

ALGEBRA 3: vector spaces and linear mappings

Vector spaces

Recall that abelian (or commutative) group is a group where group operation is commutative:

$$f \cdot g = g \cdot f$$

Group operation in abelian groups is often denoted by $+$ and called “addition”; unity is denoted by 0 in this case and is called “zero”.

Definition 3.1. **Linear** or **vector** space V over field k is an abelian group with an operation $k \times V \mapsto V$ (“multiplication of vector by element of a field”) which is in accordance with the group operation in the following sense:

- For any $\lambda \in k$, $u, v \in V$, $\lambda(u+v) = \lambda u + \lambda v$. For any $\lambda_1, \lambda_2 \in k$, $u \in V$, $(\lambda_1 + \lambda_2)u = \lambda_1 u + \lambda_2 u$ (distributivity of multiplication with respect to addition).
- For any $\lambda_1, \lambda_2 \in k$, $u \in V$, $\lambda_1(\lambda_2 u) = (\lambda_1 \lambda_2)u$ (associativity of multiplication).
- For any $v \in V$, $1v = v$ where $1 \in k$ is a unity.

Elements of vector space are called vectors and group operation is called the addition of vectors.

Exercise 3.1. Consider a field k . Prove that k is a vector space over itself.

Exercise 3.2. Prove that the group k^* of invertible elements in k with multiplication as group operation acts on any vector space over k .

Remark. This group is called the multiplicative group of field k .

Exercise 3.3. Prove that a group of parallel transports of a plane has a structure of vector space over \mathbb{R} .

Exercise 3.4. Consider a vector space V over k . Prove that $0_k(v) = 0_V$ for any $v \in V$. Here 0_k is the zero in k and 0_V is the unity in V .

Exercise 3.5. Consider a field \mathbb{F} and its subfield k . Prove that \mathbb{F} is a vector space over k .

Exercise 3.6. Consider a field k .

- Denote a set of n -tuples (a_1, a_2, \dots, a_n) of elements of k by k^n . Define a natural addition on k^n and action of k^* on it and prove that you obtained a vector space.
- Consider a set S . Denote by $k[S]$ a set of all collections of elements of k

$$\langle a_{s_1}, a_{s_2}, \dots \rangle$$

each element of a collection corresponding to precisely one element of S such that a_s are zero except a finite number of them. Introduce a structure of vector space over k on $k[S]$.

Vector space $k[S]$ is called a vector space, **generated** by a set S . Set S can be naturally embedded into $k[S]$ – every element of $s \in S$ corresponds to a vector $[s] \in k[S]$ such that all $a_{s'}$ are zeros except a_s which is 1.

Definition 3.2. Let A, B be two sets and let G act on them. It is said that a mapping $\kappa : A \rightarrow B$ is **compatible with action** G if $\kappa(g(a)) = g(\kappa(a))$.

Recall that a homomorphism of abelian groups is a mapping that preserves the group operation

$$f : G_1 \rightarrow G_2, \quad f(g + g') = f(g) + f(g') \quad (3.1)$$

Definition 3.3. Homomorphism of vector spaces over k is a mapping which preserves addition of vectors (cf. (3.1)) and is compatible with the action of k^* .

In other words a homomorphism of vector spaces is a mapping $f : V_1 \rightarrow V_2$ which satisfies conditions $f(v_1 + v_2) = f(v_1) + f(v_2)$, $f(\lambda v) = \lambda f(v)$. Monomorphism, epimorphism, isomorphism and automorphism of vector spaces are defined in the same fashion as for groups (as well as rings, fields and all other algebraic structures). When talking about vector spaces one says “linear operator” or “linear mapping” instead of “homomorphism”.

Recall that identity mapping of a vector space V is denoted by Id_V . Id_V is obviously an automorphism.

