sum of 2^n copies of k).

Hint. Prove that $k[\mathbb{Z}/2\mathbb{Z}] \cong k \oplus k$, and use the isomorphism $k[G_1 \times G_2] \cong k[G_1] \otimes k[G_2]$.

Exercise 12.12 (*). Consider the Klein group (the subgroup of order 8 in the quaternions that consists of elements $\{\pm 1, \pm I, \pm J, \pm K\}$). Prove that its group algebra over $\mathbb{R}$ is isomorphic to $\mathbb{H} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$.

Exercise 12.13 (*). Let G be a finite Abelian group, and let k be an algebraically closed field of characteristic 0. Prove that $k[G]$ is a semisimple Artinian ring over k. Deduce from this that $k[G]$ is a direct sum of $|G|$ copies of k.

Hint. Use the criterion mentioned in ALGEBRA 9: an Artinian ring R over a field of characteristic 0 is semisimple if and only if the trace defines a nondegenerate form on R.

Exercise 12.14 (*). Let G be a finite Abelian group, k be an algebraically closed field characteristic 0, and $\rho : G \longrightarrow \operatorname{End}(V)$ be a representation of G over k. Prove that V decomposes into a direct sum of one-dimensional G-invariant subspaces.

Hint. Use the previous exercise and the exercise 12.5.

Exercise 12.15 (*). Let G be a finite Abelian group, and $\mathbb{R}[G]$ its group ring over $\mathbb{R}$. Prove that $\mathbb{R}[G]$ is isomorphic to a direct sum of several copies of $\mathbb{R}$ and $\mathbb{C}$.

Exercise 12.16 (*). Let G be a finite Abelian group, and $\rho : G \longrightarrow \operatorname{End}(V)$ be a representation of G over $\mathbb{R}$. Prove that V can be decomposed into a direct sum of one-dimensional and two-dimensional G-invariant subspaces.

Exercise 12.17 (!). Let G be a finite Abelian group, and let $\rho : G \longrightarrow \operatorname{End}(V)$ be its three-dimensional representation over $\mathbb{R}$. Prove that there is a G-invariant line in V.

Semi-simple operators

Let $A \in \operatorname{End}(V)$ be a linear operator over a finite-dimensional vector space. It is easy to see that the subalgebra $\langle 1, A, A^2, A^3, \dots \langle \subset \operatorname{End}(V)$ generated by A is commutative.

Definition 12.2. The operator $A \in \operatorname{End}(V)$ is called **semi-simple** if the algebra generated by it in $\operatorname{End}(V)$ is semi-simple.

Exercise 12.18. Prove that a linear operator over an algebraically closed field is semi-simple if and only if it is diagonalizable.

Exercise 12.19 (!). Let $k \subset \overline{k}$ be two fields, moreover $\overline{k}$ is algebraically closed, and let V be a finite dimensional vector space over k. Consider $V \otimes_k \overline{k}$ as a vector space over $\overline{k}$. Prove that $\operatorname{End}(V) \otimes_k \overline{k}$ is naturally isomorphic to $\operatorname{End}_{\overline{k}}(V \otimes_k \overline{k})$. This defines a natural inclusion $\operatorname{End}(V) \longrightarrow \operatorname{End}_{\overline{k}}(V \otimes_k \overline{k})$. Prove that the linear operator $A \in \operatorname{End}(V)$ is semi-simple if and only if the corresponding linear operator in $V \otimes_k \overline{k}$ is diagonalizable.

Exercise 12.20. Let V be a two-dimensional vector space over $\mathbb{R}$ endowed with a positive definite bilinear symmetric form, and let $A \in \operatorname{End}(V)$ be an orthogonal operator. Prove that it is semi-simple.

Exercise 12.21 (*). Let V be a vector space over $\mathbb{R}$ of arbitrary finite dimension endowed with a positive definite bilinear symmetric form, and let $A \in \operatorname{End}(V)$ be an orthogonal operator. Prove that it is semi-simple.

