

# ALGEBRA 5: Algebras over a field

## Algebras over a field

From now on we work with a fixed field  $k$ .

Recall that a mapping  $(V_1 \times V_2) \xrightarrow{\mu} V_3$  of vector spaces is called **bilinear**, if for any  $v_1 \in V_1$ ,  $v_2 \in V_2$ , the mappings

$$\mu(v_1, \cdot) : V_2 \longrightarrow V_3, \quad \mu(\cdot, v_2) : V_1 \longrightarrow V_3$$

(one argument is fixed, another takes values in  $V_2, V_1$  respectively) are linear. It is said that bilinear mapping is a mapping which is “linear in both arguments”. A symbol of tensor multiplication is used to denote bilinear mappings, for example, the mapping just mentioned is denoted

$$\mu : V_1 \otimes V_2 \longrightarrow V_3.$$

**Definition 5.1.** Let  $A$  be a vector space over a field  $k$  and  $\mu : A \otimes A \longrightarrow A$  be a bilinear operation (“multiplication”).  $(A, \mu)$  is called an **algebra over  $k$** , if  $\mu$  is associative:

$$\mu(a_1, \mu(a_2, a_3)) = \mu(\mu(a_1, a_2), a_3).$$

Multiplication in an algebra is usually denoted like this:  $a \cdot b$ . If there is an element  $1$  in an algebra such that  $\mu(1, a) = \mu(a, 1) = a$  for all  $a \in A$  then this element is called a **unit**, and an algebra is called **algebra with unit**. A **homomorphism**  $r : A \longrightarrow A'$  of algebras is a linear mapping which preserves multiplication and an **isomorphism** of algebras is an invertible homomorphism. **Subalgebra** of an algebra is a linear subspace which is closed under multiplication.

**Exercise 5.1.** Consider an algebra with unit such that for all  $a, b$  it is true that  $\mu(a, b) = \mu(b, a)$ . Prove that this is a (commutative) ring.

**Exercise 5.2.** Give an example of an algebra without a unit.

**Exercise 5.3.** Prove that unit is unique.

**Exercise 5.4 (\*)**. Give an example of non-commutative algebra with unit.

**Exercise 5.5.** Let  $V$  be a vector space and  $\text{End}(V)$  be a space of linear homomorphisms from  $V$  to  $V$  with an operation of composition. Prove that  $\text{End}(V)$  is an algebra.

**Definition 5.2.**  $\text{End}(V)$  is called a **matrix algebra** and is denoted  $\text{Mat}(V)$ .

**Exercise 5.6.** Is  $\text{Mat}(V)$  commutative?

**Exercise 5.7.** Consider an isomorphism of matrix algebras  $\text{Mat}(V) \cong \text{Mat}(V')$ .

- Suppose that  $V, V'$  are finite-dimensional. Prove that  $V, V'$  are isomorphic. Find a set of all isomorphisms  $a : V \longrightarrow V'$  which are compatible with a given isomorphism  $\text{Mat}(V) \cong \text{Mat}(V')$ .
- (\*) Prove that  $V \cong V'$  for any  $V, V'$  (possibly infinite-dimensional). Use Zorn’s Lemma.<sup>1</sup>

<sup>1</sup>We mean the following statement. Suppose that a system  $S$  of subsets of  $V$  is defined, and suppose that it satisfies the following conditions:

- For any  $V_\alpha \in S$  which is not equal to all  $V$ , there is a subset  $V_{\alpha'}$  in  $S$  which contains  $V_\alpha$  but is not equal to it.
- Consider a collection  $S' \subset S$  of subsets of  $V$  such that any  $V_\alpha, V_{\alpha'} \in S'$  are contained one in another: either  $V_\alpha \subset V_{\alpha'}$ , or  $V_{\alpha'} \subset V_\alpha$ . Then a union  $V_{\alpha_i} \in S'$  also belongs to  $S$ .

If these conditions hold  $V$  is contained in  $S$ . Zorn’s lemma is a corollary of the axiom of choice.

c. (\*\*) Is it possible to prove (b) without axiom of choice?

**Exercise 5.8 (!).** Consider an algebra  $A$  with unit. Prove that  $A$  can be realized as a subalgebra of  $\text{Mat } V$  for some vector space  $V$  (possibly infinite dimensional).

**Hint.** Take  $V = A$ .

**Definition 5.3.** An algebra  $A$  with a unit is called a **division algebra**, if  $A \setminus \{0\}$  is a multiplication group. In other words  $A$  is called a division algebra, if all non-zero elements  $A$  are invertible.