Exercise 3.7. Prove that a linear mapping $\varphi : V_1 \rightarrow V_2$ is bijective if and only if it is **invertible**, i.e. there exists a linear mapping $\psi : V_2 \rightarrow V_1$ such that

$$\psi \circ \varphi = \text{Id}_{V_1}, \varphi \circ \psi = \text{Id}_{V_2}. \quad (3.2)$$

Are these conditions sufficient each on its own ?

Exercise 3.8. The fact that two vector spaces V, V' are isomorphic is denoted by $V \cong V'$. Prove that

- If $V \cong V'$ and $V' \cong V''$ then $V \cong V''$.
- If $V \cong V'$ then $V' \cong V$.
- It is always true that $V \cong V$.

Exercise 3.9. Prove that the set of homomorphisms from V_1 to V_2 form a vector space (it is often denoted by $\text{Hom}(V_1, V_2)$).

Exercise 3.10. Prove that the set of automorphisms V forms a group (this group is often denoted by $GL(V)$). Is this a commutative group?

Definition 3.4. Subgroup $V' \subset V$ of a vector space V is called a **vector subspace** or **linear subspace of V** if it is preserved under the action of k^* (in other words for any $\lambda \in k, v, v' \in V'$ it is true that $\lambda(v) \in V', v + v' \in V'$).

Vector subspace is itself a vector space over the same field.

Exercise 3.11. Consider a set S . Prove that a set of all mappings $\text{Map}(S, k)$ from S to k is a vector space.

Exercise 3.12 (*). Consider a vector space W and a set S . Prove that any mapping $S \rightarrow W$ can be extended in a unique way to a linear mapping $k[S] \rightarrow W$. Is it true if we allow for the infinite non-zero elements of a field in the definition of $k[S]$?

Exercise 3.13. Let $k[t]$ be a set of polynomials $a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$ with coefficients belonging to a field k . Prove that this is a linear space.

Exercise 3.14. Consider a set $\text{Map}(k, k)$ of all mappings from the field k to k . For any polynomial $P = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$ and every $\lambda \in k$ define $\Psi_P(\lambda) = P(\lambda)$. We obtain the mapping $\Psi : k[t] \rightarrow \text{Map}(k, k)$, $P \mapsto \Phi_P$. Prove that it is a homomorphism.

Exercise 3.15 (*). Let k be a finite field. Prove that $\Psi : k[t] \rightarrow \text{Map}(k, k)$ is not a monomorphism. Prove that it is an epimorphism.

Exercise 3.16 (*). Let k be an infinite field. Prove that $\Psi : k[t] \rightarrow \text{Map}(k, k)$ is a monomorphism.

Exercise 3.17 (*). Prove that it is not an epimorphism.

Consider a set A and a binary relation \sim on A (that is, we state for certain pairs of elements $a, b \in A$ that $a \sim b$). It is said that \sim is an **equivalence relation** if the following holds:

- a. For any element $a \in A$ it is true that $a \sim a$.
- b. If $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity).
- c. If $a \sim b$ then $b \sim a$ (symmetry).

An equivalence relation \sim having been defined an **equivalence class** of an element $a \in A$ is a set of all elements $a' \in A$ such that $a' \sim a$. It is easy to check that if $a \sim a'$ then the equivalence class of a is the same as the equivalence class of a' . So we can talk just about an equivalence class without mentioning a particular element a . A relation $a = b \pmod n$ is an example of an equivalence relation of the set of natural numbers. The problem 3.8 can be reformulated as follows: " $V \cong V'$ is an equivalence relation".

Remark. We work with equivalence relation in Geometry 1 when considering Cauchy sequences, although we do not introduce this notion explicitly.

Exercise 3.18. Let V be a vector space and $V' \subset V$ be a subspace. Consider the following equivalence relation on V : $a \sim b$ iff $a + v = b$ for some $v \in V'$. Prove that the set of equivalence classes forms linear space.

Definition 3.5. This space is called a **quotient space** and is denoted by V/V' .

Exercise 3.19. Consider a natural mapping $V \rightarrow V/V'$ which maps an element to its equivalence class. Prove that this is a homomorphism and an epimorphism.

