ALGEBRA 7: matrices and determinants

We suppose that all vector spaces are vector spaces over a field k.

Exercise 7.1. Let $v_1, \ldots, v_n \in V$, $w_1, \ldots, w_m \in W$ be bases in vector spaces V and W. Consider a homomorphism e_i^j from V to W that maps v_i to w_j and maps v_k to zero for $k \neq i$. Prove that e_j^i form a basis in the space of homomorphisms $\mathrm{Hom}(V, W)$.

Definition 7.1. In the previous problem setting consider a homomorphism $\gamma \in \mathrm{Hom}(V, W)$. Consider $\gamma = \gamma_j^i e_j^i$, $\gamma_j^i \in k$. The matrix

$$\begin{pmatrix} \gamma_1^1 & \cdots & \gamma_n^1 \\ \vdots & \ddots & \vdots \\ \gamma_1^m & \cdots & \gamma_n^m \end{pmatrix}$$

is called the **matrix of the homomorphism** γ.

Exercise 7.2. Consider homomorphisms $a \in \mathrm{Hom}(U, V)$, $b \in \mathrm{Hom}(V, W)$ defined by the matrices (a_j^i), (b_l^k). Prove that the composition of a and b is defined by the matrix $c_k^i = \sum_j a_j^i b_k^j$.

Remark. Note that the matrix product formula makes sense for matrices of elements of an arbitrary ring.

Exercise 7.3. Consider the space A of square matrices of the size $n \times n$, with the multiplication $A \times A \longrightarrow A$ defined by the formula $(a_j^i) \circ (b_l^k) \longrightarrow \sum_j a_j^i b_k^j$. Prove that this is an algebra with unit. Prove that this algebra is isomorphic to the algebra of linear operators from k^n to k^n.

Definition 7.2. This algebra is called the **matrix algebra** and is denoted $\mathrm{Mat}(n)$. The unit element of this algebra (the diagonal matrix with $a_i^i = 1$) is called the **identity matrix** and is denoted Id.

Exercise 7.4. Consider a linear operator $f \in \mathrm{Hom}(V, V)$ and let $v_1, \ldots, v_n$ be a basis of V and (f_j^i) be the matrix of f. Consider another basis $v_1', \ldots, v_n'$ of V. Prove that there exists a unique operator g that maps v_i to v_i', and g is invertible. Let (g_j^i), $((g^{-1})_j^i)$ be the matrices of g and g^{-1}. Prove that f is defined by the matrix $h_j^i := (g_j^i) \circ (f_j^i) \circ ((g^{-1})_j^i)$ in the basis $v_1', \ldots, v_n'$.

Definition 7.3. In that case the matrices (h_j^i), (f_j^i) are said to be **equivalent**.

Exercise 7.5. Find all the matrices equivalent to $c\,\mathsf{Id}$ where $c \in k$.

Exercise 7.6 (!). Consider a matrix $E(i, j)$

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

which has 1 on the position i, j and has 0 everywhere else. What are the values of i, j, i', j' that make the matrices $E(i, j)$ and $E(i', j')$ equivalent?

Exercise 7.7 (!). Consider a matrix A which is equivalent to $E(i, j)$. Prove that all rows of A are proportional. Prove that all columns of A are proportional.

Exercise 7.8 (*). Prove that if all rows and columns of A are proportional, then A is equivalent to $E(i,j)$.

Definition 7.4. Consider a vector space V and an endomorphism $A \in \operatorname{End}(V)$ over it (i.e. a homomorphism from V to itself) and its dual space V^*. An operator $A^* : V^* \longrightarrow V^*$ that maps a linear functional $\gamma \in V^*$ to the linear functional $A^*(\gamma)(v) = \gamma(A(v))$ is called a **conjugate operator** for A.

Exercise 7.9. Consider a finite-dimensional vector space V and its dual space V^*. Construct the natural isomorphism between $\Lambda^k(V)^*$ and $\Lambda^k(V^*)$.

