

ALGEBRA 7: matrices and determinants

We suppose that all vector spaces are vector spaces over a field k .

Exercise 7.1. Let $v_1, \dots, v_n \in V$, $w_1, \dots, w_m \in W$ be bases in vector spaces V and W . Consider a homomorphism e_i^j from V to W that maps v_i to w_j and maps v_k to zero for $k \neq i$. Prove that e_j^i form a basis in the space of homomorphisms $\text{Hom}(V, W)$.

Definition 7.1. In the previous problem setting consider a homomorphism $\gamma \in \text{Hom}(V, W)$. Consider $\gamma = \gamma_j^i e_j^i \in k$. The matrix

$$\begin{pmatrix} \gamma_1^1 & \cdots & \gamma_n^1 \\ \vdots & \ddots & \vdots \\ \gamma_1^m & \cdots & \gamma_n^m \end{pmatrix}$$

is called the **matrix of the homomorphism** γ .

Exercise 7.2. Consider homomorphisms $a \in \text{Hom}(U, V)$, $b \in \text{Hom}(V, W)$ defined by the matrices (a_j^i) , (b_k^j) . Prove that the composition of a and b is defined by the matrix $c_k^i = \sum_j a_j^i b_k^j$.

Remark. Note that the matrix product formula makes sense for matrices of elements of an arbitrary ring.

Exercise 7.3. Consider the space A of square matrices of the size $n \times n$, with the multiplication $A \times A \rightarrow A$ defined by the formula $(a_j^i) \circ (b_l^k) \rightarrow \sum_j a_j^i b_l^j$. Prove that this is an algebra with unit. Prove that this algebra is isomorphic to the algebra of linear operators from k^n to k^n .

Definition 7.2. This algebra is called the **matrix algebra** and is denoted $\text{Mat}(n)$. The unit element of this algebra (the diagonal matrix with $a_i^i = 1$) is called the **identity matrix** and is denoted ld .

Exercise 7.4. Consider a linear operator $f \in \text{Hom}(V, V)$ and let v_1, \dots, v_n be a basis of V and (f_j^i) be the matrix of f . Consider another basis v'_1, \dots, v'_n of V . Prove that there exists a unique operator g that maps v_i to v'_i , and g is invertible. Let (g_j^i) , $((g^{-1})_j^i)$ be the matrices of g and g^{-1} . Prove that f is defined by the matrix $h_j^i := (g_j^i) \circ (f_j^i) \circ ((g^{-1})_j^i)$ in the basis v'_1, \dots, v'_n .

Definition 7.3. In that case the matrices (h_j^i) , (f_j^i) are said to be **equivalent**.

Exercise 7.5. Find all the matrices equivalent to $c \text{ld}$ where $c \in k$.

Exercise 7.6 (!). Consider a matrix $E(i, j)$

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

which has 1 on the position i, j and has 0 everywhere else. What are the values of i, j, i', j' that make the matrices $E(i, j)$ and $E(i', j')$ equivalent?

Exercise 7.7 (!). Consider a matrix A which is equivalent to $E(i, j)$. Prove that all rows of A are proportional. Prove that all columns of A are proportional.

Exercise 7.8 (*). Prove that if all rows and columns of A are proportional, then A is equivalent to $E(i, j)$.

Definition 7.4. Consider a vector space V and an endomorphism $A \in \text{End}(V)$ over it (i.e. a homomorphism from V to itself) and its dual space V^* . An operator $A^* : V^* \rightarrow V^*$ that maps a linear functional $\gamma \in V^*$ to the linear functional $A^*(\gamma)(v) = \gamma(A(v))$ is called a **conjugate operator** for A .

Exercise 7.9. Consider a finite-dimensional vector space V and its dual space V^* . Construct the natural isomorphism between $\Lambda^k(V)^*$ and $\Lambda^k(V^*)$.

Remark. “Natural” means that it does not require any extra choice (choice of base, for example). In this situation, a natural isomorphism $\Lambda^k(V)^* \cong \Lambda^k(V^*)$ is permutable with the standard action of $GL(V)$ on $\Lambda^k(V)^*$, $\Lambda^k(V^*)$. The spaces V and V^* are isomorphic, but one can prove that there is no $GL(V)$ -invariant isomorphism $V \cong V^*$. In other words it is *impossible* to construct a natural homomorphism $V \cong V^*$.

