# ALGEBRA 5: Algebras over a field

## Algebras over a field

From now on we work with a fixed field $k$.

Recall that a mapping $(V_1 \times V_2) \xrightarrow{\mu} V_2$ of vector spaces is called **bilinear**, if for any $v_1 \in V_1$, $v_2 \in V_2$, the mappings

$$\mu(v_1, \cdot) : V_2 \longrightarrow V_3, \ \mu(\cdot, v_2) : V_1 \longrightarrow V_3$$

(one argument is fixed, another takes values in $V_2$, $V_1$ respectively) are linear. It is said that bilinear mapping is a mapping which is "linear in both arguments". A symbol of tensor multiplication is used to denote bilinear mappings, for example, the mapping just mentioned is denoted

$$\mu : V_1 \otimes V_2 \longrightarrow V_3.$$

**Definition 5.1.** Let $A$ be a vector space over a field $k$ and $\mu : A \otimes A \longrightarrow A$ be a bilinear operation ("multiplication"). $(A, \mu)$ is called an **algebra over** $k$, if $\mu$ is associative:

$$\mu(a_1, \mu(a_2, a_3)) = \mu(\mu(a_1, a_2), a_3)).$$

Multiplication in an algebra is usually denoted like this: $a \cdot b$. If there is an element 1 in an algebra such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$ then this element is called a **unit**, and an algebra is called **algebra with unit**. A **homomorphism** $r : A \longrightarrow A'$ of algebras is a linear mapping which preserves multiplication and an **isomorphism** of algebras is an invertible homomorphism. **Subalgebra** of an algebra is a linear subspace which is closed under multiplication.

**Exercise 5.1.** Consider an algebra with unit such that for all $a, b$ it is true that $\mu(a, b) = \mu(b, a)$. Prove that this is a (commutative) ring.

**Exercise 5.2.** Give an example of an algebra without a unit.

**Exercise 5.3.** Prove that unit is unique.

**Exercise 5.4 (*).** Give an example of non-commutative algebra with unit.

**Exercise 5.5.** Let $V$ be a vector space and $\mathrm{End}(V)$ be a space of linear homomorphisms from $V$ to $V$ with an operation of composition. Prove that $\mathrm{End}(V)$ is an algebra.

**Definition 5.2.** $\mathrm{End}(V)$ is called a **matrix algebra** and is denoted $\mathrm{Mat}(V)$.

**Exercise 5.6.** Is $\mathrm{Mat}(V)$ commutative?

**Exercise 5.7.** Consider an isomorphism of matrix algebras $\mathrm{Mat}(V) \cong \mathrm{Mat}(V')$.

a. Suppose that $V$, $V'$ are finite-dimensional. Prove that $V$, $V'$ are isomorphic. Find a set of all isomorphisms $a : V \longrightarrow V'$ which are compatible with a given isomorphism $\mathrm{Mat}(V) \cong \mathrm{Mat}(V')$.

b. **(*)**Prove that $V \cong V'$ for any $V$, $V'$ (possibly infinite- dimensional). Use Zorn's Lemma. [1]

[1] We mean the following statement. Suppose that a system $S$ of subsets of $V$ is defined, and suppose that it satisfies the following conditions:

(a) For any $V_\alpha \in S$ which is not equal to all $V$, there is a subset $V_{\alpha'}$ in $S$ which contains $V_\alpha$ but is not equal to it.

(b) Consider a collection $S' \subset S$ of subsets of $V$ such that any $V_\alpha, V_{\alpha'} \in S'$ are contained one in another: either $V_\alpha \subset V_{\alpha'}$, or $V_{\alpha'} \subset V_\alpha$. Then a union $V_{\alpha_i} \in S'$ also belongs to $S$

If these conditions hold $V$ is contained in $S$. Zorn's lemma is a corollary of the axiom of choice.

c. **(**)** Is it possible to prove (b) without axiom of choice?

**Exercise 5.8 (!).** Consider an algebra $A$ with unit. Prove that $A$ can be realized as a subalgebra of $\operatorname{Mat} V$ for some vector space $V$ (possibly infinite dimensional).

**Hint.** Take $V = A$.

**Definition 5.3.** An algebra $A$ with a unit is called a **division algebra**, if $A\backslash 0$ is a multiplication group. In other words $A$ is called a division algebra, if all non-zero elements $A$ are invertible.