Exercise 12.22 (*). Let V be a vector space over $\mathbb{R}$ endowed with a non-degenerate bilinear symmetric form, not necessarily positive definite, and let $A \in \operatorname{End}(V)$ be an orthogonal operator. Is it always semi-simple?

Definition 12.3. An element of an Artinian ring over k is called **semi-simple** if it generates a semi-simple subalgebra in R.

Exercise 12.23. Let R be an Artinian ring over k and let $r \in R$ be a semi-simple element. Prove that in any representation of $R \longrightarrow \operatorname{End}(V)$, r is mapped to a semi-simple endomorphism of V.

Hint. Use the exercise 12.5.

Exercise 12.24 (!). Let V be a finite dimensional vector space over an algebraically closed field, and let $A \in \operatorname{End}(V)$ be a linear operator. Prove that A decomposes into a direct sum of a semi-simple and a nilpotent operator, $A = A_{ss} + A_n$, which commute. Prove that this decomposition is unique and A_{ss}, A_n can be expressed as polynomials of A.

Hint. Use exercise 12.3.

Exercise 12.25 (*). Is this true if the base field k is not algebraically closed?

Exercise 12.26 (!). Let A be a upper-triangular matrix, A_δ be its diagonal part. Prove that A and A_δ commute.

Exercise 12.27 (**). Let (V, g) be a vector space endowed with a bilinear skew-symmetric form, and let A be anti-symmetric operator and $A = A_{ss} + A_n$ be its decomposition into a semi-simple and nilpotent part. Prove that A_{ss}, A_n are anti-symmetric.

Exercise 12.28 (*). Is it possible that an antisymmetric operator over $\mathbb{C}$ be nilpotent?

Hamilton-Cayley theorem

Let k be any field and let $k(t)$ be the field of rational functions over k, and V be an n-dimensional vector space over k, and $B(t) \in \operatorname{End}(V)[t]$ a polynomial with coefficients in $\operatorname{End}(V)$. Recall that in this situation $\det(B(t))$ is a polynomial of t (see ALGEBRA 8). Let us consider $B(t)$ as a $k(t)$-linear endomorphism of $V \otimes k(t)$. Consider the endomorphism $\Lambda^{n-1}(V \otimes k(t))$ induced by $B(t)$ and let $\check{B}(t)$ be the adjoint endoorphim of $V \otimes k(t)$ with respect to the natural pairing

$$\Lambda^{n-1}(V \otimes k(t)) \otimes V \otimes k(t) \longrightarrow \det V \otimes k(t)$$

It is shown in ALGEBRA 7 that $B(t)\check{B}(t) = \check{B}(t)B(t) = \det(B(t)) \operatorname{\mathsf{Id}}_V$.

Exercise 12.29. In this situation show that $\check{B}(t)$ is $\operatorname{End}(V)$-valued polynomial: $\check{B}(t) \in \operatorname{End}(V)[t]$.

Hint. Express $\check{B}(t)$ via the minors of $B(t)$.

Exercise 12.30. Let $A \in \operatorname{End}(V)$. Applying the argument from the Remark to $B = t - A$ prove that $(t-A)\check{(t-A)} = \mathsf{Chpoly}_A(t)$. Prove the the coefficients of the polynomial $\check{(t-A)} \in \operatorname{End}(V)[t]$ commute with A.

Exercise 12.31. Let $R \subset \operatorname{End}(V)$ be a subset. Denote by $Z(R)$ the set of all operators $A' \in \operatorname{End}(V)$ that commute with all operators $r \in R$ (this set is called the **centralizer** of R). Prove that $Z(R)$ is a subalgebra of $\operatorname{End}(V)$.