Exercise 7.10 (!). Consider a vector space V , an endomorphism $A \in \text{End}(V)$ and the conjugate operator A^* . Prove that $\det A^* = \det A$.

Hint. Use the previous problem.

Definition 7.5. Consider a square matrix (A_j^i) and a matrix (B_j^i) , that is constructed from (A_j^i) by reflecting it over the diagonal: $B_j^i = A_i^j$. Then (B_j^i) is called the **transposed matrix** of (A_j^i) , and is denoted $(A_j^i)^\perp$.

Exercise 7.11 (!). Consider a basis v_1, \dots, v_n in V and a dual basis v^1, \dots, v^n in V^* (v^i maps v_i to 1 and maps other v_j s to zero). Consider an operator $A \in \text{End}(V)$ and its matrix (A_j^i) . Prove that A^* is given as the matrix $(A_j^i)^\perp$.

Definition 7.6. Consider a nondegenerate bilinear symmetric form g defined on a vector space V . An operator $A \in \text{End}(V)$ is called **orthogonal with respect to g** (or simply **orthogonal**) iff $g(Av, Av) = g(v, v)$ for any $v \in V$.

Exercise 7.12 (!). Prove that any orthogonal operator is invertible.

Exercise 7.13 (!). Consider a linear operator $A \in \text{End}(V)$ on a vector space endowed with a nondegenerate bilinear symmetric form g . Identify V and V^* using g . Then the dual operator A^* can be considered as an endomorphism of V . Prove that a linear operator A is orthogonal iff $A^{-1} = A^*$.

Exercise 7.14 (!). Prove that the determinant of an orthogonal operator equals ± 1 .

Definition 7.7. A nondegenerate bilinear antisymmetric form (see ALGEBRA 3) is called a **symplectic form**.

Exercise 7.15 (*). Consider a vector space V with a symplectic form ω defined on it. An operator $A \in \text{End}(V)$ is called **symplectic**, if it preserves ω , i.e. if $\omega(Av, Av) = \omega(v, v)$. Prove that any symplectic operator has the determinant 1.

Exercise 7.20. Prove that the row-wise Gauss transformation can be described in terms of the following matrix operations: (B_j^i) is obtained from (A_j^i) by permuting of rows or by adding the j -th row multiplied by λ to the i -th. What operations can be used to describe the column-wise Gauss transformation?

Exercise 7.21. Prove that a matrix of the form (7.1) has determinant -1 and that a matrix of the form (7.2) has determinant 1.

Exercise 7.22 (!). Prove that a Gauss transformation of the form (7.2) does not change the determinant but a transformation of the form (7.1) multiplies it by -1.

Definition 7.9. A matrix (A_j^i) is called **upper triangular**, if $A_j^i = 0$ when $i < j$:

$$\begin{pmatrix} * & * & * & \dots & * & * & * \\ 0 & * & * & \dots & * & * & * \\ 0 & 0 & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & * \\ 0 & 0 & 0 & \dots & 0 & * & * \\ 0 & 0 & 0 & \dots & 0 & 0 & * \end{pmatrix}.$$

A matrix is called **diagonal**, if $A_j^i = 0$ when $i \neq j$.

Exercise 7.23 (!). Consider an upper triangular matrix (A_j^i) of the size $n \times n$. Prove that $\det(A_j^i)$ equals to the product of all the diagonal coefficients:

$$\det(A_j^i) = \prod_i A_i^i.$$

Exercise 7.24. a. Prove that any matrix can be brought into upper triangular form using row-wise Gauss transformations;

b. Prove that any matrix can be brought into diagonal form using row-wise and column-wise Gauss transformations.

Remark. Since Gauss transformations preserve the determinant (up to ± 1), one can compute the determinant of a square matrix by bringing it to diagonal form and multiplying the coefficients on the diagonal.

Exercise 7.25 (*). Consider a Euclidean ring A (cf. ALGEBRA 2) such that any element $a \in A$ admits decomposition into prime factors. Solve the Problem 7.24 for matrices with elements from A .

