ALGEBRA 8: Linear algebra: characteristic polynomial

Characteristic polynomial

Definition 8.1. Consider a linear operator $A \in \text{End} V$ over a vector space $V$. Consider a vector $v \in V$ such that $A(v) = \lambda v$. This vector is called an eigenvector and $\lambda$ is called an eigenvalue of the operator $A$.

Exercise 8.1. Consider a 2-dimensional vector space $V$ over $\mathbb{R}$, endowed with non-degenerate bilinear symmetric form $g$, and let $A \in \text{End} V$ be an orthogonal automorphism that is not equal to $\pm \text{Id}$. Prove that if $g$ is positive definite or negative definite (such forms are called definite forms) then $A$ does not have eigenvectors. Prove that if $g$ is not definite then $A$ has two linearly independent eigenvectors. What eigenvalues can $A$ have in that case?

Exercise 8.2. Consider a set of fractions on the form $\frac{P(t)}{Q(t)}$ where $P, Q$ are polynomials over $k$ and $Q \neq 0$. Consider an equivalence relation generated by the relation defined as follows: $\frac{P(t)}{Q(t)} \sim \frac{P'(t)}{Q'(t)}$, if

$$P(t) = Z(t)P'(t), \quad Q(t) = Z(t)Q'(t)$$

Define addition and multiplication on equivalence classes in the usual manner:

$$\frac{P(t)}{Q(t)} + \frac{P'(t)}{Q'(t)} = \frac{P(t)Q'(t) + P'(t)Q(t)}{Q(t)Q'(t)}, \quad \frac{P(t)}{Q(t)} \cdot \frac{P'(t)}{Q'(t)} = \frac{P(t)P'(t)}{Q(t)Q'(t)}$$

Prove that this structure is a field.

Definition 8.2. This field is called the field of rational functions of one variable or just the field of rational fractions. It is denoted $k(t)$.

Exercise 8.3. Prove that this field is not an algebraic extension of $k$.

Exercise 8.4. Consider a $n$-dimensional vector space $V$ over $k$ and some other field $K \ni k$. Consider the tensor product $K \otimes_k V$ endowed with the natural action of the multiplicative group $K^*$. Prove that this is a vector space. Prove that this vector space is finite-dimensional over $K$ if $V$ is finite-dimensional over $k$. Find the dimension of $K \otimes_k V$ over $K$ assuming the dimension of $V$ over $k$ is known.

Consider a vector space $V$ over $k$ and a linear operator $A \in \text{End} V$ on it. Consider the tensor product of $V$ by the vector space $k(t)$ over $k$, $V \otimes_k k(t)$. The $A$ action can be naturally extended to a linear operator on $V \otimes k(t)$. We will abuse the notation and denote the corresponding operator $A \in \text{End}_{k(t)}(V \otimes_k k(t))$ as $A$.

Exercise 8.5 (!). Consider a linear operator $A \in \text{End} V$ on a $n$-dimensional vector space $V$ over $k$, and let $\det(t \cdot \text{Id} - A) \in k(t)$ be the determinant of the operator $t \cdot \text{Id} - A$ that acts on $V \otimes_k k(t)$. Prove that this is a polynomial over $k$ of degree $n$ with the leading coefficient 1.

Definition 8.3. This polynomial is called the characteristic polynomial of the operator $A$ and is denoted $\text{Chpoly}_A(t)$.

Exercise 8.6 (!). Let $\lambda$ be a root of the characteristic polynomial of $A$. Prove that it is an eigenvalue of $A$. Prove that all $A$ eigenvalues are the roots of $\text{Chpoly}_A(t)$.
Hint. An operator $\lambda Id - A$ has a non-trivial kernel iff $\lambda$ is a root of $\text{Chpoly}_A(t)$.

Exercise 8.7. Consider eigenvectors $v_1, \ldots, v_n$ that correspond to distinct eigenvalues. Prove that $v_1, \ldots, v_n$ are linearly independent.

Exercise 8.8. Consider a linear operator $A \in \text{End} V$ on a $n$-dimensional vector space. Suppose that the characteristic polynomial has $n$ distinct roots. Prove that $A$ is diagonalisable, that is its matrix is diagonal in some basis.

Exercise 8.9 (*). Consider a finite-dimensional vector space $V$ over $\mathbb{C}$. Consider the set of all linear operators on $V$ as a vector space with the natural topology on it. Prove that the set of diagonalisable operators is dense in $\text{End} V$. Prove that the set of non-diagonalisable operators is nowhere dense.

Exercise 8.10 (!). Prove that $\text{Chpoly}_A(t) = \text{Chpoly}_{B A B^{-1}}(t)$ for any invertible linear operator $B$.

Definition 8.4. Consider a linear operator $A \in \text{End} V$ on an $n$-dimensional vector space and his characteristic polynomial $\text{Chpoly}_A(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots$. The coefficient $a_{n-1}$ is called the trace of $A$ and is denoted $\text{tr} A$.