Exercise 12.32. Let $R \in \operatorname{End}(V)$ be a subalgebra and let $A_1 \in Z(A)$ be an element of the centralizer R, $R[t]$ be the algebra of R-valued polynomials, and $R[t] \xrightarrow{\varphi} R'$ be a homomorphism of algebras. Denote by $R[A_1]$ the subalgebra $\operatorname{End}(V)$, generated by R and A_1. Prove that there exists a homomorphism $\varphi_0 : \ R[A_1] \longrightarrow R'$ such that $\varphi_0\big|_R = \varphi\big|_R$ and $\varphi_0(A_1) = \varphi(t)$. Prove that these conditions determine φ_0 uniquely.

Exercise 12.33. Let $A \in \operatorname{End}(V)$ be a linear operator, apply the previous exercise and construct a homomorphism $Z(A)[t] \xrightarrow{\Psi} Z(A)$ that maps t to A, and which is identity on $Z(A)$.

Exercise 12.34 (!). (Cayley-Hamilton theorem) Consider the equality $(t-A)(t \check{-} A) = \mathsf{Chpoly}_A(t)$ in $Z(A)[t]$. Apply the homomorphism Ψ constructed above to both left and right parts of the equality. Prove that this results in the following equality in the algebra $\operatorname{End}(V)$:

$$\mathsf{Chpoly}_A(A) = 0.$$

Exercise 12.35 (*). Let $A, B \in \operatorname{End} V$ be linear operators. Consider a function of two variables $Q(t_1, t_2) = \det(t_1 A + t_2 B)$, where $t_1 A + t_2 B$ is considered as a linear operator on $V \otimes_k k(t_1, t_2)$, and $k(t_1, t_2) = k(t_1)(t_2)$ is the field of rational functions over $k(t_1)$. Prove that $Q(t_1, t_2)$ is a polynomial with coefficients in k. Prove that in the ring $\operatorname{End} V$ the equality $Q(-B, A) = 0$ holds.

Exercise 12.36 (!). Let $A \in \operatorname{End}(V)$ be a linear operator that acts on a finite dimensional vector space over an algebraically closed field k. Let $\{\lambda_1, \dots, \lambda_n\}$ be the roots of the characteristic polynomial of the operator A. Consider the space V_{λ_i} of all $v \in V$ such that $(A - \lambda_i)^{m_i}(v) = 0$ where m_i is the multiplicity of the root λ_i of the polynomial $\mathsf{Chpoly}_A(t)$. Prove that $V = \oplus V_{\lambda_i}$, where summation is over all roots λ_i of the characteristic polynomial A.

Hint. Use the Cayley-Hamilton theorem.

Remark. The space V_{λ_i} is called an **generalized eigenspace** of operator A.

Minimal polynomial and characteristic polynomial

Definition 12.4. Let $A \in \operatorname{End}(V)$ be a linear operator that acts on a vector space of finite dimension over k. a sequence of endomorphisms $1, A, A^2, \dots \in \operatorname{End}(V)$. Since the space $\langle 1, A, A^2, \dots \rangle$ is finite dimensional, then starting from some - i all A^i can be expressed as a sum of the form: $A^N = \sum_{i=0}^{l-1} \lambda_i A^i$, $\lambda_i \in k$, $l = \dim\langle 1, A, A^2, \dots \rangle$. Let us write down such an equation for A^l: $A^l + \sum_{i=0}^{l-1} \lambda_i A^i = 0$. Recall that the polynomial $P(t) = t^l + \lambda_{l-1} t^{l-1} + \dots + \lambda_0$ is called the **minimal polynomial** of A and is denoted $\mathsf{Minpoly}_A(t)$.

Exercise 12.37 (!). Prove that the following identity holds in the algebra $\operatorname{End}(V)$:

$$\mathsf{Minpoly}_A(A) = 0.$$

Prove that any polynomial $Q(t) = t^m + \mu_{m-1} t^{l-1} + \dots + \mu_0$, such that $Q(A) = 0$, is divided by $\mathsf{Minpoly}_A(t)$.

Exercise 12.38. Prove that the characteristic polynomial of the operator is divided by its minimal polynomial.

