ALGEBRA 8: Linear algebra: characteristic polynomial

Characteristic polynomial

Definition 8.1. Consider a linear operator $A \in \operatorname{End} V$ over a vector space V. Consider a vector $v \in V$ such that $A(v) = \lambda v$. This vector is called an **eigenvector** and λ is called an **eigenvalue** of the operator A.

Exercise 8.1. Consider a 2-dimensional vector space V over $\mathbb{R}$, endowed with non-degenerate bilinear symmetric form g, nd let $A \in \operatorname{End} V$ be an orthogonal automorphism that is not equal to $\pm Id$. Prove that if g is positive definite or negative definite (such forms are called **definite forms**) then A does not have eigenvectors. Prove that if g is not definite then A has two linearly independent eigenvectors. What eigenvalues can A have in that case?

Exercise 8.2. Consider a set of fractions on the form $\frac{P(t)}{Q(t)}$ where P, Q are polynomials over k and $Q \neq 0$. Consider an equivalence relation generated by the relation defined as follows: $\frac{P(t)}{Q(t)} \sim \frac{P'(t)}{Q'(t)}$, if

$$P(t) = Z(t)P'(t), \qquad Q(t) = Z(t)Q'(t)$$

Define addition and multiplication on equivalence classes in the usual manner:

$$\frac{P(t)}{Q(t)} + \frac{P'(t)}{Q'(t)} = \frac{P(t)Q'(t) + P'(t)Q(t)}{Q(t)Q'(t)}, \qquad \frac{P(t)}{Q(t)}\frac{P'(t)}{Q'(t)} = \frac{P(t)P'(t))}{Q(t)Q'(t)}$$

Prove that this structure is a field.

Definition 8.2. This field is called the **field of rational functions of one variable** or just the **field of rational fractions**. It is denoted $k(t)$.

Exercise 8.3. Prove that this field is not an algebraic extension of k.

Exercise 8.4. Consider a n-dimensional vector space V over k and some other field $K \supset k$. Consider the tensor product $K \otimes_k V$ endowed with the natural action of the multiplicative group K^*. Prove that this is a vector space. Prove that this vector space is finite-dimensional over K if V is finite-dimensional over k. Find the dimension of $K \otimes_k V$ over K assuming the dimension of V over k is known.

Consider a vector space V over k and a linear operator $A \in \operatorname{End} V$ on it. Consider the tensor product of V by the vector space $k(t)$ over k, $V \otimes_k k(t)$. The A action can be naturally extended to a linear operator on $V \otimes k(t)$. We will abuse the notation and denote the corresponding operator $A \in \operatorname{End}_{k(t)}(V \otimes_k k(t))$ as A.

Exercise 8.5 (!). Consider a linear operator $A \in \operatorname{End} V$ on a n-dimensional vector space V over k, and let $\det(t \cdot Id - A) \in k(t)$ be the determinant of the operator $t \cdot Id - A$ that acts on $V \otimes_k k(t)$. Prove that this is a polynomial over k of degree n with the leading coefficient 1.

Definition 8.3. This polynomial is called the **characteristic polynomial of the operator** A and is denoted $\mathsf{Chpoly}_A(t)$.

Exercise 8.6 (!). Let λ be a root of the characteristic polynomial of A. Prove that it is an eigenvalue of A. Prove that all A eigenvalues are the roots of $\mathsf{Chpoly}_A(t)$.

Hint. An operator $\lambda Id - A$ has a non-trivial kernel iff λ is a root of $\mathsf{Chpoly}_A(t)$.

Exercise 8.7. Consider eigenvectors $v_1, \dots, v_n$ that correspond to distinct eigenvalues. Prove that $v_1, \dots, v_n$ are linearly independent.

Exercise 8.8. Consider a linear operator $A \in \operatorname{End} V$ on a n-dimensional vector space. Suppose that the characteristic polynomial has n distinct roots. Prove that A is **diagonalisable**, that is its matrix is diagonal in some basis.

Exercise 8.9 (*). Consider a finite-dimensional vector space V over $\mathbb{C}$. Consider the set of all linear operators on V as a vector space with the natural topology on it. Prove that the set of diagonalisable operators is dense in $\operatorname{End} V$. Prove that the set of non-diagonalisable operators is nowhere dense.

Exercise 8.10 (!). Prove that $\mathsf{Chpoly}_A(t) = \mathsf{Chpoly}_{BAB^{-1}}(t)$ for any invertible linear operator B.

Definition 8.4. Consider a linear operator $A \in \operatorname{End} V$ on an n-dimensional vector space and his characteristic polynomial $\mathsf{Chpoly}_A(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \dots$. The coefficient a_{n-1} is called the **trace** of A and is denoted $\mathsf{tr}\, A$.

