

ALGEBRA 4: algebraic numbers

Algebraic numbers

Definition 4.1. Let $k \subset K$ be a field contained in the field K (one says that k is a **subfield** of K and K is an **extension** of k). Element $x \in K$ is **algebraic over** k if x is a root of a non-zero polynomial with coefficients from k .

One often means complex numbers which are algebraic over \mathbb{Q} (that is, roots of polynomials with rational coefficients) when saying simply “algebraic numbers” .

Exercise 4.1. Let k be a subfield in K and x be an element in K . Consider K as a linear space over k . Let $K_x \subset K$ be a linear subspace of K generated by the powers of x . Prove that K_x is finite dimensional iff x is algebraic.

Exercise 4.2. Let k be a subfield in K , x be an algebraic element of K and $K_x \subset K$ be a linear subspace generated by powers of x . Consider an operation m_v of multiplication by a non-zero vector $v \in K_x$ defined on K . Prove that m_v is a k -linear mapping that preserves a subspace $K_x \subset K$.

Exercise 4.3. Consider the previous problem, prove that the restriction of m_v on $K_x \subset K$ is invertible.

Exercise 4.4 (!). Conclude that K_x is a subfield of K .

Definition 4.2. Finite extension of a field k is a field $K \supset k$ which is finite dimensional vector subspace over k .

Exercise 4.5. Let $K_1 \supset K_2 \supset K_3$ be fields such that K_1 is finite dimensional over K_2 which is finite dimensional over K_3 . Prove that K_1 is a finite extension of K_3 .

Exercise 4.6 (!). Conclude that the sum, the product and the factor of elements which are algebraic over k are also algebraic over k .

Exercise 4.7. Prove that any finite field is a finite extension of a field of remainders modulo p for some prime p . Conclude that a finite field has p^n elements (for some p, n, p is prime).

Exercise 4.8 (*). Prove that there exists a non-algebraic complex number.

Exercise 4.9 ().** Prove that the number $0,0100100001000000001\dots$ (there are 2^i zeros after the i th one) is non-algebraic.

Exercise 4.10 (*). Let the complex number x be algebraic. Prove that its conjugate \bar{x} is also algebraic.

Hint. Use the fact that complex conjugation is an automorphism of \mathbb{C} that preserves \mathbb{Q} .

Exercise 4.11 (*). Let the complex number $x = a + b\sqrt{-1}$ be algebraic. Prove that real numbers a and b are algebraic.

Algebraic closure

Exercise 4.12. Let $P(t), Q(t) \in k[t]$ be polynomials of a positive degree over a field k which are co-prime. Prove that 1 can be represented as a linear combination of P and Q over $k[t]$:

$$1 = Q(t)A(t) + P(t)B(t).$$

Hint. Use the algorithm of Euclid for polynomials.

Exercise 4.13. Let $P(t)$ be an irreducible polynomial (it cannot be represented as a product of polynomials of a positive degree with coefficients from k) and a product $Q(t)Q_1(t)$ is divisible by $P(t)$ where $Q(t), Q_1(t)$ are non-zero polynomials. Prove that either $Q(t)$ or $Q_1(t)$ is divisible by $P(t)$.

Hint. Suppose $Q(t)$ is not divisible by $P(t)$. Use the previous exercise to represent 1 as a linear combination of $Q(t)$ and $P(t)$:

$$1 = Q(t)A(t) + P(t)B(t).$$

Then $1 \cdot Q_1(t) = Q(t)Q_1(t)A(t) + P(t)B(t)Q_1(t)$ is divisible by $P(t)$.

Exercise 4.14. Let $P(t)$ be a polynomial over k . Consider a ring $k[t]$ of polynomials of t and a quotient space $k[t]/Pk[t]$ of all polynomials factored by polynomials that are divisible by P . Prove that $k[t]/Pk[t]$ is a ring (with respect to naturally defined multiplication and addition).

Exercise 4.15. Prove that multiplication by a polynomial $Q(t)$ acts on $k[t]/Pk[t]$ as an endomorphism (an endomorphism is a homomorphism from a space to itself).