Exercise 8.11 (!). Consider an operator $A$ defined by a matrix $A_{ij}$. Prove that $\text{tr} A = \sum A_{ii}$ (the sum of all numbers standing on the diagonal of the matrix).

Exercise 8.12 (*). Prove that $\text{tr} AB = \text{tr} BA$ for any linear operators $A$, $B$.

Remark. If $B$ is invertible, this follows from 8.10.

Exercise 8.13. Consider a finite-dimensional vector space $V$. Consider the homomorphism $V \otimes V^* \rightarrow \text{Hom}(V, V)$ that maps $v \otimes \lambda \in V \otimes V^*$ to $v' \rightarrow \lambda(v') \otimes v \in \text{Hom}(V, V)$. Prove that it is an isomorphism.

Exercise 8.14 (*). Consider $A \in \text{End} V$ a linear operator on a finite-dimensional vector space and $A \otimes A^*$, an operator induced by $A$ on $V \otimes V^*$. Consider the tensor $\text{Id} \in V \otimes V^*$ that corresponds to the identity operator under the isomorphism $\text{Hom}(V, V) \cong V \otimes V^*$ and the natural pairing $V \otimes V^* \rightarrow k$. Prove that $\text{tr} A = \mu(A \otimes A^*(\text{Id}))$.

Upper triangular matrices

Exercise 8.15. Let $V' \subset V$ a $k$-dimensional subspace of a vector space and $A \in \text{End} V$ be an operator that preserves $V'$ (that is, $A$ maps $V'$ to itself). Choose a basis $e_1, \ldots, e_n$ in $V$ such that $e_1, \ldots, e_k \in V'$. Prove that $A$ has the following form in this basis:

$$
\begin{pmatrix}
* & * & * & \ldots & * & * & * \\
: & : & : & \ddots & : & : & : \\
* & * & * & \ldots & * & * & * \\
0 & 0 & 0 & \ldots & * & * & * \\
: & : & : & \ddots & : & : & : \\
0 & 0 & 0 & \ldots & * & * & * 
\end{pmatrix}
$$

(lower left rectangle $k \times (n - k)$ is filled with zeroes and other coefficients are arbitrary).
Definition 8.5. Consider an \( n \)-dimensional vector space \( V \). A sequence of subspaces \( 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \) is called a flag (or a full flag), if \( \dim V_i = i \). The basis \( e_1, \ldots, e_n \) is called adapted to the flag, if \( e_i \in V_i \). We say that a linear operator \( A \in \text{End} \ V \) preserves the flag \( \{V_i\} \), if \( A(V_i) \subset V_i \).

Exercise 8.16 (!). Let \( A \in \text{End} \ V \) be a linear operator. Prove that \( A \) preserves some flag \( \{V_i\} \) iff \( A \) can be represented by an upper-triangular matrix in in a basis \( e_1, \ldots, e_n \) adapted to \( \{V_i\} \).

Exercise 8.17 (!). Let \( V \) be a vector space over an algebraically closed field. Prove that \( A \in \text{End} \ V \) preserves a flag \( 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V \) (and consequently can be represented by an upper-triangular matrix in some basis).

Hint. Take as \( V_1 \) a vector subspace spanned by an eigenvector and apply induction.

Exercise 8.18 (*). Consider an invertible linear operator \( A \in \text{End} \ V \) on an \( n \)-dimensional space that has \( n \) pairwise disjoint eigenvalues. Consider a subalgebra \( R_A \) in \( \text{End} \ V \) generated by \( A \). Prove that \( \dim R_A = n \).

Hint. Use the Vandermonde determinant.

Exercise 8.19 (*). Consider two commuting linear operators. Prove that the can be represented by two upper-triangular matrices in the same basis \( e_1, \ldots, e_n \).

Exercise 8.20 (*). Consider \( l \) pairwise commuting linear operators. Prove that they all can be represented by upper-triangular matrices in the same basis \( e_1, \ldots, e_n \).

Symmetric and skew-symmetric matrices

Definition 8.6. A matrix is called symmetric if it is equal to its transpose: \( A = A^\top \). A matrix is called skew-symmetric, or antisymmetric, if \( A = -A^\top \).

Definition 8.7. Consider a vector space \( V \) together with a non-degenerate bilinear symmetric form \( g \) and a linear operator \( A \in \text{End} \ V \). The operator \( A \) is called symmetric if for any \( x, y \in V \) we have \( g(Ax, y) = g(x, Ay) \); it is called skew-symmetric, if we have \( g(Ax, y) = -g(x, Ay) \).

Definition 8.8. Let \( V \) be a vector space endowed with a non-degenerate bilinear symmetric form \( g \). Recall that a basis \( e_1, \ldots, e_n \) in \( V \) is called orthonormal if \( e_i \)-s are pairwise orthogonal and \( g(e_i, e_i) = 1 \).

Exercise 8.21. Let \( V \) be a vector space endowed with a non-degenerate bilinear symmetric form \( g \) and \( e_1, \ldots, e_n \) be an orthonormal basis. Consider a linear operator \( A \in \text{End} \ V \). Prove that \( A \) is symmetric iff its matrix is symmetric, and antisymmetric iff its matrix is antisymmetric.