Exercise 3.20. Let $\varphi : V_1 \rightarrow V_2$ be a linear mapping and $V_0 \subset V_1$ be a set of elements that are mapped to zero.

- a. Prove that V_0 is a linear subspace in V_1 .
- b. Prove that an image of φ – that is, a subset of all $v \in V_2$ of the form $\varphi(v')$, $v' \in V_1$ – is a linear subspace in V_2 .

Definition 3.6. V_0 is called a **kernel of φ** .

Exercise 3.21. Let $\varphi : V_1 \rightarrow V_2$ be a linear operator. Prove that an image of φ is isomorphic to a quotient space V_1/V_0 where V_0 is a kernel of φ .

Linear hull, basis, dimension

Exercise 3.22. Let V be a vector space over the field k and x_1, \dots, x_n be vectors from V . Any vector of the form $v = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$ is called **linear combination** of vectors x_1, \dots, x_n where λ_i are arbitrary elements from k . Prove that linear combinations of vectors x_1, \dots, x_n form a linear subspace of V .

Definition 3.7. This subspace is called a **linear hull** of x_1, \dots, x_n and is denoted $\langle x_1, x_2, \dots, x_n \rangle$. $\langle x_1, x_2, \dots, x_n \rangle$ is called a subspace **generated** by vectors x_1, \dots, x_n .

Exercise 3.23. Construct an epimorphism from k^n into $\langle x_1, x_2, \dots, x_n \rangle$.

Hint. Map an n -tuple $(0, 0, 0, \dots, 1, \dots, 0) \in k^n$ (unity is in the l -th position) to x_l .

Definition 3.8. Vectors x_1, \dots, x_n are called **linear independent** vectors, if for any linear combination $v = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$ such that there is at least one $\lambda_i \neq 0$, it is true that $v \neq 0$.

Exercise 3.24. Let x_1, \dots, x_n be vectors from vector space V and φ be an epimorphism constructed in the exercise 3.23. Prove that φ is injective iff x_1, \dots, x_n are linearly independent vectors.

Definition 3.9. Let x_1, \dots, x_n be linearly independent vectors from a vector space V such that $V = \langle x_1, x_2, \dots, x_n \rangle$. x_1, \dots, x_n is called a **basis** of the vector space V .

Exercise 3.25. Construct a basis of k^n .

Exercise 3.26. Prove that a vector space with a basis x_1, \dots, x_n is isomorphic to k^n .

Exercise 3.27. Let v be a non-zero vector from $V \cong k^n$, $\langle v \rangle$ be a subspace generated by it and $V/\langle v \rangle$ is a quotient space. Prove that $V/\langle v \rangle$ is isomorphic to k^{n-1} .

Hint. Consider a subspace $V_l \subset V \cong k^n$, generated by n -tuples of the form

$$\langle \lambda_1, \lambda_2, \dots, \lambda_{l-1}, 0, \lambda_{l+1}, \dots, \lambda_n \rangle$$

(there is a 0 on the l -th position). This space is isomorphic to k^{n-1} . Prove that for some $l = 1, 2, \dots, n$ the natural projection $V_l \rightarrow V/\langle v \rangle$ is an isomorphism.

Exercise 3.28. Let x_1, \dots, x_l be linearly independent vectors from $V \cong k^n$. Prove that $V/\langle x_1, x_2, \dots, x_l \rangle$ is isomorphic to k^{n-l} .

Hint. Use an inductive argument.

Exercise 3.29. Let $V_1 \subset V_2$ be a subspace of $V_2 \cong k^n$. Suppose that $V_1 \cong k^m$. Prove that $m \leq n$.

Exercise 3.30. Let x_1, \dots, x_l be a basis of $V \cong k^n$. Prove that $l = n$.

Exercise 3.31 (!). Let vector spaces k^l and k^m be isomorphic. Prove that $l = m$.

