Algebra 9: Artinian rings and idempotents

Definition 9.1. Consider a commutative algebra \( R \) with unity over a field \( k \). One says that \( R \) is a finitely generated Artinian ring over the field \( k \) if \( R \) is finite-dimensional as a vector space.

Exercise 9.1. Consider a linear operator \( A \in \text{End} V \). Consider a subalgebra of \( \text{End} V \) generated by \( k \) and \( A \). Prove that this in an Artinian ring over \( k \).

Definition 9.2. An element \( r \in R \) of an algebra (or ring) \( R \) is called nilpotent if \( r^k = 0 \) for some \( k \in \mathbb{N} \).

Exercise 9.2. Let \( r, r' \) be nilpotent elements in an Artinian ring over a field. Prove that any linear combination \( r, r' \) is nilpotent.

Exercise 9.3. Let \( r, r' \) be nilpotent elements in the algebra \( \text{Mat}(V) \). Is \( r + r' \) always nilpotent?

Remark. A nilpotent element in the matrix algebra is called a nilpotent operator.

Exercise 9.4. Let \( A \in \text{End} V \) be a nilpotent operator. Prove that there is a chain of subspaces \( V \supset V_1 \supset V_2 \supset \cdots \supset V_k = 0 \) in \( V \) such that \( A(V_i) = V_{i+1} \).

Exercise 9.5 (!). Consider a nilpotent operator \( A \in \text{End} V \). Prove that in some basis \( A \) has the form:

\[
\begin{pmatrix}
0 & * & * & \cdots & * & * & *
0 & 0 & * & \cdots & * & * & *
0 & 0 & 0 & \cdots & * & * & *
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & * & *
0 & 0 & 0 & \cdots & 0 & 0 & *
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix}
\]

(that is, an upper-triangular matrix with 0 on the diagonal). Prove that any matrix of this form is nilpotent.

Hint. Use the previous problem.

Exercise 9.6 (!). Let \( A \in \text{End} V \) be nilpotent operator. Prove that \( \text{tr}(A) = \det(A) = 0 \) and \( \text{Chpoly}_A(t) = t^{\dim V} \).

Definition 9.3. Let \( R \) be a ring. A subset \( m \subset R \) is called an ideal if the following it has the following properties:

(i) \( m \) is closed under addition (that is, the sum of two elements from \( m \) belongs to \( m \))

(ii) For any \( m \in m, a \in R \) the product \( am \) belongs to \( R \).

Exercise 9.7. Consider a homomorphism of rings \( R \longrightarrow R' \). Prove that the kernel of this homomorphism is an ideal.

Exercise 9.8. Consider a surjective homomorphism \( f : R_1 \rightarrow R_2 \) of algebras over a field \( k \) and let \( R_1 \) be a field. Prove that either \( R_2 = 0 \), or \( f \) is an isomorphism.

Exercise 9.9. Consider an ideal \( m \subset R \). Consider the quotient \( R/m \), that is the set of cosets of the form \( r + m \). Define on \( R/m \) the natural ring structure.
Definition 9.4. A ring \( R/m \) is called a quotient ring of the ring \( R \). An ideal is called prime, if the corresponding quotient ring is non-zero and has no zero divisors. An ideal is called maximal if, moreover, the quotient is a field.

Exercise 9.10. Prove that any prime ideal in an Artinian ring is maximal.

Exercise 9.11 (*). Describe all maximal ideals in the ring of polynomials \( k[t] \).

Exercise 9.12. Consider the set of all nilpotent elements in the ring \( R \). Prove that it is an ideal.

Definition 9.5. This ideal is called the nilradical of the ring \( R \).

Exercise 9.13 (!). Consider the quotient ring \( R/n \) of a ring by its nilradical. Prove that \( R/n \) has no nilpotent elements.

Exercise 9.14. Consider an ideal in an Artinian ring that does not coincide with the whole ring. Prove that it is contained in a maximal one.

Exercise 9.15 (*). Consider an ideal in a ring (not necessary Artinian) that does not coincide with the whole ring. Prove that it is contained in a maximal one.