Remark. "Natural" means that it does not require any extra choice (choice of base, for example). In this situation, a natural isomorphism $\Lambda^k(V)^* \cong \Lambda^k(V^*)$ is permutable with the standard action of $GL(V)$ on $\Lambda^k(V)^*$, $\Lambda^k(V^*)$. The spaces V and V^* are isomorphic, but one can prove that there is no $GL(V)$-invariant isomorphism $V \cong V^*$. In other words it is *impossible* to construct a natural homomorphism $V \cong V^*$.

Exercise 7.10 (!). Consider a vector space V, an endomorphism $A \in \operatorname{End}(V)$ and the conjugate operator A^*. Prove that $\det A^* = \det A$.

Hint. Use the previous problem.

Definition 7.5. Consider a square matrix (A_j^i) and a matrix (B_j^i), that is constructed from (A_j^i) by reflecting it over the diagonal: $B_j^i = A_i^j$. Then (B_j^i) is called the **transposed matrix** of (A_j^i), and is denoted $(A_j^i)^\perp$.

Exercise 7.11 (!). Consider a basis $v_1, \dots, v_n$ in V and a dual basis $v^1, \dots, v^n$ in V^* (v^i maps v_i to 1 and maps other v_js to zero). Consider an operator $A \in \operatorname{End}(V)$ and its matrix (A_j^i). Prove that A^* is given as the matrix $(A_j^i)^\perp$.

Definition 7.6. Consider a nondegenerate bilinear symmetric form g defined on a vector space V. An operator $A \in \operatorname{End}(V)$ is called **orthogonal with respect to** g (or simply **orthogonal**) iff $g(Av, Av) = g(v, v)$ for any $v \in V$.

Exercise 7.12 (!). Prove that any orthogonal operator is invertible.

Exercise 7.13 (!). Consider a linear operator $A \in \operatorname{End}(V)$ on a vector space endowed with a nondegenerate bilinear symmetric form g. Identify V and V^* using g. Then the dual operator A^* can be considered as an endomorphism of V. Prove that a linear operator A is orthogonal iff $A^{-1} = A^*$.

Exercise 7.14 (!). Prove that the determinant of an orthogonal operator equals ± 1.

Definition 7.7. A nondegenerate bilinear antisymmetric form (see ALGEBRA 3) is called a **symplectic form**.

Exercise 7.15 (*). Consider a vector space V with a symplectic form ω defined on it. An operator $A \in \operatorname{End}(V)$ is called **symplectic**, if it preserves ω, i.e. if $\omega(Av, Av) = \omega(v, v)$. Prove that any symplectic operator has the determinant 1.

Exercise 7.16 (!). Consider a two-dimensional vector space V over $\mathbb{R}$ and let A be the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Consider the matrix

$$A' = \begin{pmatrix} d & b \\ -c & a \end{pmatrix}.$$

Prove that $AA' = \Delta\,\mathsf{Id}$, where $\Delta \in k$ is the number $ad - bc$. Prove that A is invertible iff $\Delta \neq 0$.

Exercise 7.17 (!). In the previous problem setting prove that Δ equals to A determinant .

Exercise 7.18. Consider a two-dimensional vector space V over $\mathbb{R}$ endowed with a positive bilinear symmetric form. Let A be an orthogonal operator and its matrix in an orthonormal basis has a form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It follows from the Problem 7.14 that $\det A = \pm 1$.

a. Suppose $\det A = 1$. Prove that $b = -c$, $a = d$ and $a^2 + b^2 = 1$.

b. Suppose $\det A = -1$. Prove that $b = c$, $a = -d$ and $a^2 + b^2 = 1$.

Exercise 7.19 (*). Use the statement of the previous problem to describe (in terms of 2x2 matrices) the group of movements of a plane which preserve the origin. Prove that this is a dihedral group (cf. ALGEBRA 1).