Exercise 12.39. Let $A \in \operatorname{End}(V)$ be a linear operator.

a. Prove that A is nilpotent if and only if $\mathsf{Minpoly}_A(t) = t^n$.

b. Prove that A is a non-identity idempotent if and only if $\mathsf{Minpoly}_A(t) = t^2 - t$.

Exercise 12.40. Let $A \in \operatorname{End}(V)$ be a linear operator that acts on a finite dimensional vector space over an algebraically closed field k and let $V = \oplus V_{\lambda_i}$ be a decomposition of V into a direct sum of generalized eigenspaces. Let $P(t)$ be a minimal polynomial of A and let $P_i(t)$ be minimal polynomials of restrictions of A to V_{λ_i}. Prove that $P(t) = P_1(t)P_2(t)\dots$. Prove that $P_i(t) = (t - \lambda_i)^k$ where $k \leqslant \dim V_{\lambda_i}$.

Hint. It is clear that $P_i(t) = (t - \lambda_i)^k$, since the operator $A - \lambda_i$ on V_{λ_i} is nilpotent. That $P(t) = P_1(t)P_2(t)\dots$ follows easily from the fact that all $P_j(A)$ $(j \neq i)$ are invertible on V_{λ_i}.

Remark. The characteristic polynomial also has this muliplicativity property, as can be easily observed.

Exercise 12.41. Let $A \in \operatorname{End}(V)$ be a linear operator in a n-dimensional vector space. Prove that $\mathsf{Minpoly}_A(t) = (t - \lambda)^n$ if and only if in some basis A has the form

$$\begin{pmatrix}
\lambda & 1 & 0 & \dots\dots\dots & & & 0 \\
0 & \lambda & 1 & 0 & \dots\dots & & 0 \\
0 & 0 & \lambda & 1 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \dots\dots & & \lambda & 1 & 0 \\
0 & 0 & \dots\dots\dots & & & \lambda & 1 \\
0 & 0 & \dots\dots\dots\dots & & & & \lambda
\end{pmatrix} \tag{12.1}$$

Hint. Replacing A by $A - \lambda\,\mathsf{Id}_V$ one may assume that $\mathsf{Minpoly}_A(t) = t^n$. Take a vector $v \in V$ such that $(A - \lambda)^{n-1}(v) \neq 0$. Prove that $v, A(v), A^2(v), \dots, A^{n-1}(v)$ constitute a basis in V, and in this basis A hase the form (12.1).

Remark. Such a matrix is called a **Jordan block**. We will denote it by $J(n, \lambda)$.

Definition 12.5. Let $e_1, \dots, e_n$ be a basis in a vector space and let A_i^j be a matrix of a linear operator A in this basis. Assume that $e_1, \dots, e_n$ are divided into groups (blocks) $[e_1, \dots, e_{k_1}][e_{k_1+1}, \dots, e_{k_2}]\dots$, in such a way that A maps every e_i into a linear combination of vectors that belong to the same block. In this case A consists of square pieces of size $k_i - k_{i-1}$, and is zero outside of these square pieces:

$$\begin{pmatrix}
* & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
* & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 \\
0 & \dots & 0 & * & \dots & * & 0 & \dots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & * & \dots & * & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & * & \dots & * \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & \dots & 0 & * & \dots & *
\end{pmatrix}$$

A matrix of this form is called **block diagonal**.

Exercise 12.42. Let $A \in \operatorname{End}(V)$ be a linear operator that acts on a finite dimensional vector space over an algebraically closed field k. Assume that the minimal polynomial $\mathsf{Minpoly}_A(t)$ equals characteristic polynomial $\mathsf{Chpoly}_A(t)$. Prove that in some basis A can be represented as a block diagonal matrix that consists of Jordan blocks $J(n_i, \lambda_i)$ where all λ_i are distinct.

Hint. Use the multiplicativity of $\mathsf{Minpoly}$ and Chpoly with respect to decomposition of V into a direct sum of generalized eigenspaces, and reduce the problem to the case $V = V_{\lambda_i}$. Now apply exercise 12.41.