Hint. Use Zorn’s lemma.

Definition 9.6. An Artinian ring \( R \) is called semisimple, if it does not have non-zero nilpotents.

Definition 9.7. Consider a direct sum \( \oplus R_i \) with the natural (coordinate-wise) multiplication and addition. The resulting algebra is called the direct sum of \( R_i \) and is denoted \( \oplus R_i \) too.

Exercise 9.16. Prove that the direct sum of semisimple Artinian rings is semisimple.

Exercise 9.17. Let \( v \) be an element of a finite-dimensional algebra \( R \) over \( k \). Consider a subspace \( R \) generated by \( 1, v, v^2, v^3, \ldots \) (for all powers of \( v \)). Supposed this space has dimension \( n \). Prove that \( P(v) = 0 \) for some polynomial \( P = t^{n+1} + a_n t^n + \ldots \) with coefficients in \( k \). Prove that this polynomial is unique.

Definition 9.8. This polynomial is called the minimal polynomial of the element \( v \) and is denoted \( \text{Minpoly}(v) \).

Exercise 9.18. Let \( v \in R \) be an element of an Artinian ring over \( k \), and \( P(t) \) be its minimal polynomial. \( R_v, v k, R_v k[t]/P P \).

Definition 9.9. Let \( v \in R \) be an element of an algebra \( R \) such that \( v^2 = v \). Then \( v \) is called an idempotent.

Exercise 9.19. Let \( e \in R \) be an idempotent in a ring. Prove that \( 1 - e \) is an idempotent too. Prove that a product of idempotents is and idempotent.

Exercise 9.20. Let \( e \in R \) be an idempotent in a ring. Consider the space \( eR \subset R \) (the image of the multiplication by \( e \)). Prove that \( eR \) is a subalgebra in \( R \), that \( e \) is an identity in \( eR \), and that \( R = eR \oplus (1 - e)R \).

Exercise 9.21 (!). Let \( R = k(t)/P \) where \( P \) is a polynomial that decomposes into a product of pairwise co-prime polynomials \( P = P_1 P_2 \ldots P_n \). Prove that \( R \) has \( m \) idempotents \( e_1, \ldots, e_n \subset R \), and that \( e_i R \cong k[t]/P_i \).
Hint. Find polynomials $Q(t)$, $Q'(t)$ such that $QP_1 + Q'P_1P_3 \ldots P_n = 1$. Let $e = Q'P_1P_3 \ldots P_n$. Prove that $e^2 = e(\mod P)$, $eP_1(t) = 0(\mod P)$. Deduce that $k[z]/P(z) \cong eR$, and the isomorphism is given by $z \mapsto et$.

Exercise 9.22. Let $R$ be a semisimple Artinian ring without non-identity idempotents. Prove that it is a field.

Hint. Let $R$ be a field. Consider the subalgebra $k(x) \subset R$ generated by a non-invertible element $x \in R$, and apply the previous problem.

Definition 9.10. Two idempotents $e_1, e_2 \in R$ in a commutative algebra $R$ are called orthogonal if $e_1e_2 = 0$.

Exercise 9.23. Let $e_1, e_2, e_3 \in R$ be idempotents in an Artinian ring $R$ over a field $k$ and let $e_1 = e_2 + e_3$, let $e_2$ and $e_3$ be orthogonal. Prove that $e_2, e_3 \in e_1R$ and $e_1R = e_2R \oplus e_3R$.

Exercise 9.24. Let $\text{char} k \neq 2$. Suppose that $e_1, e_2, e_3$ be idempotents in an Artinian ring $R$ over a ring $k$ and $e_1 = e_2 + e_3$. Prove that $e_2$ and $e_3$ are orthogonal.

Definition 9.11. Let $R$ be an Artinian ring over a field $k$. An idempotent $e$ in $R$ is called indecomposable if there are no such non-zero orthogonal idempotents $e_2, e_3$ such that $e_1 = e_2 + e_3$.