Definition 7.8. Consider a matrix (A^i_j). It is said that the matrix (B^i_j) is obtained from (A^i_j) using the **row-wise Gauss transformation**, if $(B^i_j) = (A^i_j) \circ E$, where E is the matrix either of the following form:

$$\begin{pmatrix}
1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
. & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & 0 & \dots & 1 & \dots & \dots & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\dots & \dots & \dots & 1 & \dots & 0 & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & 1 & . \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 1
\end{pmatrix}, \tag{7.1}$$

or of the following form:

$$\begin{pmatrix}
1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
. & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & 1 & \dots & \lambda & \dots & \dots & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\dots & \dots & \dots & 0 & \dots & 1 & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & 1 & . \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 1
\end{pmatrix} \tag{7.2}$$

(dots denote zeroes). If $(B^i_j) = E \circ (A^i_j)$, where E is as above, then one says that (B^i_j) is obtained from (A^i_j) using the **column-wise Gauss transformation**.

Exercise 7.20. Prove that the row-wise Gauss transformation can be described in terms of the following matrix operations: (B^i_j) is obtained from (A^i_j) by permuting of rows or by adding the j-th row multiplied by λ to the i-th. What operations can be used to describe the column-wise Gauss transformation?

Exercise 7.21. Prove that a matrix of the form (7.1) has determinant -1 and that a matrix of the form (7.2) has determinant 1.

Exercise 7.22 (!). Prove that a Gauss transformation of the form (7.2) does not change the determinant but a transformation of the form (7.1) multiplies it by -1.

Definition 7.9. A matrix (A^i_j) is called **upper triangular**, if $A^i_j = 0$ when $i < j$:

$$\begin{pmatrix} * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & * & * & * \\ 0 & 0 & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \\ 0 & 0 & 0 & \dots & 0 & 0 & * \end{pmatrix}.$$

A matrix is called **diagonal**, if $A^i_j = 0$ when $i \neq j$.

Exercise 7.23 (!). Consider an upper triangular matrix (A^i_j) of the size $n \times n$. Prove that $\det(A^i_j)$ equals to the product of all the diagonal coefficients:

$$\det(A^i_j) = \prod_i A^i_i.$$

Exercise 7.24. a. Prove that any matrix can be brought into upper triangular form using row-wise Gauss transformations;

b. Prove that any matrix can be brought into diagonal form using row-wise and column-wise Gauss transformations.

Remark. Since Gauss transformations preserve the determinant (up to ± 1), one can compute the determinant of a square matrix by bringing it to diagonal form and multiplying the coefficients on the diagonal.

Exercise 7.25 (*). Consider a Euclidean ring A (cf. ALGEBRA 2) such that any element $a \in A$ admits decomposition into prime factors. Solve the Problem 7.24 for matrices with elements from A.

Hint. First consider matrices (a^i_j) of the size 1×2, then prove the statement by induction for matrices of the size $1 \times n$ (which is the same as $n \times 1$). Prove that after the matrix is brought into upper triangular form, the only non-zero element will be $\mathrm{GCD}(a^1_1, \dots, a^1_n)$. Consider now an arbitrary matrix of the size $m \times n$ and permute the columns and the rows in such a way that a^1_1 be non-zero. To prove (b) apply the Gauss transformation to rows, columns then once more to rows, once more to columns etc. and obtain a matrix where $a^1_1 \neq 0$ and such that all other elements of the first column and the first row are zeroes.

Grassmann algebra and minors of matrices

Exercise 7.26 (!). Consider a basis $v_1, \dots, v_n$ of a vector space V, then $v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k}$, $i_1 < i_2 < \dots < i_k$ is the corresponding basis in $\Lambda^k(V)$. Consider a matrix $A \in \operatorname{End} V$, and $A(i_1, i_2, \dots, i_k; i'_1, i'_2, \dots i'_k)$, the coefficients of the matrix of the endomorphism induced by A on $\Lambda^k(V)$ in the basis described above. Prove that $A(i_1, i_2, \dots, i_k; i'_1, i'_2, \dots, i'_k)$ is the determinant of the matrix which is obtained from A after all rows except i_1-th, i_2-th, $\dots, i_k$-th and all columns except i'_1-th, i'_2-th, $\dots, i'_k$-th has been removed from it.