Exercise 4.16. Suppose that multiplication by $Q(t)$ maps all elements $k[t]/Pk[t]$ to zero. Prove that Q is divisible by P in the ring $k[t]$.

Exercise 4.17. Suppose that $P(t)$ is irreducible. Suppose that $Q(t)$ is a polynomial that is not divisible by $P(t)$. Prove that the operator m_Q of multiplication by $Q(t)$ on the space $k[t]/Pk[t]$ is a monomorphism.

Hint. Suppose v belongs to the kernel of m_Q and $Q_1(t)$ is a polynomial representing v . Then QQ_1 is divisible by P by the previous exercise statement. Use the algorithm of Euclid for polynomials to deduce that either Q is divisible by P or Q_1 is divisible by P .

Exercise 4.18 (*). Let $A : V \rightarrow V$ be a linear operator. Prove that there exists a polynomial $P(t) = t^n + a_n t^{n-1} + \dots$ such that $P(A) = 0$. Is it possible in general to find an irreducible polynomial $P(t)$ such that $P(A) = 0$?

Exercise 4.19 (!). Let $P(t)$ be irreducible. Prove that $k[t]/Pk[t]$ is a field.

Hint. Use the previous exercise to prove that if Q is not divisible by P then multiplication by $Q(t)$ defines an invertible linear operator on $k[t]/Pk[t]$.

Definition 4.3. Let $P(t)$ be an irreducible polynomial. We say that the field $k[t]/Pk[t]$ is an **extension obtained by adding the root $P(t)$** .

Definition 4.4. **Algebraic extension** of a field k is a field $K \supset k$ such that all elements of K are algebraic over k .

Exercise 4.20. Prove that any finite extension is algebraic.

Exercise 4.21 (*). Prove that not every algebraic extension is finite.

Definition 4.5. Let k be a field. The field k is called **algebraically complete** if any polynomial of a positive degree $P \in k[t]$ has a root in k .

Definition 4.6. **Algebraic closure of a field** k is an algebraic extension $\bar{k} \supset k$ which is algebraically complete.

Exercise 4.22 (*). Let K be an extension of the field k and $z \in K$ is a root of a non-zero polynomial $P(t)$ with coefficients which are algebraic over k . Prove that z is algebraic over k .

Exercise 4.23 (*). Suppose K is an algebraic extension of the field k such that any polynomial $P(t) \in k[t]$ has a root in K . Prove that any polynomial $P(t) \in k[t]$ can be represented as a product of linear polynomials from $K[t]$.

Exercise 4.24 (*). Take the statement of the previous exercise and prove that K is algebraically complete.

Hint. Let $P \in K[t]$ be an irreducible polynomial with coefficients K . Add its root α to K . Using the exercise 4.22 we obtain that α is algebraic over k . Then α is a root of a polynomial from $k[t]$. Every such polynomial can be represented as a product $\prod(t - \alpha_i)$, $\alpha_i \in K$ as follows from the previous exercise. Deduce that $\alpha \in K$.

Exercise 4.25 (*). Prove that any field k has an algebraic closure.

Hint. Take any algebraic extension of the field k . If it is algebraically complete then the proof is over. Otherwise there exists a polynomial $P(t) \in k[t]$ which has no roots in K . Add its root to K and obtain a field K_1 . Now consider K_1 instead of K and prove the statement for it. After having applied this argument as many times as it would be necessary consider the union of all algebraic extensions of k . We have obtained a field that contains a root of any polynomial from $k[t]$. Use the previous exercise to ensure that this field is algebraically closed.

Exercise 4.26 ()**. In the proof sketch for the previous exercise we have used implicitly the Zorn's lemma. Find a proof for a countable field k that does not use Zorn's lemma and therefore does not depend on the axiom of choice.

Exercise 4.27 ()**. Can you prove existence of an algebraic closure for an arbitrary field without using the axiom of choice?

Exercise 4.28 ()**. Prove that algebraic closure of a field is unique up to isomorphism.