ALGEBRA 6: Grassmann algebra and determinant

Grassmann algebra

Definition 6.1. An algebra A is called **graded**, if A can be represented in the form $A = \oplus_{i=1}^{\mathbb{Z}} A^i$ and the multiplication satisfies the following condition: $A^i \cdot A^j \subset A^{i+j}$. $\oplus_i A^i$ is often written as $A^\bullet$, which means a direct sum over all possible indices. Some A^i subspaces can be empty. Algebra unit (if it exists) always belongs to A^0.

Exercise 6.1. What is the natural grading of $T(V)$?

Definition 6.2. A subspace $W \subset A^\bullet$ of a graded algebra is called **graded** or **homogeneous**, if W is a direct sum of components of the form $W^i \subset A^i$.

Exercise 6.2 (!). Consider a graded subspace $W \subset T(V)$. Prove that algebra defined by the relations space W is graded.

Exercise 6.3. Consider a vector space V and its basis $\langle x_i \rangle$. Consider a subspace $W \subset V \otimes V$ generated by vectors of the form $x \otimes y - y \otimes x$. Prove that an algebra of polynomials $k[x_1, \dots, x_n]$ is defined by generators V and relations W. Describe a natural grading inherited from $T(V)$.

Definition 6.3. The algebra obtained is called **symmetric algebra of space** V, and is denoted as $\mathsf{Sym}^\bullet(V)$.

Exercise 6.4. Let $\dim V > 1$. Are there an injective algebra homomorphism $\mathsf{Sym}^\bullet(V) \longrightarrow T(V)$.

Definition 6.4. Consider a vector space V and a graded subspace $W \subset V \otimes V$ generated by vectors of the form $x \otimes y + y \otimes x$ and vectors of the form $x \otimes x$. The graded algebra defined by the generators space V and relations space W is called a **Grassmann algebra** and is denoted as $\Lambda^\bullet(V)$. The space $\Lambda^i(V)$ is called an ***i*-th exterior power** of the space V and the operation of multiplication in Grassmann algebra is called **exterior multiplication**. Exterior multiplication is usually denoted as $\wedge$.

Remark. Elements of Grassmmann algebra can be thought of as "anticommutative polinomials" on V.

Remark. Grassmann algebra is a particular case of Clifford algebra defined in Algebra 5.

Exercise 6.5. Prove that $\Lambda^1 V$ is isomorphic to V.

Exercise 6.6. Consider a finite-dimensional space V. Prove that $\Lambda^2(V)^*$ is isomorphic to a space of bilinear antisymmetric forms on V.

Exercise 6.7. Consider a subalgebra $\Lambda^{2\bullet}(V) \subset \Lambda^\bullet(V)$ that consists of linear combinations of vectors of even grading. Prove that this subalgebra is commutative.

Definition 6.5. Vector $\Lambda^\bullet(V)$ is called **even**, if it belongs to an even grading component and **odd** if it belongs to an odd component. A **parity** $\tilde{x}$ of a vector x is defined to be zero for an even x and 1 for an odd x. .

Exercise 6.8 (!). Prove that $xy = (-1)^{\tilde{x}\tilde{y}} yx$.

Exercise 6.9 (*). Find all elements $\eta \in \Lambda^2(V)$ such that $\eta^2 = 0$.

Exercise 6.10 (!). Let $x_1, x_2, \ldots$ be a basis in $V \cong \Lambda^1 V$. Denote the product of vectors that belong to the basis in $\Lambda^\bullet(V)$ as $x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \wedge \cdots$. Prove that vectors of the form $x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \wedge \cdots$ where $i_1 < i_2 < i_3 < \ldots$, define a basis in $\Lambda^\bullet(V)$.

Exercise 6.11 (!). Let V be a d-dimensional vector space. Find $\dim \Lambda^i(V)$. Prove that $\Lambda^d V$ is one-dimensional.

Definition 6.6. The space $\Lambda^d V$ is called a **space of determinant vectors in** V.

Exercise 6.12 (!). Let V be a d-dimensional vector space, $x_1, x_2, \ldots, x_d$ be its basis and $\det := x_1 \wedge x_2 \wedge x_3 \cdots \wedge x_d$ be a determinant vector in $\Lambda^d V$. Consider a permutation $I = (i_1, i_2, \ldots, i_d)$ and consider a vector $I(\det) := x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \cdots \wedge x_{i_d}$. Prove that $I(\det) = \pm \det$. Prove that this correspondence defines a homomorphism from a permutation group S_n into $\{\pm 1\}$. Prove that this homomorphism maps a product of an odd number of transpositions to -1 and a product of even number of transpositions to 1.