Remark. This determinant is called the **minor** of the matrix A.

Hint. Take the composition of A with an operator that maps v_{i_l} to $v_{i'_l}$, and reduce the problem to the case $i_l = i'_l$. Prove that the coefficients $A(i_1, i_2, \dots, i_k; i'_1, i'_2, \dots, i'_k)$ do not depend on rows except the i_1-th, i_2-th, $\dots, i_k$-th rows, and on columns except the i'_1-th, i'_2-th, $\dots, i'_k$-th columns. Then put $A^i_j = 0$ if i and j do not belong to $\{i_1, i_2, \dots, i_k\}$. Thus you have reduced the problem to the case when $V = V_1 \oplus V_2$ and A is of the form $B \oplus 0_{V_2}$ where $B \in \operatorname{End}(V_1)$ and 0_{V_2} acts on V_2 by mapping all vectors to 0. In this situation one can apply the formula $\Lambda^*(V) = \Lambda^*(V_1) \otimes \Lambda^*(V_2)$ to get the desired result.

Definition 7.10. Consider a linear operator $A \in \operatorname{End}(V)$. Consider the endomorphism induced by A on $\Lambda^*(V)$. Consider the biggest number N such that this endomorphism is non-zero on $\Lambda^N(V)$. This number N is called the **rank of the linear operator** A (denoted $\operatorname{rk} A$). If A is represented by a matrix (A^i_j) then $\operatorname{rk} A$ is called the rank of this matrix.

Exercise 7.27 (!). Consider an operator A that induces the zero action on $\Lambda^k(V)$. Prove that A induces the zero action on $\Lambda^l(V)$ for any $l > k$.

Exercise 7.28. Prove that the rank of a matrix is the size of its biggest non-zero minor.

Exercise 7.29. Prove that the rank of an operator A is the biggest number N such that there are vectors $v_1, \dots, v_N$ such that $A(v_1), \dots, A(v_N)$ are linearly independent.

Exercise 7.30 (!). Prove that the rank of an operator A is the dimension of its image.

Exercise 7.31. Consider a matrix of rank 1. Prove that all its rows are proportional. Prove that all its columns are proportional.

Exercise 7.32. Prove that $\operatorname{rk} A = \operatorname{rk} A^*$.

Hint. Use the Problem 7.9.

Definition 7.11. A bilinear form $\mu: V_1 \otimes V_2 \longrightarrow k$ is called **nondegenerate pairing** if for every non-zero $v_1 \in V_1$ there is a vector $v'_1 \in V_2$ such that $\mu(v_1, v'_1) \neq 0$ and for any non-zero $v_2 \in V_2$ there is a vector $v'_2 \in V_1$ such that $\mu(v_2, v'_2) \neq 0$.

Exercise 7.33. Consider finite-dimensional vector spaces V_1, V_2. Prove that a nondegenerate pairing $\mu: V_1 \otimes V_2 \longrightarrow k$ defines an isomorphism between V_1 and V_2^* and any isomorphism between those spaces is defined in this way.

Exercise 7.34 (!). Consider an n-dimensional vector space V. Construct the natural isomorphism

$$\Lambda^k(V)^* \cong \Lambda^{n-k}(V) \otimes \det V^*$$

($\det V$ denotes the one-dimensional vector space $\Lambda^n(V)$).

Hint. Use the previous problem.