Exercise 8.11 (!). Consider an operator A defined by a matrix A^i_j. Prove that $\mathsf{tr}\, A = \sum A^i_i$ (the sum of all numbers standing on the diagonal of the matrix).

Exercise 8.12 (*). Prove that $\mathsf{tr}\, AB = \mathsf{tr}\, BA$ for any linear operators A, B.

Remark. If B is invertible, this follows from 8.10.

Exercise 8.13. Consider a finite-dimensional vector space V. Consider the homomorphism $V \otimes V^* \longrightarrow \operatorname{Hom}(V, V)$ that maps $v \otimes \lambda \in V \otimes V^*$ to $v' \longrightarrow \lambda(v') \otimes v \in \operatorname{Hom}(V, V)$. Prove that it is an isomorphism.

Exercise 8.14 (*). Consider $A \in \operatorname{End} V$ a linear operator on a finite-dimensional vector space and $A \otimes A^*$, an operator induced by A on $V \otimes V^*$. Consider the tensor $\mathsf{Id} \in V \otimes V^*$ that corresponds to the identity operator under the isomorphism $\operatorname{Hom}(V, V) \cong V \otimes V^*$ and the natural pairing $V \otimes V^* \xrightarrow{\mu} k$. Prove that $\mathsf{tr}\, A = \mu(A \otimes A^*(\mathsf{Id}))$.

Upper triangular matrices

Exercise 8.15. Let $V' \subset V$ b k-dimensional subspace of a vector space and $A \in \operatorname{End} V$ be an operator that preserves V' (that is, A maps V' to itself). Choose a basis $e_1, \dots, e_n$ in V such that $e_1, \dots, e_k \in V'$. Prove that A has the following form in this basis:

$$\begin{pmatrix} * & * & * & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & * & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * & * \end{pmatrix}.$$

(lower left rectangle $k \times (n-k)$ is filled with zeroes and other coefficients are arbitrary).

Definition 8.5. Consider an n-dimensional vector space V. A sequence of subspaces $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ is called a **flag** (or a **full flag**), if $\dim V_i = i$. The basis $e_1, \ldots, e_n$ is called **adapted to the flab**, if $e_i \in V_i$. We say that a linear operator $A \in \operatorname{End} V$ **preserves the flag** $\{V_i\}$, if $A(V_i) \subset V_i$.

Exercise 8.16 (!). Let $A \in \operatorname{End} V$ be a linear operator. Prove that A preserves some flag $\{V_i\}$ iff A can be represented by an upper-triangular matrix in in a basis $e_1, \ldots, e_n$ adapted to $\{V_i\}$.

Exercise 8.17 (!). Let V be a vector space over an algebraically closed field. Prove that $A \in \operatorname{End} V$ preserves a flag $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V$ (and consequently can be represented by an upper-triangular matrix in some basis).

Hint. Take as V_1 a vector subspace spanned by an eigenvector and apply induction.

Exercise 8.18 (*). Consider an invertible linear operator $A \in \operatorname{End} V$ on an n-dimensional space that has n pairwise disjoint eigenvalues. Consider a subalgebra R_A in $\operatorname{End} V$ generated by A. Prove that $\dim R_A = n$.

Hint. Use the Vandermonde determinant.

Exercise 8.19 (*). Consider two commuting linear operators. Prove that the can be represented by two upper-triangular matrices in the same basis $e_1, \ldots, e_n$.

Exercise 8.20 (*). Consider l pairwise commuting linear operators. Prove that they all can be represented by upper-triangular matrices in the same basis $e_1, \ldots, e_n$.

Symmetric and skew-symmetric matrices

Definition 8.6. A matrix is called **symmetric** if it is equal to its transpose: $A = A^\perp$. A matrix is called **skew-symmetric**, or **antisymmetric**, if $A = -A^\perp$.

Definition 8.7. Consider a vector space V together with a non-degenerate bilinear symmetric form g and a linear operator $A \in \operatorname{End} V$. The operator A is called **symmetric** if for any $x, y \in V$ we have $g(Ax, y) = g(x, Ay)$; it is called **skew-symmetric**, if we have $g(Ax, y) = -g(x, Ay)$.

Definition 8.8. Let V be a vector space endowed with a non-degenerate bilinear symmetric form g. Recall that a basis $e_1, \ldots, e_n \in V$ is called **orthonormal** if e_i-s are pairwise orthogonal and $g(e_i, e_i) = 1$.

Exercise 8.21. Let V be a vector space endowed with a non-degenerate bilinear symmetric form g and $e_1, \ldots, e_n$ be an orthonormal basis. Consider a linear operator $A \in \operatorname{End} V$. Prove that A is symmetric iff its matrix is symmetric, and antisymmetric iff its matrix is antisymmetric.