Hint. First consider matrices (a_j^i) of the size 1×2 , then prove the statement by induction for matrices of the size $1 \times n$ (which is the same as $n \times 1$). Prove that after the matrix is brought into upper triangular form, the only non-zero element will be $\text{GCD}(a_1^1, \dots, a_n^1)$. Consider now an arbitrary matrix of the size $m \times n$ and permute the columns and the rows in such a way that a_1^1 be non-zero. To prove (b) apply the Gauss transformation to rows, columns then once more to rows, once more to columns etc. and obtain a matrix where $a_1^1 \neq 0$ and such that all other elements of the first column and the first row are zeroes.

Grassmann algebra and minors of matrices

Exercise 7.26 (!). Consider a basis v_1, \dots, v_n of a vector space V , then $v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k}$, $i_1 < i_2 < \dots < i_k$ is the corresponding basis in $\Lambda^k(V)$. Consider a matrix $A \in \text{End } V$, and $A(i_1, i_2, \dots, i_k; i'_1, i'_2, \dots, i'_k)$, the coefficients of the matrix of the endomorphism induced by A on $\Lambda^k(V)$ in the basis described above. Prove that $A(i_1, i_2, \dots, i_k; i'_1, i'_2, \dots, i'_k)$ is the determinant of the matrix which is obtained from A after all rows except i_1 -th, i_2 -th, \dots , i_k -th and all columns except i'_1 -th, i'_2 -th, \dots , i'_k -th has been removed from it.

Remark. This determinant is called the **minor** of the matrix A .

Hint. Take the composition of A with an operator that maps v_{i_l} to $v_{i'_l}$, and reduce the problem to the case $i_l = i'_l$. Prove that the coefficients $A(i_1, i_2, \dots, i_k; i'_1, i'_2, \dots, i'_k)$ do not depend on rows except the i_1 -th, i_2 -th, \dots , i_k -th rows, and on columns except the i'_1 -th, i'_2 -th, \dots , i'_k -th columns. Then put $A_j^i = 0$ if i and j do not belong to $\{i_1, i_2, \dots, i_k\}$. Thus you have reduced the problem to the case when $V = V_1 \oplus V_2$ and A is of the form $B \oplus 0_{V_2}$ where $B \in \text{End}(V_1)$ and 0_{V_2} acts on V_2 by mapping all vectors to 0. In this situation one can apply the formula $\Lambda^*(V) = \Lambda^*(V_1) \otimes \Lambda^*(V_2)$ to get the desired result.

Definition 7.10. Consider a linear operator $A \in \text{End}(V)$. Consider the endomorphism induced by A on $\Lambda^*(V)$. Consider the biggest number N such that this endomorphism is non-zero on $\Lambda^N(V)$. This number N is called the **rank of the linear operator** A (denoted $\text{rk } A$). If A is represented by a matrix (A_j^i) then $\text{rk } A$ is called the rank of this matrix.

Exercise 7.27 (!). Consider an operator A that induces the zero action on $\Lambda^k(V)$. Prove that A induces the zero action on $\Lambda^l(V)$ for any $l > k$.

Exercise 7.28. Prove that the rank of a matrix is the size of its biggest non-zero minor.

Exercise 7.29. Prove that the rank of an operator A is the biggest number N such that there are vectors v_1, \dots, v_N such that $A(v_1), \dots, A(v_N)$ are linearly independent.

Exercise 7.30 (!). Prove that the rank of an operator A is the dimension of its image.

Exercise 7.31. Consider a matrix of rank 1. Prove that all its rows are proportional. Prove that all its columns are proportional.

Exercise 7.32. Prove that $\text{rk } A = \text{rk } A^*$.

Hint. Use the Problem 7.9.

Definition 7.11. A bilinear form $\mu : V_1 \otimes V_2 \longrightarrow k$ is called **nondegenerate pairing** if for every non-zero $v_1 \in V_1$ there is a vector $v'_1 \in V_2$ such that $\mu(v_1, v'_1) \neq 0$ and for any non-zero $v_2 \in V_2$ there is a vector $v'_2 \in V_1$ such that $\mu(v_2, v'_2) \neq 0$.

Exercise 7.33. Consider finite-dimensional vector spaces V_1, V_2 . Prove that a nondegenerate pairing $\mu : V_1 \otimes V_2 \longrightarrow k$ defines an isomorphism between V_1 and V_2^* and any isomorphism between those spaces is defined in this way.