Exercise 7.35. Consider an n-dimensional vector space V with the basis $v_1, v_2, \ldots, v_n$ and an operator $A \in \operatorname{End} V$. Consider the basis $w_1, w_2, \ldots, w_n$ in $\Lambda^{n-1}(V)$ where $w_k = v_1 \wedge v_2 \wedge \ldots v_{k-1} \wedge v_{k+1} \wedge \ldots$ (there are all v_i in the product except one). Consider the matrix (A^i_j) of A and consider $\check{A}^i_j$, the minor that is obtained from A after i-th row and j-th column have been removed. Prove that A acts on $\Lambda^{n-1}(V)$ as the matrix $(\check{A}^i_j)$.

Exercise 7.36. In the previous problem setting consider a nondegenerate bilinear pairing

$$V \otimes \Lambda^{n-1}(V) \longrightarrow \det V,$$

defined by the form $v \otimes w \longrightarrow v \wedge w$. Choose the isomorphism $k \cong \det V$ such that $v_1 \wedge v_2 \wedge \cdots \wedge v_n$ is mapped to 1. This gives a nondegenerate pairing defined on V and $\Lambda^{n-1}(V)$. Prove that the basis in $\Lambda^{n-1}(V)$ dual to $v_1, v_2, \ldots, v_n$ is $w_1, -w_2, w_3, -w_4, \ldots$. Prove that A acts on $\Lambda^{n-1}(V)$ by the matrix $((-1)^{i+j}\check{A}^i_j)$ in this basis.

Exercise 7.37. Consider a nondegenerate bilinear pairing $\mu: \ V \otimes V' \longrightarrow k$ and endomorphisms $A \in \operatorname{End} V$ and $B \in \operatorname{End} V'$ such that $\mu(Av, Bv') = \mu(v, v')$ for all $v, v' \in V, V'$. Choose dual bases in V, V' and suppose (α^i_j) and (β^i_j) are the matrices of A and B. Prove that $(\alpha^i_j) \circ (\beta^i_j)^\perp = \mathsf{Id}$.

Exercise 7.38 (!). Consider an $A \in \operatorname{End} V$ where V is an n-dimensional vector space with a basis $v_1, v_2, \ldots, v_n$ and (A^i_j) is the matrix of the operator A. Prove that A is invertible iff $\det A \neq 0$. Prove that

$$A^{-1} = \frac{1}{\det A}((-1)^{i+j}\check{A}^i_j)^\perp.$$

Hint. Prove that for the natural pairing form

$$V \otimes \Lambda^{n-1}(V) \xrightarrow{\mu} \det V,$$

it holds that $\mu(A(v), A(w)) = \det A\mu(v, w)$, where $A(w)$ denote the natural action of A on $\Lambda^{n-1}(V)$. Then use the previous problem for $(A^i_j) = (\alpha^i_j)$, $\frac{1}{\det A}((-1)^i\check{A}^i_j) = (\beta^i_j)^\perp$.

Remark. We have obtained the well-known formula for calculation of the inverse matrix by expansion by minors. The geometric meaning of this formula can be explained as follows: minors of a matrix are (by definition) the matrix coefficients of the action of this matrix on $\Lambda^{n-1}(V)$ and the natural pairing between V and $\Lambda^{n-1}(V)$ is multiplied by $\det A$ by the action of A. This allows for the calculation of A^{-1} using $\det A$ and $\check{A}$.

Calculation of determinant

Exercise 7.39 (!). Consider the matrix (A^i_j) of a linear operator A. Prove that $\det A$ is equal to

$$\sum_{\sigma \in S_n} \mathsf{sgn}(\sigma) A^1_{\sigma_1} A^2_{\sigma_2} \ldots A^n_{\sigma_n}$$

where $(\sigma_1, \sigma_2, \ldots, \sigma_n) \in S_n$ is a permutation, the sum is over the elements of the group of all permutations and sgn is the sign of the permutation σ.

Hint. Use the explicit formula (one that uses the sum over the elements of S_n) from ALGEBRA 6 for the tensor $v_1 \wedge v_2 \wedge \cdots \wedge v_n$

Remark. The determinant is usually defined using this formula.