**Exercise 5.9.** Let  $\mathbb{H}$  be a four dimensional vector space over  $\mathbb{R}$  with a basis  $1, I, J, K$ . Prove that there exists a unique algebra structure on  $\mathbb{H}$  such that

1.  $1 \cdot a = a$  for all  $a \in \mathbb{H}$ ,
2.  $I^2 = J^2 = K^2 = -1$ ,
3.  $I \cdot J = -J \cdot I = K$ .

**Definition 5.4.** Algebra  $\mathbb{H}$  is called a **quaternion algebra**.

**Exercise 5.10 (!).** Consider “complex conjugation” map  $z \rightarrow \bar{z}$ , defined on  $\mathbb{H}$  as follows

$$a + bI + cJ + dK \rightarrow a - bI - cJ - dK.$$

Prove that  $\overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$ .

**Exercise 5.11.** Prove that  $z\bar{z} = a^2 + b^2 + c^2 + d^2$ , if  $z = a + bI + cJ + dK$ .

**Exercise 5.12 (!).** Prove that  $\mathbb{H}$  is a division algebra.

**Hint.** Use the argument that was used to prove invertibility of complex numbers.

**Exercise 5.13.** Replace the equality  $I^2 = J^2 = K^2 = -1$  with  $I^2 = -1, J^2 = K^2 = 1$  in the statement of the problem 5.9, replace the second equality with  $I \cdot J \cdot K = -1$ . Prove that you still get an algebra structure on  $\mathbb{R}^4$  (this algebra is called the **algebra of para-quaternions**). Is it a division algebra?

**Exercise 5.14 (\*).** Prove that the algebra of para-quaternions is isomorphic to  $\text{Mat}(\mathbb{R}^2)$ .

**Exercise 5.15.** Prove that a finite-dimensional algebra  $A$  with unity is a division algebra iff it has no divisors of zero.

## Algebras defined by generators and relations

Consider a vector space  $V$  over  $k$ . A multilinear form  $\varphi$  on  $V$  is a mapping  $V \times V \times V \times \dots \rightarrow k$  which is linear in each of its arguments. We denote it like this:

$$\varphi : V \otimes V \otimes V \otimes \dots \rightarrow k.$$

If  $\varphi, \psi$  are multilinear  $i$ -form and multilinear  $j$ -form then the mapping

$$\varphi \otimes \psi : \underbrace{V \times V \times V \times \dots}_{i+j} \rightarrow k,$$

defined as

$$(\varphi \otimes \psi)(v_1, v_2, \dots, v_{i+j}) = \varphi(v_1, \dots, v_i) \varphi(v_{i+1}, \dots, v_{i+j})$$

is apparently multilinear. This defines multiplication on the space of multilinear forms.

**Exercise 5.16.** Prove that a direct sum  $\bigoplus_i \mathcal{M}^i V$  of spaces of  $i$ -linear forms  $\mathcal{M}^i V$  forms an algebra with respect to multiplication as defined above.

**Exercise 5.17.** Let  $V$  be finite-dimensional. Prove that any element of the algebra of multilinear forms can be represented as a linear combination of products of elements of  $V^*$  (we say that the algebra is **generated by**  $V^*$ ).

**Definition 5.5.** Let  $V, W$  be vector spaces over  $k$ . Consider a space  $U = \langle V \times W \rangle$ , freely generated by pairs of vectors  $v, w \in V, W$ . We will denote vectors from  $U$  which correspond to  $v, w$  as  $v \otimes w$ . Let us take a quotient of  $U$  by a subspace generated by the following elements:

$$\begin{aligned} &(\lambda v) \otimes w - \lambda(v \otimes w), & v \otimes (\lambda w) - \lambda(v \otimes w), & \lambda \in k \\ &(v + v') \otimes w - v \otimes w - v' \otimes w, & v \otimes (w + w') - v \otimes w - v \otimes w'. \end{aligned}$$

The quotient space we obtain is called a **tensor product of  $V$  and  $W$**  and is denoted  $V \otimes W$ .

**Exercise 5.18 (!).** Prove that  $(V \otimes W)^*$  is naturally isomorphic to a space of bilinear forms over  $(V, W)$ .

**Exercise 5.19.** Find a number of dimensions of  $V \otimes W$  when  $\dim V = n, \dim W = m$ . Prove that  $V \otimes W^*$  is naturally isomorphic to a space of homomorphisms from  $W$  to  $V$ .

**Exercise 5.20 (!).** Prove that  $U \otimes (V \otimes W)$  is canonically isomorphic to  $(U \otimes V) \otimes W$ .

**Remark.** This statement allows to omit brackets: we write  $U \otimes V \otimes W$  which can be interpreted with any possible bracketing.