Exercise 7.34 (!). Consider an n -dimensional vector space V . Construct the natural isomorphism

$$\Lambda^k(V)^* \cong \Lambda^{n-k}(V) \otimes \det V^*$$

($\det V$ denotes the one-dimensional vector space $\Lambda^n(V)$).

Hint. Use the previous problem.

Exercise 7.35. Consider an n -dimensional vector space V with the basis v_1, v_2, \dots, v_n and an operator $A \in \text{End } V$. Consider the basis w_1, w_2, \dots, w_n in $\Lambda^{n-1}(V)$ where $w_k = v_1 \wedge v_2 \wedge \dots \wedge v_{k-1} \wedge v_{k+1} \wedge \dots$ (there are all v_i in the product except one). Consider the matrix (A_j^i) of A and consider \check{A}_j^i , the minor that is obtained from A after i -th row and j -th column have been removed. Prove that A acts on $\Lambda^{n-1}(V)$ as the matrix (\check{A}_j^i) .

Exercise 7.36. In the previous problem setting consider a nondegenerate bilinear pairing

$$V \otimes \Lambda^{n-1}(V) \longrightarrow \det V,$$

defined by the form $v \otimes w \longrightarrow v \wedge w$. Choose the isomorphism $k \cong \det V$ such that $v_1 \wedge v_2 \wedge \dots \wedge v_n$ is mapped to 1. This gives a nondegenerate pairing defined on V and $\Lambda^{n-1}(V)$. Prove that the basis in $\Lambda^{n-1}(V)$ dual to v_1, v_2, \dots, v_n is $w_1, -w_2, w_3, -w_4, \dots$. Prove that A acts on $\Lambda^{n-1}(V)$ by the matrix $((-1)^{i+j} \check{A}_j^i)$ in this basis.

Exercise 7.37. Consider a nondegenerate bilinear pairing $\mu : V \otimes V' \longrightarrow k$ and endomorphisms $A \in \text{End } V$ and $B \in \text{End } V'$ such that $\mu(Av, Bv') = \mu(v, v')$ for all $v, v' \in V, V'$. Choose dual bases in V, V' and suppose (α_j^i) and (β_j^i) are the matrices of A and B . Prove that $(\alpha_j^i) \circ (\beta_j^i)^\perp = \text{Id}$.

Exercise 7.38 (!). Consider an $A \in \text{End } V$ where V is an n -dimensional vector space with a basis v_1, v_2, \dots, v_n and (A_j^i) is the matrix of the operator A . Prove that A is invertible iff $\det A \neq 0$. Prove that

$$A^{-1} = \frac{1}{\det A} ((-1)^{i+j} \check{A}_j^i)^\perp.$$

Hint. Prove that for the natural pairing form

$$V \otimes \Lambda^{n-1}(V) \xrightarrow{\mu} \det V,$$

it holds that $\mu(A(v), A(w)) = \det A \mu(v, w)$, where $A(w)$ denote the natural action of A on $\Lambda^{n-1}(V)$. Then use the previous problem for $(A_j^i) = (\alpha_j^i)$, $\frac{1}{\det A} ((-1)^i \check{A}_j^i) = (\beta_j^i)^\perp$.

Remark. We have obtained the well-known formula for calculation of the inverse matrix by expansion by minors. The geometric meaning of this formula can be explained as follows: minors of a matrix are (by definition) the matrix coefficients of the action of this matrix on $\Lambda^{n-1}(V)$ and the natural pairing between V and $\Lambda^{n-1}(V)$ is multiplied by $\det A$ by the action of A . This allows for the calculation of A^{-1} using $\det A$ and \check{A} .

Calculation of determinant

Exercise 7.39 (!). Consider the matrix (A_j^i) of a linear operator A . Prove that $\det A$ is equal to

$$\sum_{\sigma \in S_n} \text{sgn}(\sigma) A_{\sigma_1}^1 A_{\sigma_2}^2 \dots A_{\sigma_n}^n$$

where $(\sigma_1, \sigma_2, \dots, \sigma_n) \in S_n$ is a permutation, the sum is over the elements of the group of all permutations and sgn is the sign of the permutation σ .