Exercise 7.40. Consider the matrix (A^i_j) of a linear operator A. Prove that $\det A$ can be calculated as follows:

$$A^1_1 \check{A}^1_1 - A^1_2 \check{A}^1_2 + A^1_3 \check{A}^1_3 \dots$$

where $\check{A}^i_j$ are minors that are obtained after removing the i-the row and j-th column.

Remark. This procedure is called **determinant expansion along a row**.

Exercise 7.41 (*). (Vandermonde determinant) Consider the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ t_1 & t_2 & t_3 & \dots & t_n \\ t_1^2 & t_2^2 & t_3^2 & \dots & t_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ t_1^{n-1} & t_2^{n-1} & t_3^{n-1} & \dots & t_n^{n-1} \end{pmatrix},$$

where $n > 1$. Prove that its determinant is $\prod_{i<j}(t_i - t_j)$.

Exercise 7.42 (*). Consider the matrix

$$\begin{pmatrix} t & x_1 & x_2 & x_3 & \dots & x_n \\ t^2 & x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ t^4 & x_1^4 & x_2^4 & x_3^4 & \dots & x_n^4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ t^{2^n} & x_1^{2^n} & x_2^{2^n} & x_3^{2^n} & \dots & x_n^{2^n} \end{pmatrix},$$

and denote its determinant as $P_n(t, x_1, ...x_n)$. Suppose that this matrix is over the field $\mathbb{Z}/2\mathbb{Z}$. Prove that $P_n(t, x_1, \dots, x_n)$ becomes zero if one takes $t = \sum \alpha_i x_i$ to be an arbitrary linear combination of x_i. Deduce from Bézout's theorem that

$$P_n(t, x_1, \dots, x_n) = Q(x_1, \dots, x_n) \prod (t - \sum \alpha_i x_i),$$

where $\alpha_i \in \mathbb{Z}/2\mathbb{Z}$, and $Q \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ is a polynomial.

Hint. Use long division of P_n by $t - \sum \alpha_i x_i$. If you get a non-zero value, then if you substitute t for $t = \sum \alpha_i x_i$ in $P(t)$ then you will also get a non-zero value.

Exercise 7.43 (*). Prove in the previous problem setting that $Q = P_{n-1}(x_n)$.

Exercise 7.44 (*). Deduce from the previous problem that $Q(x_1, \dots, x_n) \neq 0$.

Exercise 7.45 (*). (Dickson's theorem) Consider the polynomial

$$F_n(t) = \prod (t - \sum \alpha_i x_i) \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n].$$

Prove that

$$F_n(t) = t^{2^n} + \sum_{i=0}^{n-1} c_{n,i} t^{2^i},$$

where $c_{n,i} \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ are polynomials in $x_1, \dots, x_n$.

Hint. Use the previous problem and problem 7.42.

Remark. Polynomials $c_{n,i} \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ are called **Dickson's invariants**.

Exercise 7.46 (*). Consider the coefficients Q_r (which are $c_{n,i}$ according to Dickson's theorem) of the polynomial $F_n(t)$ as elements of the symmetric algebra $S^*(V)$ where V is the vector space over the field $\mathbb{Z}/2\mathbb{Z}$ with basis $x_1, \dots, x_n$. Consider the action of the group $GL(V)$ of invertible linear operators on V and extend it naturally (by multiplicativity) over the symmetric algebra. Prove that Q_r is invariant with respect to $GL(V)$:

$$Q_r(x_1, x_2, \dots, x_n) = Q_r(h(x_1), h(x_2), \dots, h(x_n))$$

where $h \in GL(V)$ is an arbitrary invertible endomorphism.

Remark. Consider the subring of $GL(V)$-invariant polynomials in the polynomials ring $S^*(V)$. Dickson (1911) proved that this ring is the ring of polynomials with generators $c_{n,i}$. Consult

> A PRIMER ON THE DICKSON INVARIANTS, Contemporary Mathematics 19 (1983), 421-434. `http://www.math.purdue.edu/~ wilker/papers/dickson.pdf`

for details.