Definition 3.10. Vector space V over k is called a **finite-dimensional** if it is isomorphic to k^n . Number n is called a **dimension** of V . This is denoted by $\dim V = n$. It follows from the previous exercise that n is uniquely defined.

Exercise 3.32. Let x_1, x_2, \dots, x_l be linearly independent vectors in a linear space V such that $V' := V/\langle x_1, x_2, \dots, x_l \rangle$ is a non-zero space. Let $x_{l+1} \in V$ be a vector such that its natural projection on V' is non-zero. Prove that $x_1, x_2, \dots, x_l, x_{l+1}$ are linearly independent vector.

Exercise 3.33 (!). Let V be a linear space which is not finite-dimensional. Prove that there exists an infinite sequence of linearly independent vectors $x_1, x_2, \dots, x_l, \dots \in V$.

Hint. Use the previous exercise.

Exercise 3.34 (!). Let $V \subset V'$ be a subspace of a finite dimensional vector space. Prove that V is finite-dimensional. Deduce that $\dim V \leq \dim V'$

Hint. Use the previous exercise.

Exercise 3.35. Let $V = V'/V''$ be a quotient space of finite dimensional space. Prove that V' is finite-dimensional. Prove that $\dim V \leq \dim V'$. Deduce that $\dim V' = \dim V + \dim V''$.

Hint. Use a proof by contradiction: consider an infinite sequence of linearly independent vectors from V and lift them to V' .

Exercise 3.36. Let $f : V \rightarrow V'$ be homomorphism of vector spaces of the same dimension n . Suppose that f is an injection or surjection. Prove that f is an isomorphism.

Linear forms, bilinear forms

Definition 3.11. Let V be a linear space over k . A **linear form** or **bilinear form** over V is a homomorphism of linear spaces V and k . The space of linear forms over V is denoted by V^* .

Exercise 3.37. Consider a finite-dimensional linear space V . Prove that $\dim V = \dim V^*$.

Exercise 3.38. Let k be a field and S be an arbitrary set. Is it true that $(k[S])^* \cong \text{Map}(S, k)$?

Exercise 3.39 (!). Consider a natural map $V \xrightarrow{ev} V^{**}$, $v \mapsto (\lambda \mapsto \lambda(v))$, the vector $v \in V$ maps a form $\lambda \mapsto \lambda(v)$ to V^* . Let V be finite dimensional. Prove that $V \xrightarrow{ev} V^{**}$ is an isomorphism.

Hint. Prove that this is an injection and use the exercise 3.36.

Exercise 3.40 ().** Consider a infinite-dimensional linear space V . Prove that $V \xrightarrow{ev} V^{**}$ is not an isomorphism.

Definition 3.12. Let U, V, W be linear spaces over a field k . A mapping $U \times V \xrightarrow{\mu} W$, $u, v \mapsto \mu(u, v)$ is called **bilinear** if for every u the mappings $\mu(u, \cdot) : V \rightarrow W$ and $\mu(\cdot, u) : U \rightarrow W$ are linear.

Exercise 3.41. Prove that a sum of bilinear mappings is bilinear. Prove that a structure of vector space can be defined on the set of bilinear mappings $U \times V \rightarrow W$.

A space of bilinear mappings is denoted $\text{Hom}(U \otimes V, W)$. The reason for that is the following:

Exercise 3.42 (*). Consider vector spaces U and V . Consider the set $U \times V$ and the vector space generated by it $k[U \times V]$. Let us denote the element of $k[U \times V]$ corresponding to the pair $\langle u, v \rangle \in U \times V$ by $u \otimes v$. Consider the subspace generated by the vectors of the form $au \otimes v - a(u \otimes v)$, $u \otimes av - a(u \otimes v)$, $(u_1 + u_2) \otimes v - u_1 \otimes v - u_2 \otimes v$, $u \otimes (v_1 + v_2) - u \otimes v_1 - u \otimes v_2$ and denote the quotient space by $U \otimes V$. Prove that for any W the subspace $\text{Hom}(U \otimes V, W)$ is isomorphic to the set of bilinear mappings from $U \times V$ to W .