Hint. Use the explicit formula (one that uses the sum over the elements of S_n) from ALGEBRA 6 for the tensor $v_1 \wedge v_2 \wedge \cdots \wedge v_n$

Remark. The determinant is usually defined using this formula.

Exercise 7.40. Consider the matrix (A_j^i) of a linear operator A . Prove that $\det A$ can be calculated as follows:

$$A_1^1 \check{A}_1^1 - A_2^1 \check{A}_2^1 + A_3^1 \check{A}_3^1 \dots$$

where \check{A}_j^i are minors that are obtained after removing the i -th row and j -th column.

Remark. This procedure is called **determinant expansion along a row**.

Exercise 7.41 (*). (Vandermonde determinant) Consider the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ t_1 & t_2 & t_3 & \dots & t_n \\ t_1^2 & t_2^2 & t_3^2 & \dots & t_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ t_1^{n-1} & t_2^{n-1} & t_3^{n-1} & \dots & t_n^{n-1} \end{pmatrix},$$

where $n > 1$. Prove that its determinant is $\prod_{i < j} (t_i - t_j)$.

Exercise 7.42 (*). Consider the matrix

$$\begin{pmatrix} t & x_1 & x_2 & x_3 & \dots & x_n \\ t^2 & x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ t^4 & x_1^4 & x_2^4 & x_3^4 & \dots & x_n^4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ t^{2^n} & x_1^{2^n} & x_2^{2^n} & x_3^{2^n} & \dots & x_n^{2^n} \end{pmatrix},$$

and denote its determinant as $P_n(t, x_1, \dots, x_n)$. Suppose that this matrix is over the field $\mathbb{Z}/2\mathbb{Z}$. Prove that $P_n(t, x_1, \dots, x_n)$ becomes zero if one takes $t = \sum \alpha_i x_i$ to be an arbitrary linear combination of x_i . Deduce from Bézout's theorem that

$$P_n(t, x_1, \dots, x_n) = Q(x_1, \dots, x_n) \prod (t - \sum \alpha_i x_i),$$

where $\alpha_i \in \mathbb{Z}/2\mathbb{Z}$, and $Q \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ is a polynomial.

Hint. Use long division of P_n by $t - \sum \alpha_i x_i$. If you get a non-zero value, then if you substitute t for $t = \sum \alpha_i x_i$ in $P(t)$ then you will also get a non-zero value.

Exercise 7.43 (*). Prove in the previous problem setting that $Q = P_{n-1}(x_n)$.

Exercise 7.44 (*). Deduce from the previous problem that $Q(x_1, \dots, x_n) \neq 0$.

Exercise 7.45 (*). (Dickson's theorem) Consider the polynomial

$$F_n(t) = \prod (t - \sum \alpha_i x_i) \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n].$$

Prove that

$$F_n(t) = t^{2^n} + \sum_{i=0}^{n-1} c_{n,i} t^{2^i},$$

where $c_{n,i} \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ are polynomials in x_1, \dots, x_n .

Hint. Use the previous problem and problem 7.42.

Remark. Polynomials $c_{n,i} \in \mathbb{Z}/2\mathbb{Z}[x_1, \dots, x_n]$ are called **Dickson's invariants**.

Exercise 7.46 (*). Consider the coefficients Q_r (which are $c_{n,i}$ according to Dickson's theorem) of the polynomial $F_n(t)$ as elements of the symmetric algebra $S^*(V)$ where V is the vector space over the field $\mathbb{Z}/2\mathbb{Z}$ with basis x_1, \dots, x_n . Consider the action of the group $GL(V)$ of invertible linear operators on V and extend it naturally (by multiplicativity) over the symmetric algebra. Prove that Q_r is invariant with respect to $GL(V)$:

$$Q_r(x_1, x_2, \dots, x_n) = Q_r(h(x_1), h(x_2), \dots, h(x_n))$$

where $h \in GL(V)$ is an arbitrary invertible endomorphism.

Remark. Consider the subring of $GL(V)$ -invariant polynomials in the polynomials ring $S^*(V)$. Dickson (1911) proved that this ring is the ring of polynomials with generators $c_{n,i}$. Consult

A PRIMER ON THE DICKSON INVARIANTS, Contemporary Mathematics 19 (1983), 421-434. <http://www.math.purdue.edu/~wilker/papers/dickson.pdf>

for details.