Definition 12.6. Assume that operator A can be represented in some basis as a block diagonal matrix that consists of Jordan blocks. The operator A is said then to be in a **Jordan normal form**.

We will now show the unicity of the Jordan normal form, and then we will show its existence. We work assuming the base field to be algebraically closed.

Exercise 12.43. Let $A \in \operatorname{End}(V)$ be a nilpotent operator with the Jordan normal form that consists of Jordan blocks $J(0, n_1), \dots, J(0, n_k)$. Prove that the number of blocks in the Jordan normal form of A equals the dimension of the space V/AV. Prove that $A^jV/A^{j+1}V$ is the number of blocks $J(0, n_i)$ with $n_j \geqslant j$. Deduce that Jordan normal form of a nilpotent operator is determined uniquely, up to permutation of the blocks.

Exercise 12.44 (!). Prove that Jordan normal form of any operator is unique, up to permutation of the blocks.

Hint. Decompose V into a direct sum of generalized eigenspaces, and reduce the problem to the case $V = V_{\lambda_i}$. Replacing A by $A - \lambda_i$, one can content oneself with nilpotent operators. Now the statement follows from the previous exercise.

Definition 12.7. Let $A \in \operatorname{End}(V)$ be a linear operator. We say that A **acts cyclically** on V if there exists an element v such that $v, Av, A^2v, A^3v, \dots$ generates V.

Exercise 12.45. Let $A \in \operatorname{End}(V)$ be a linear operator that acts cyclically on V. Prove that $\mathsf{Minpoly}_A(t) = \mathsf{Chpoly}_A(t)$.

Hint. If A act cyclically then the degree of $\mathsf{Minpoly}_A(t)$ equals $\dim V$ equals the degree of $\mathsf{Chpoly}_A(t)$.

Exercise 12.46. Let $A \in \operatorname{End}(V)$ be a linear operator such that V decomposes as a sum of A-invariant subspaces on which A acts cyclically. Prove that A can be represented in some basis in a Jordan normal form.

Hint. Use the exercise 12.42.

Modules over a ring and Jordan normal form

Definition 12.8. Let R be a ring. A **module** over R is an Abelian group endowed with an operation $R \times M \longrightarrow M$ that is compatible with the addition in the following sense

(i) For any $\lambda \in R$, $u, v \in M$ we have $\lambda(u + v) = \lambda u + \lambda v$. For any $\lambda_1, \lambda_2 \in R$, $u \in M$ we have $(\lambda_1 + \lambda_2)u = \lambda_1 u + \lambda_2 u$ (distributivity of multiplication over addition).

(ii) For any $\lambda_1, \lambda_2 \in R$, $u \in M$ we have $\lambda_1(\lambda_2 u) = (\lambda_1\lambda_2)u$ (associativity of multiplication).

(iii) For any $v \in M$ we have $1v = v$ where 1 denotes the identity in R.

Remark. This definition repeats almost verbatim the definition of a vector space over a field. Many notions that were defined for vector spaces (for example, homomorphism, monomorphism, epimorphism, kernel, image, quotient space) can be redefined without modification for modules over a ring.

Exercise 12.47. Let R be an algebra over a field k. For any module M over R consider M as a vector space over $k \subset R$. Consider each the operation of multiplication by an elements of R as an endomorphism of M. Prove that this defines a homomorphism $R \longrightarrow \operatorname{End}_k(M)$. Prove that all representations can be obtained in this way.

Exercise 12.48. Prove that any Abelian group has a unique structure of a module over $\mathbb{Z}$.

Definition 12.9. Consider the group R^n as a module over R, with the action given by $r\cdot(x_1, \dots, x_n) = (rx_1, \dots, rx_n)$. This module is called **free**. The quotient of R^n by a submodule is called **finitely generated**. If M can be represented as a quotient of a free module by a finitely generated submodule then M is called **finitely presented**.

Definition 12.10. Let $\varphi : M \to M'$ be a homomorphism of modules over an algebra R. The **cokernel** φ (denoted by $\operatorname{Coker}\varphi$) is the quotient of M' by the image of φ.