**Exercise 5.9.** Let $\mathbb{H}$ be a four dimensional vector space over $\mathbb{R}$ with a basis $1, I, J, K$. Prove that there exists a unique algebra structure on $\mathbb{H}$ such that

1. $1 \cdot a = a$ for all $a \in \mathbb{H}$,
2. $I^2 = J^2 = K^2 = -1$,
3. $I \cdot J = -J \cdot I = K$.

**Definition 5.4.** Algebra $\mathbb{H}$ is called a **quaternion algebra**.

**Exercise 5.10 (!).** Consider "complex conjunction" map $z \longrightarrow \overline{z}$, defined on $\mathbb{H}$ as follows

$$a + bI + cJ + dK \longrightarrow a - bI - cJ - dK.$$

Prove that $\overline{z_1 z_2} = \overline{z}_2 \overline{z}_1$.

**Exercise 5.11.** Prove that $z\overline{z} = a^2 + b^2 + c^2 + d^2$, if $z = a + bI + cJ + dK$.

**Exercise 5.12 (!).** Prove that $\mathbb{H}$ is a division algebra.

**Hint.** Use the argument that was used to prove invertibility of complex numbers.

**Exercise 5.13.** Replace the equality $I^2 = J^2 = K^2 = -1$ with $I^2 = -1$, $J^2 = K^2 = 1$ in the statement of the problem 5.9, replace the second equality with $I \cdot J \cdot K = -1$. Prove that you still get an algebra structure on $\mathbb{R}^4$ (this algebra is called the **algebra of para-quaternions**). Is it a division algebra?

**Exercise 5.14 (*).** Prove that the algebra of para-quaternions is isomorphic to $\operatorname{Mat}(\mathbb{R}^2)$.

**Exercise 5.15.** Prove that a finite-dimensional algebra $A$ with unity is a division algebra iff it has no divisors of zero.

## Algebras defined by generators and relations

Consider a vector space $V$ over $k$. A multilinear form $\varphi$ on $V$ is a mapping $V \times V \times V \times \ldots \longrightarrow k$ which is linear in each of its arguments. We denote it like this:

$$\varphi : \ V \otimes V \otimes V \otimes \ldots \longrightarrow k.$$

If $\varphi$, $\psi$ are multilinear $i$-form and multilinear $j$-form then the mapping

$$\varphi \otimes \psi : \ \underbrace{V \times V \times V \times \ldots}_{i+j} \longrightarrow k,$$

defined as

$$(\varphi \otimes \psi)(v_1, v_2, \ldots, v_{i+j}) = \varphi(v_1, \ldots, v_i)\varphi(v_{i+1}, \ldots, v_{i+j})$$

is apparently multilinear. This defines multiplication on the space of multilinear forms.

**Exercise 5.16.** Prove that a direct sum $\oplus_i \mathcal{M}^i V$ of spaces of $i$-linear forms $\mathcal{M}^i V$ forms an algebra with respect to multiplication as defined above.

**Exercise 5.17.** Let $V$ be finite-dimensional. Prove that any element of the algebra of multilinear forms can be represented as a linear combination of products of elements of $V^*$ (we say that the algebra is **generated by** $V^*$).

**Definition 5.5.** Let $V, W$ be vector spaces over $k$. Consider a space $U = \langle V \times W \rangle$, freely generated by pairs of vectors $v, w \in V, W$. We will denote vectors from $U$ which correspond to $v, w$ as $v \otimes w$. Let us take a quotient of $U$ by a subspace generated by the following elements:

$$\begin{array}{ccc} (\lambda v) \otimes w - \lambda(v \otimes w), & v \otimes (\lambda w) - \lambda(v \otimes w), & \lambda \in k \\ (v + v') \otimes w - v \otimes w - v' \otimes w, & v \otimes (w + w') - v \otimes w - v \otimes w'. & \end{array}$$

The quotient space we obtain is called a **tensor product of** $V$ **and** $W$ and is denoted $V \otimes W$.

**Exercise 5.18 (!).** Prove that $(V \otimes W)^*$ is naturally isomorphic to a space of bilinear forms over $(V, W)$.

**Exercise 5.19.** Find a number of dimensions of $V \otimes W$ when $\dim V = n$, $\dim W = m$. Prove that $V \otimes W^*$ is naturally isomorphic to a space of homomorphisms from $W$ to $V$.