The space $U \otimes V$ is called a **tensor product** of spaces U and V .

Exercise 3.43 (*). The dimensions of U, V, W are a, b, c . Find the dimension of $\text{Hom}(U \otimes V, W)$.

Definition 3.13. Let V be a vector space over k . Bilinear form over V is a bilinear mapping $V \times V \xrightarrow{\mu} k$. A bilinear symmetric form is a form that satisfies the equality $\mu(x, y) = \mu(y, x)$. Bilinear antisymmetric form is a form that satisfies the equality $\mu(x, y) = -\mu(y, x)$. We will denote the space of bilinear symmetric forms by S^2V^* , the space of bilinear antisymmetric forms by Λ^2V^* and the space of all bilinear forms by $(V \otimes V)^*$.

Definition 3.14. It is said that the characteristic of a field k is not 2 if the number $2 = 1 + 1$ is invertible in k .

Remark. Apparently, this is not true in a field of two elements.

Up to the end of this section we suppose that the characteristic of a field we are talking about is not 2.

Definition 3.15. If U, V are vector spaces then the product $U \times V$ of sets U and V is endowed with a natural structure of vector space. This product considered as a vector space is called a **direct sum of U and V** and is denoted $U \oplus V$.

Exercise 3.44. Consider two subspaces U, V of a vector space W such that intersection of U and V contains only $0 \in V$ and the linear hull over U and V is W . Prove that W is isomorphic to $U \oplus V$.

Remark. Notation that is used in that case: $W = U \oplus V$.

Exercise 3.45. Consider the symmetrization mapping that turns any bilinear form into symmetric form $\text{Sym}(\mu)(x, y) = \frac{1}{2}(\mu(x, y) + \mu(y, x))$ and alternation mapping $\text{Alt}(\mu)(x, y) = \frac{1}{2}(\mu(x, y) - \mu(y, x))$. Prove that these mappings are linear operators

$$(V \otimes V)^* \xrightarrow{\text{Sym}} S^2V^*, \quad (V \otimes V)^* \xrightarrow{\text{Alt}} \Lambda^2V^*$$

Prove that the sum

$$\text{Sym} \oplus \text{Alt} : (V \otimes V)^* \rightarrow S^2V^* \oplus \Lambda^2V^*$$

is an isomorphism.

Exercise 3.46 (*). Let $\dim V = n$. Find dimension of S^2V^* and Λ^2V^* .

Exercise 3.47. Let μ be a bilinear symmetric form. Prove that $\mu(u+v, u+v) = \mu(v, v) + \mu(u, u) + 2\mu(u, v)$.

Exercise 3.48 (!). Let μ be a non-zero bilinear symmetric form over V . Prove that $\mu(x, x) \neq 0$ for some $x \in V$.

Hint. Use the previous exercise.

Definition 3.16. Consider V a linear space with a symmetric or antisymmetric bilinear form $\mu : V \times V \rightarrow k$ defined on it. For any $v \in V$, μ defines a linear form $\mu(v, \cdot) : V \rightarrow k$. We say that v belongs to the radical **radical** of μ if this form equals zero.

Exercise 3.49 (*). Prove that a radical is a linear subspace of V .

Radical is denoted by $\text{rad } \mu$.

Exercise 3.50 (*). Prove that $\mu(v + r, v' + r') = \mu(v, v')$ where $r, r' \in \text{rad } \mu$.

Remark. It follows that μ is naturally defined on the quotient space $V/\text{rad } \mu$.

Definition 3.17. A symmetric (or antisymmetric) form μ is called **non-degenerate** if its radical is zero. Non-degenerate bilinear antisymmetric form is called **symplectic**.