Exercise 12.49. Let M be a module over R. Prove that M is finitely generated if and only if it has a collection of elements $m_1, \dots, m_N$ such that any element of M can be represented as a linear combination , $m = r_1m_1 + \dots + r_Nm_N$, $r_1, \dots, r_N \in R$.

Exercise 12.50. Let M be a module over R. Prove that M is finitely presented if and only if it is isomorphic to a cokernel of a homomorphism $\varphi : R^N \to R^M$ of free R-modules.

Exercise 12.51 (!). Let k be a field and M be a module over $k[t]$ that has a finite dimension over k. Prove that M is finitely generated and finitely presented over $k[t]$.

Hint. Consider M as a vector space over k, pick a basis $m_1, \dots, m_M \in M$, and consider these elements as generators. Then prove that the kernel of the map $\varphi : M \otimes_k k[t] \to M$ is generated by elements of the form $m_i \otimes t - tm_i \otimes 1$.

Exercise 12.52 (!). Let M be a finite Abelian group. Prove that M is finitely generated and finitely presented as a module over $\mathbb{Z}$.

Hint. Take all elements of M as the set of generators $m_1, \dots, m_N \in M$.

Exercise 12.53. Let R be a ring and V a module over R represented as the cokernel of the homomorphism $(R)^n \xrightarrow{\varphi} (R)^m$. Write down φ as a matrix A^i_j with coefficients in R. Let B^i_j be a matrix obtained from R using elementary (Gaussian) row and column transformations (see ALGEBRA 7). Prove that V is isomorphic to a cokernel of a homomorphism that corresponds to B^i_j.

Definition 12.11. Let R be a ring and $a \in R$ be an element. Consider aR as a module over R. A **cyclic module** over R is a quotient module R/aR.

Exercise 12.54. Let M be a Z-module. Prove that M is cyclic if and only if the corresponding Abelian group is cyclic.

Exercise 12.55. Let M be a $k[t]$-module. Prove that M is cyclic if and only if for some $v \in M$, $v, tv, t^2v, t^3v, \dots$ generated M.

Exercise 12.56 (!). Let R be a ring such that any $n \times m$ matrix with coefficients in R can be brought into a diagonal form using elementary row and column operations. Prove that any finitely generated and finitely presented module over R is isomorphic to a direct sum of cyclic ones.

Hint. If $(R)^n \xrightarrow{\varphi} (R)^n$ is represented by a diagonal matrix with a_i^i on the diagonal then the cokernel of this homomorphism has the form $\oplus_i R/a_i^i R$.

Exercise 12.57 (!). Let R be a Euclidean ring (see ALGEBRA 2). Prove that any matrix of size $n \times m$ with coefficients in R can be brought to a diagonal form using elementary row and column operations. Deduce that any finitely generated and finitely presented module over R is a direct sum of cyclic ones.

Hint. A similar problem is in ALGEBRA 7.

Exercise 12.58 (!). Let G be a finite Abelian group. Prove that G is a finite sum of cyclic groups.

Exercise 12.59 (!). Let V be a module over $k[t]$ which is finite dimensional over k. Prove that V is a direct sum of cyclic modules.

Hint. The ring $k[t]$ is Euclidean and V is finitely generated and finitely presented as fallows from exercise 12.51.

Exercise 12.60 (!). Let $A \in \mathrm{End}(V)$ be a linear operator. Prove that V can be decomposed into a direct sum of A-invariant subspaces so that on each of them A acts cyclically. Deduce that if the field k is algebraically closed then A can be brought into Jordan normal form.

Hint. Consider the action of $k[t]$ on V given by $P(t)(v) = P(A)v$. Prove that V is a $k[t]$-module. Decompose V into a direct sum of cyclic submodules: $V = \oplus V_i$. Prove that all V_i are A-invariant, and A acts on them cyclically.

Exercise 12.61 (*). Find a commutative ring and a module over it that does not decompose into a direct sum of cyclic ones.