**Exercise 5.20 (!).** Prove that $U \otimes (V \otimes W)$ is canonically isomorphic to $(U \otimes V) \otimes W$.

**Remark.** This statement allows to omit brackets: we write $U \otimes V \otimes W$ which can be interpreted with any possible bracketing.

**Remark.** A tensor product $V$ with itself $i$ times is denoted as $V^{\otimes i}$. An isomorphism of associativity constructed above allows to endow $\bigoplus_i V^{\otimes i}$ with associative multiplication

$$V^{\otimes i} \times V^{\otimes j} \longrightarrow V^{\otimes j+j}$$

**Definition 5.6.** The **free** (or **tensor**) algebra, generated by $V$ is an algebra $\bigoplus_i V^{\otimes i}$ with multiplication as defined above. This algebra is denoted as $T(V)$. $V^{\otimes 0}$ is naturally interpreted as $k$. It follows that $T(V)$ is an algebra with unit.

**Exercise 5.21 (!).** Let $V$ be a finite-dimensional vector space. Prove that $T(V)$ is isomorphic to the algebra of multilinear forms on $V^*$.

**Exercise 5.22 (*).** Consider a linear mapping from $V$ into some algebra $A$. Prove that is can be uniquely extended to a homomorphism of algebras $T(V) \longrightarrow A$.

**Exercise 5.23 (!).** Let $\langle x_i \rangle$ be a basis in $V$. Prove that all the monomials of the form $x_{i_1} x_{i_2} x_{i_3} \cdots$ define a basis in $T(V)$.

**Exercise 5.24 (!).** Consider a vector space $V$ over $k$ (a "generator space") and a subspace $W \subset T(V)$ (a "relations space"). Consider a quotient space of $T(V)$ over the space $T(V)WT(V)$ generated by the vectors of the form $vwv'$ where $w \in W$. Let this space be nonempty. Prove that this quotient space carries a natural structure of an algebra with unit.

**Definition 5.7.** In the previous problem setting let $x_i$ be a basis in $V$ and $w_i$ be a basis in $W$. Every relation $w_i = 0$ can written down using a non-commutative polynomial of the form

$$\sum_I \alpha_{i_1,\dots,i_n} x_{i_1} x_{i_2} \dots = 0$$

where the sum is taken over some set of multiindices and $\alpha_{i_1,\dots,i_n}$ are coefficients from the field $k$. An algebra $T(V)/T(V)WT(V)$ is called an **algebra defined by generators** $v_i$ **and relations** $w_i$.

**Exercise 5.25.** Prove that any algebra with unit $A$ can be defined by generators and relations. Prove that if $A$ is finite-dimensional then generator space $V$ and relation space $W$ can be made finite-dimensional too.

**Hint.** Take $A = V$.

**Definition 5.8.** An algebra is called **finitely generated** if it can be defined by generators and relations in such a way that relations generator space is finite-dimensional $V$. An algebra is called **finitely presented** if it can be defined in such a way that relations space $W$ is also finite-dimensional.

**Exercise 5.26.** Give an example of an algebra that is not finitely presented.

**Exercise 5.27 (*).** Is it true that any finitely generated algebra is finitely presented?

**Exercise 5.28.** Prove that algebra $\mathrm{Mat}(\mathbb{R}^2)$ is finitely presented.

**Exercise 5.29.** Define an algebra of Laurent polynomials $k[t, t^{-1}]$ by generators and relations.

**Definition 5.9.** Let $V$ be a vector space with a bilinear symmetric form $g: V \otimes V \longrightarrow \mathbb{R}$. Consider an algebra $Cl(V)$, generated by $V$ and defined by relations of the form

$$v_1 \cdot v_2 + v_2 \cdot v_1 = g(v_1, v_2) \cdot 1,$$

where $v_1, v_2$ passes through $V$. This algebra is called a **Clifford algebra** over the field $k$.

**Exercise 5.30.** Define complex numbers as a Clifford algebra over $\mathbb{R}$.

**Exercise 5.31.** Find all Clifford algebras over $\mathbb{R}$ for $\dim V = 1, 2$.

**Exercise 5.32 (!).** Define quaternions and para-quaternions as a Clifford algebra over $\mathbb{R}$.

**Exercise 5.33 (*).** Let the dimension of $V$ is $n$. What is the dimension of $Cl(V)$ as a vector space?

**Exercise 5.34 (**).** Define an algebra $\mathrm{Mat}(\mathbb{R}^{2^n})$ as a Clifford algebra.