Definition 6.7. A homomorphism constructed above $S_n \xrightarrow{\sigma} \mathbb{Z}/2\mathbb{Z}$ is called a **sign** of a permutation. The additive notation is used here for historical reasons. It is said that a permutation is **even** if its sign is 0 and is **odd** if its sign is 1.

Exercise 6.13. Consider a permutation decomposed into cycles as follows:

$$I = (i_{1,1}, i_{2,1} \ldots i_{k_1,1})(i_{1,2}, i_{2,2} \ldots i_{k_2,2}) \ldots$$

where cycles are of length k_1, k_2 etc. Prove that I is even iff there is an even number of even k_i-s.

From now till the end of the section we suppose that the field k we are using is of characteristic 0.

Definition 6.8. Let $\eta \in V^{\otimes i}$ be a vector of a i-th tensor power of the space V. Consider a natural action of S_i on $V^{\otimes i}$. Define $\mathsf{Alt}(\eta)$ as

$$\mathsf{Alt}(\eta) := \frac{1}{i!} \sum_{I \in S_i} (-1)^{\sigma(I)} I(\eta).$$

This operation is called **alternation**. It is said that a vector $\eta \in V^{\otimes i}$ is **totally antisymmetric** if $\eta = \mathsf{Alt}(\eta)$.

Exercise 6.14. Let $\eta = \frac{1}{i!} \sum_{I \in S_i} I(\eta)$. Prove that $I(\eta) = \eta$ for any permutation $I \in S_i$.

Exercise 6.15 (!). Consider a totally antisymmetric vector $\eta \in V^{\otimes i}$. Prove that $I(\eta) = (-1)^{\sigma(I)}\eta$ for any permutation $I \in S_i$.

Exercise 6.16 (!). Prove that $\mathsf{Alt}(\mathsf{Alt}(\eta)) = \mathsf{Alt}(\eta)$ for any η.

Exercise 6.17. Consider a tensor $x_{i_1} x_{i_2} \cdots x_{i_k} \in V^{\otimes i}$. Prove that

$$\mathsf{Alt}(x_{i_1} x_{i_2} \ldots x_{i_k}) = -\mathsf{Alt}(x_{i_1} x_{i_2} \ldots x_{i_l} x_{i_l - 1} \ldots x_{i_k})$$

(x_{i_l} is permuted with $x_{i_l - 1}$ in the second expression).

Exercise 6.18. Prove that the map $x_{i_1}x_{i_2}\dots x_{i_k} \longrightarrow \mathsf{Alt}(x_{i_1}x_{i_2}\dots x_{i_k})$ vanishes on all tensors of the form awb, where w belongs to the relations space of $\Lambda^\bullet(V)$. Deduce that there exists a natural map $\Lambda^i(V) \longrightarrow R^i$ from $\Lambda^i(V)$ to the space of totally antisymmetric tensors.

Exercise 6.19 (!). Prove that the natural map constructed above $\Lambda^i(V) \longrightarrow R^i$ is a bijection.

Exercise 6.20 (!). We put $\Lambda^i(V)$ into one-to-one correspondence with the space of totally antisymmetric tensors. It defines a multiplicative structure on antisymmetric tensors. Prove that this multiplicative structure can be defined like this: take two totally antisymmetric tensors $\alpha, \beta \in T(V)$, multiply them in $T(V)$ and apply Alt to the result.

Exercise 6.21. Consider two algebras A and B over a field k. Define a multiplicative structure on $A \otimes B$ like this: $a \otimes b \cdot a' \otimes b' = aa' \otimes bb'$. Prove that this multiplication indeed defines an algebra structure on $A \otimes B$.

Definition 6.9. A tensor product of algebras A and B is a space $A \otimes B$ with multiplicative structure defined above. If the algebras are graded, then the grading on the tensor product is defined by the formula $(A \otimes B)^n = \oplus_{i+j=n} A^i \otimes B^j$.