**Remark.** A tensor product  $V$  with itself  $i$  times is denoted as  $V^{\otimes i}$ . An isomorphism of associativity constructed above allows to endow  $\bigoplus_i V^{\otimes i}$  with associative multiplication

$$V^{\otimes i} \times V^{\otimes j} \longrightarrow V^{\otimes j+i}$$

**Definition 5.6.** The **free** (or **tensor**) algebra, generated by  $V$  is an algebra  $\bigoplus_i V^{\otimes i}$  with multiplication as defined above. This algebra is denoted as  $T(V)$ .  $V^{\otimes 0}$  is naturally interpreted as  $k$ . It follows that  $T(V)$  is an algebra with unit.

**Exercise 5.21 (!).** Let  $V$  be a finite-dimensional vector space. Prove that  $T(V)$  is isomorphic to the algebra of multilinear forms on  $V^*$ .

**Exercise 5.22 (\*).** Consider a linear mapping from  $V$  into some algebra  $A$ . Prove that it can be uniquely extended to a homomorphism of algebras  $T(V) \longrightarrow A$ .

**Exercise 5.23 (!).** Let  $\langle x_i \rangle$  be a basis in  $V$ . Prove that all the monomials of the form  $x_{i_1} x_{i_2} x_{i_3} \cdots$  define a basis in  $T(V)$ .

**Exercise 5.24 (!).** Consider a vector space  $V$  over  $k$  (a “generator space”) and a subspace  $W \subset T(V)$  (a “relations space”). Consider a quotient space of  $T(V)$  over the space  $T(V)WT(V)$  generated by the vectors of the form  $vwv'$  where  $w \in W$ . Let this space be nonempty. Prove that this quotient space carries a natural structure of an algebra with unit.

**Definition 5.7.** In the previous problem setting let  $x_i$  be a basis in  $V$  and  $w_i$  be a basis in  $W$ . Every relation  $w_i = 0$  can be written down using a non-commutative polynomial of the form

$$\sum_I \alpha_{i_1, \dots, i_n} x_{i_1} x_{i_2} \cdots = 0$$

where the sum is taken over some set of multiindices and  $\alpha_{i_1, \dots, i_n}$  are coefficients from the field  $k$ . An algebra  $T(V)/T(V)WT(V)$  is called an **algebra defined by generators  $v_i$  and relations  $w_i$** .

**Exercise 5.25.** Prove that any algebra with unit  $A$  can be defined by generators and relations. Prove that if  $A$  is finite-dimensional then generator space  $V$  and relation space  $W$  can be made finite-dimensional too.

**Hint.** Take  $A = V$ .

**Definition 5.8.** An algebra is called **finitely generated** if it can be defined by generators and relations in such a way that relations generator space is finite-dimensional  $V$ . An algebra is called **finitely presented** if it can be defined in such a way that relations space  $W$  is also finite-dimensional.

**Exercise 5.26.** Give an example of an algebra that is not finitely presented.

**Exercise 5.27 (\*).** Is it true that any finitely generated algebra is finitely presented?

**Exercise 5.28.** Prove that algebra  $\text{Mat}(\mathbb{R}^2)$  is finitely presented.

**Exercise 5.29.** Define an algebra of Laurent polynomials  $k[t, t^{-1}]$  by generators and relations.

**Definition 5.9.** Let  $V$  be a vector space with a bilinear symmetric form  $g : V \otimes V \rightarrow \mathbb{R}$ . Consider an algebra  $Cl(V)$ , generated by  $V$  and defined by relations of the form

$$v_1 \cdot v_2 + v_2 \cdot v_1 = g(v_1, v_2) \cdot 1,$$

where  $v_1, v_2$  passes through  $V$ . This algebra is called a **Clifford algebra** over the field  $k$ .

**Exercise 5.30.** Define complex numbers as a Clifford algebra over  $\mathbb{R}$ .

**Exercise 5.31.** Find all Clifford algebras over  $\mathbb{R}$  for  $\dim V = 1, 2$ .

**Exercise 5.32 (!).** Define quaternions and para-quaternions as a Clifford algebra over  $\mathbb{R}$ .

**Exercise 5.33 (\*).** Let the dimension of  $V$  is  $n$ . What is the dimension of  $Cl(V)$  as a vector space?

**Exercise 5.34 (\*\*).** Define an algebra  $\text{Mat}(\mathbb{R}^{2^n})$  as a Clifford algebra.