Exercise 3.51. Consider a non-degenerate symmetric (or antisymmetric) bilinear form μ defined on a finite-dimensional vector space V . Define the mapping $V \rightarrow V^*$, $v \mapsto \mu(v, \cdot)$ that maps v to the form $t \mapsto \mu(v, t)$. Prove that it is an isomorphism.

Hint. Prove that it is a monomorphism of spaces of the same dimension.

Exercise 3.52. Let μ be non-degenerate symmetric (or antisymmetric) bilinear form on the finite-dimensional space V and $\lambda : V \rightarrow k$ be a linear functional. Prove that there exists a vector $v \in V$ such that $\lambda(t) = \mu(v, t)$.

Hint. Use the previous exercise.

Definition 3.18. Let V be a space with a symmetric (or antisymmetric) bilinear form μ and be $V_1 \subset V$ its linear subspace. Define an **orthogonal complement** V_1^\perp to be a set of all vectors $v \in V$ such that $\mu(v, v_1) = 0$ for all $v_1 \in V_1$.

Exercise 3.53 (!). Let μ be non-degenerate on V and on V_1 . Suppose that V_1 is finite-dimensional. Then $V = V_1 \oplus V_1^\perp$.

Hint. That V_1 and V_1^\perp do not intersect can be shown explicitly. It remains to prove that every vector $v \in V$ can be represented as the sum of vectors from V_1 and V_1^\perp . Consider $\mu(v, \cdot)$ as a functional over V_1 . Use the previous exercise and find the vector $v_1 \in V_1$ such that a form $\mu(v - v_1, \cdot)$ is zero on V_1 . It follows that $v - v_1 \in V_1^\perp$.

Exercise 3.54 (!). Deduce the following statement from the previous exercise. Let μ be a symmetric bilinear non-degenerate form on a vector space V . Then there exists a basis x_1, \dots, x_n in V such that $\mu(x_i, x_j) = 0$ for all $i \neq j$ and $\mu(x_i, x_i) \neq 0$ for all i .

Hint. Find a vector x such that $\mu(x, x) \neq 0$. Use the decomposition $V = \langle x \rangle \oplus \langle x \rangle^\perp$ from the previous exercise and apply an inductive argument.

Exercise 3.55 (*). Let μ be a bilinear symmetric form on V . Then there exists a basis x_1, \dots, x_n in V such that $\mu(x_i, x_j) = 0$ for all $i \neq j$. Such basis is called an **orthogonal basis**.

Exercise 3.56 (*). Let μ be a symplectic form on the space V . Prove that V dimensions is even. Prove a basis x_1, \dots, x_{2n} in V such that

$$\mu(x_{2r-1}, x_{2r}) = -\mu(x_{2r}, x_{2r-1}) = 1$$

when $r = 1, 2, \dots, n$ and $\mu(x_i, x_j) = 0$ for all other pairs (i, j) .

Hint. The proof is analogous to the symmetric case .

Definition 3.19. Let V be a vector space over \mathbb{R} and μ be a bilinear symmetric form on it. A form μ is called **positive** if $\mu(x, x) > 0$ for all non-zero vector x .

Exercise 3.57. Let μ be a positive bilinear form on V . Then there is a basis x_1, \dots, x_n in V such that $\mu(x_i, x_j) = 0$ for all $i \neq j$ and $\mu(x_i, x_i) = 1$ for all i .

Definition 3.20. Such basis is called **orthonormal**.

Exercise 3.58 (*). Let x, y be arbitrary vectors in a space V and μ be a positive bilinear form. Prove the inequality

$$\frac{\mu(x, x) + \mu(y, y)}{2} \geq \mu(x, y).$$

Exercise 3.59 (*). Prove the **Cauchy inequality**:

$$\sqrt{\mu(x, x)\mu(y, y)} \geq \mu(x, y).$$

Exercise 3.60 (*). Prove the **triangle inequality**

$$\sqrt{\mu(x, x)} + \sqrt{\mu(y, y)} \geq \sqrt{\mu(x + y, x + y)}.$$