Exercise 6.22 (!). Let V_1, V_2 be vector spaces. Prove that $\mathsf{Sym}^\bullet(V)$ is isomorphic (as an algebra) to $\mathsf{Sym}^\bullet(V_1) \otimes \mathsf{Sym}^\bullet(V_1)$. Prove that $\Lambda^\bullet(V_1 \oplus V_2)$ and $\Lambda^\bullet(V_1) \otimes \Lambda^\bullet(V_2)$ are isomorphic as vector spaces. Is this isomorphism an isomorphism of algebras?

Exercise 6.23. Prove that $\dim \Lambda^\bullet(V) = 2^{\dim V}$.

Hint. Use the previous problem.

Exercise 6.24 (*). Consider a mapping

$$V \otimes \Lambda^i(V) \xrightarrow{\wedge} \Lambda^{i+1}(V),$$

defined by the formula $x \otimes \eta \mapsto x \wedge \eta$. For some fixed η we get a linear operator $L_\eta : V \longrightarrow \Lambda^{i+1}(V)$. Prove that for $\eta \neq 0$ an inequality $\dim \ker L_\eta \leqslant i$ holds.

Exercise 6.25 (*). Suppose in the previous problem setting an equality $\dim \ker L_\eta = i$ holds. Prove that in this case η can be represented as $\eta = x_1 \wedge x_2 \wedge \dots \wedge x_i$ for some vectors $x_1, \dots, x_i \in V$.

Exercise 6.26 (*). Let $P \in \mathsf{Sym}^i(V^*)$ be a symmetric i-form on V. Suppose that $P(v, v, v, \dots) = 0$ for all $v \in V$. Is it possible that P is non-zero?

Determinant

Exercise 6.27. Consider a one-dimensional vector space V over k. Prove that $\operatorname{End} V$ is naturally isomorphic to k.

Exercise 6.28 (!). Consider a linear space V and a linear operator $A \in \operatorname{End}(V)$. Prove that A on $V \cong \Lambda^1 V$ can be uniquely extended to a grading preserving homomorphism from $\Lambda^\bullet V$ to itself.

Definition 6.10. Consider a d-dimensional vector space V over k and a linear operator $A \in \operatorname{End}(V)$. Consider a endomorphism induced by A defined on a space of determinant vectors:

$$\det A \in \operatorname{End}(\Lambda^d(V))$$

Since $\Lambda^d(V)$ is one-dimensional, $\operatorname{End}(\Lambda^d(V))$ is naturally isomorphic to k. This allows to treat $\det A$ as a number, i.e. an element of k. This number is called a **determinant** of a linear operator A.

Exercise 6.29 (!). Consider a set of d vectors $x_1, \dots, x_d$ in a vector space V. Prove that their product $x_1 \wedge x_2 \wedge \dots$ in $\Lambda^\bullet(V)$ is zero iff these vectors are linearly dependent.

Exercise 6.30. Consider an operator $A \in \operatorname{End}(V)$ which has a non-zero kernel (such an operator is called **singular** of **degenerate**). Prove that $\det A = 0$.

Exercise 6.31. Let an operator $A \in \operatorname{End}(V)$ be invertible (such an operator is called **nonsingular** or **nondegenerate**). Prove that $\det A \neq 0$.

Exercise 6.32 (!). Prove that det defines a homomorphism from a group $GL(V)$ of invertible matrices to k^*, a multiplicative group of all nonzero elements of k.

Exercise 6.33 (!). Consider vector spaces V and V', and endomorphisms A, A'. Then $A \oplus A'$ defines an endomorphism $V \oplus V'$. Prove that $\det(A \oplus A') = \det A \det A'$.

Exercise 6.34. Consider a finite-dimensional vector space V, endowed with a positive bilinear symmetric form g. Recall that an endomorphism $A \in \operatorname{End} V$ is called **orthogonal** if it preserves g, i.e. for any $x, y \in V$ it is true that $g(Ax, Ay) = g(x, y)$. Prove that an orthogonal operator is always invertible. Consider an orthogonal operator in $\mathbb{R}^2$. What values can $\det A$ can take?

Exercise 6.35 (*). Consider a vector space V endowed with

a. nondegenerate bilinear symmetric form g

b. nondegenerate bilinear antisymmetric form g

c. **(**)** nondegenerate bilinear form (i.e. an isomorphism $g: V \longrightarrow V^*$).

Consider a linear operator $A \in \operatorname{End}(V)$ that preserves g. Prove that A is invertible in any of the aforementioned cases and find all the values that $\det A$ can take.