Exercise 8.22. Let \( V \) be a finite-dimensional vector space endowed with a bilinear non-degenerate form \( g \). Prove that any bilinear form can be a represented as \( g(Ax, y) \) for some linear operator \( A \) and that such an operator is unique.

Remark. In the previous problem setting assume that \( g \) is symmetric. Obviously, The form \( g(Ax, y) \) is symmetric iff \( A \) is symmetric, and antisymmetric iff \( A \) is antisymmetric.
Exercise 8.23. Let $V$ be a finite-dimensional vector space. The space of bilinear forms is naturally isomorphic to $V^* \otimes V^*$ and the space $\text{End} V$ is naturally isomorphic to $V \otimes V^*$. A form $g$ induces an isomorphism between $V$ and $V^*$. This gives an isomorphism between $V^* \otimes V^*$ and $V \otimes V^*$, i.e. between the bilinear forms and the linear operators. Prove that this isomorphism coincides with the one constructed in the Problem 8.22.

Exercise 8.24 (!). Let $V$ be a finite-dimensional vector space endowed with non-degenerate bilinear symmetric form $g$ and let $A$ by a symmetric operator. Suppose $A$ preserve the subspace $V' \subset V$. Prove that $A$ preserves the orthogonal complement to $V'$.

Definition 8.9. Let $V$ be a vector space over $\mathbb{R}$ and $V \otimes \mathbb{C}$ it the tensor product of the latter with $\mathbb{C}$. Since $\mathbb{C} \cong \mathbb{R} \oplus \sqrt{-1} \mathbb{R}$, there is an isomorphism $V \otimes \mathbb{C} \cong V \oplus \sqrt{-1} V$. That means that one can consider a real $(\text{Re} v)$ and imaginary part $(\text{Im} v)$ of any vector $v \in V \otimes \mathbb{C}$.

Exercise 8.25. Let $V$ be a vector space over $\mathbb{R}$ endowed with a bilinear symmetric form $g$. Consider a complex vector space $V \otimes \mathbb{C}$ and continue $g$ to $V \otimes \mathbb{C}$ using the linearity of the bilinear complex-valued form. For any vector $v \in V \otimes \mathbb{C}$ denote by $\overline{v}$ the vector $\text{Re}(v) - \sqrt{-1} \text{Im}(v)$ (this vector is called the complex conjugate to $v$). Prove that $g(v, \overline{v}) = g(\text{Re}(v), \text{Re}(v)) + g(\text{Im}(v), \text{Im}(v))$.

Exercise 8.26 (!). Let $V$ by a finite-dimensional vector space over $\mathbb{R}$ of dimension $n$ endowed with a positive definite bilinear symmetric form $g$ (such space is called Euclidean), and let $A$ be a symmetric operator and $P(t)$ be his characteristic polynomial. Prove that $P(t)$ has exactly $n$ real roots.

Hint. Consider the action of $A$ on $V \otimes \mathbb{C}$, and let $v$ be the eigenvector corresponding to a non-real eigenvalue. Prove that $g(v, \overline{v}) = 0$. Use the Problem 8.25.

Exercise 8.27 (!). Let $V$ be a Euclidean space and $A \in V$ be a symmetric operator. Prove that $V$ has an orthogonal basis of eigenvectors of $A$. In other words, $A$ is diagonalisable in an orthonormal basis.

Hint. Use the Problems 8.26 and 8.24.

Exercise 8.28 (*). Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ endowed with a non-degenerate but non necessary positive definite bilinear symmetric form. Is any symmetric operator diagonalisable?

Exercise 8.29 (*). Let $V$ be a Euclidean space and $A \in V$ be a skew-symmetric operator. Denote by $\omega$ the skew-symmetric form $g(A, \cdot)$. Let $v$ be an eigenvector of the operator $A^2$ (with a non-zero eigenvalue). Prove that $\omega$ is non-degenerate on the linear span $\langle v, A(v) \rangle$.

Exercise 8.30 (*). In the previous problem setting prove that in some orthonormal basis $e_1, \ldots, e_{2m}, e_{2m+1}$ $\omega$ is of the form

$$
\sum_{i=0}^{m-1} \alpha_i e_i e_i+1 \wedge e_i+2.
$$

Exercise 8.31 (*). Let $A$ be a skew-symmetric operator defined on a Euclidean space and $\det A$ be its determinant. Consider $\det A$ as a polynomial of matrix coefficients of $A$ in some basis. Prove that in a odd-dimensional space $V$ this determinant polynomial is identically zero. Prove that $\det A$ is a full square of some other polynomial of matrix coefficients. This polynomial is called the Pfaffian of $A$. 


**Hint.** Let $2m = \dim V$. Consider the bilinear form $\omega$ represented in the form above. Prove that $\omega^m$ (considered as an element of the Grassmann algebra $\Lambda^* (V^*)$) is proportional to $e^1 \wedge e^2 \wedge \cdots \wedge e^{2m}$ with a polynomial coefficient $Q$, moreover $Q^2 = \det A$. 