Exercise 8.22. Let V be a finite-dimensional vector space endowed with a bilinear non-degenerate form g. Prove that any bilinear form can be a represented as $g(Ax, y)$ for some linear operator A and that such an operator is unique.

Remark. In the previous problem setting assume that g is symmetric. Obviously, The form $g(Ax, y)$ is symmetric iff A is symmetric, and antisymmetric iff A is antisymmetric.

Exercise 8.23. Let V be a finite-dimensional vector space. The space of bilinear forms is naturally isomorphic to $V^* \otimes V^*$ and the space $\operatorname{End} V$ is naturally isomorphic to $V \otimes V^*$. A form g induces an isomorphism between V and V^*. This gives an isomorphism between $V^* \otimes V^*$ and $V \otimes V^*$, i.e. between the bilinear forms and the linear operators. Prove that this isomorphism coincides with the one constructed in the Problem 8.22.

Exercise 8.24 (!). Let V be a finite-dimensional vector space endowed with non-degenerate bilinear symmetric form g and let A by a symmetric operator. Suppose A preserve the subspace $V' \subset V$. Prove that A preserves the orthogonal complement to V'.

Definition 8.9. Let V be a vector space over $\mathbb{R}$ and $V \otimes \mathbb{C}$ it the tensor product of the latter with $\mathbb{C}$. Since $\mathbb{C} \cong \mathbb{R} \oplus \sqrt{-1}\,\mathbb{R}$, there is an isomorphism $V \otimes \mathbb{C} \cong V \oplus \sqrt{-1}\,V$. That means that one can consider a **real** ($\operatorname{Re} v$) and **imaginary** part ($\operatorname{Im} v$) of any vector $v \in V \otimes \mathbb{C}$.

Exercise 8.25. Let V be a vector space over $\mathbb{R}$ endowed with a bilinear symmetric form g. Consider a complex vector space $V \otimes \mathbb{C}$ and continue g to $V \otimes \mathbb{C}$ using the linearity of the bilinear complex-valued form. For any vector $v \in V \otimes \mathbb{C}$ denote by $\overline{v}$ the vector $\operatorname{Re}(v) - \sqrt{-1}\operatorname{Im}(v)$ (this vector is called the **complex conjugate to** v). Prove that $g(v, \overline{v}) = g(\operatorname{Re}(v), \operatorname{Re}(v)) + g(\operatorname{Im}(v), \operatorname{Im}(v))$.

Exercise 8.26 (!). Let V by a finite-dimensional vector space over $\mathbb{R}$ of dimension n endowed with a positive definite bilinear symmetric form g (such space is called **Euclidean**), and let A be a symmetric operator and $P(t)$ be his characteristic polynomial. Prove that $P(t)$ has exactly n real roots.

Hint. Consider the action of A on $V \otimes \mathbb{C}$, and let v be the eigenvector corresponding to a non-real eigenvalue. Prove that $g(v, \overline{v}) = 0$. Use the Problem 8.25.

Exercise 8.27 (!). Let V be a Euclidean space and $A \in V$ be a symmetric operator. Prove that V has an orthogonal basis of eigenvectors of A. In other words, A is diagonalisable in an orthonormal basis.

Hint. Use the Problems 8.26 and 8.24.

Exercise 8.28 (*). Let V be a finite-dimensional vector space over $\mathbb{R}$ endowed with a non-degenerate but non necessary positive definite bilinear symmetric form. Is any symmetric operator diagonalisable?

Exercise 8.29 (*). Let V be a Euclidean space and $A \in V$ be a skew-symmetric operator. Denote by ω the skew-symmetric form $g(A\cdot, \cdot)$. Let v be an eigenvector of the operator A^2 (with a non-zero eigenvalue). Prove that ω is non-degenerate on the linear span $\langle v, A(v) \rangle$.

Exercise 8.30 (*). In the previous problem setting prove that in some orthonormal basis $e_1, \ldots, e_{2m}, e_{2m+1}$ ω is of the form

$$\sum_{i=0}^{m-1} \alpha_i e^{i+1} \wedge e^{i+2}.$$

Exercise 8.31 (*). Let A be a skew-symmetric operator defined on a Euclidean space and $\det A$ be its determinant. Consider $\det A$ as a polynomial of matrix coefficients of A in some basis. Prove that in a odd-dimensional space V this determinant polynomial is identically zero. Prove that $\det A$ is a full square of some other polynomial of matrix coefficients. This polynomial is called the **Pfaffian of** A.

Hint. Let $2m = \dim V$. Consider the bilinear form ω represented in the form above. Prove that ω^m (considered as an element of the Grassmann algebra $\Lambda^*(V^*)$) is proportional to $e^1 \wedge e^2 \wedge \cdots \wedge e^{2m}$ with a polynomial coefficient Q, moreover $Q^2 = \det A$.