Exercise 9.25 (!). Let $R$ be a semisimple Artinian ring and $e$ be an indecomposable idempotent. Prove that $eR$ is a ring.

Exercise 9.26 (!). Let $R$ be a semisimple Artinian ring over a field $k$. Prove that 1 decomposes into a sum of indecomposable orthogonal idempotents: $1 = \sum e_i$. Prove that this decomposition is unique.

Hint. For existence take some idempotent $e \in R$ and decompose $R = eR \oplus (1 - e)R$ then use induction. For uniqueness, consider the product of two possible decompositions of 1.

Exercise 9.27 (!). Let $R$ be a semisimple Artinian ring over a ring $k$. Prove that $R$ is isomorphic to a direct sum of fields.

Hint. Use the previous problem.

Exercise 9.28 (!). Let $R_1 \xrightarrow{\psi} R_2$ be a surjective homomorphism of Artinian rings, moreover, let $R_1$ be semisimple and thus decomposed into a direct sum of fields over some set of indices $I$, $R_1 = \bigoplus_{i \in I} K_i$. Prove that $R_2 = \bigoplus_{i \in I'} K_i$, where $I'$ is some subset of $I$ and $\psi$ is the natural projection (that is, $\psi$ acts identically on $K_i$, $i \in I'$ and is zero on $K_i$, $i \notin I'$).

Hint. Decompose $1 \in R_1$ into the sum of indecomposable idempotents $e_i$, $i \in I$, prove that $f : e_iR \rightarrow f(e_i)R_2$ is surjective for all $i \in I$, and apply Problem 9.8.

Exercise 9.29 (*). Let $R = k[t]/P$ and the polynomial $P$ has multiple roots over the algebraic closure $\overline{k}$. Can $R$ be semisimple? Analyse the cases $\text{char} k = 0$, $\text{char} k \neq 0$.

Exercise 9.30 (*). Let $R$ be a semisimple Artinian ring over a field $k$, and $1 = e_1 + \cdots + e_n$ be the decomposition of 1 into the sum of indecomposable orthogonal idempotents. Prove that $R$ has exactly $n$ prime ideals. Describe these ideals in terms of $e_i$. 

3
Exercise 9.31 (*). Let $R$ be an Artinian ring over a field $k$ (of any characteristic). Prove that the intersection of all simple ideals $R$ is the nilradical of $R$.

Definition 9.12. Let $R$ be an algebra over a field $k$, and $g$ be a bilinear form on $R$. The form $g$ is called invariant, if $g(x, yz) = g(xy, z)$ for any $x, y, z$.

Exercise 9.32. Let $R$ be an Artinian ring endowed with a bilinear invariant form, and $m$ be an ideal in $R$. Prove that $m^\perp$ is an ideal too.

Exercise 9.33 (*). Find an Artinian ring that does not admit a non-degenerate invariant bilinear form.

Exercise 9.34 (!). Let $R$ be an Artinian ring over a field $k$. Consider a the bilinear form $a, b \rightarrow \text{tr}(ab)$, where $\text{tr}(ab)$ is the trace of the endomorphism $L_{ab} \in \text{End} R, x \mapsto abx$. Prove that if this form is non-degenerate then $R$ is semisimple. Prove that if $R$ is semisimple and $\text{char} k = 0$ then the form is non-degenerate.

Hint. One direction can be proved using the Problem 9.6. For the other direction consider first the case when $R$ is a field.

Exercise 9.35. Let $V, V'$ be vector spaces over $k$ endowed with bilinear forms $g, g'$. Define on $V \otimes V'$ the bilinear form $g \otimes g'$ that would satisfy

$$g \otimes g'(v \otimes v', w \otimes w') = g(v, w)g'(v', w')$$

Prove that the bilinear form on $V \otimes V'$ is well-defined and unique.

Exercise 9.36. Let $R, R'$ be commutative algebras over $k$. Consider a tensor product $R \otimes R'$. Endow $R \otimes R'$ with a multiplicative structure such that $v \otimes v' \cdot w \otimes w = vw \otimes v'w'$. Prove that the ring structure on $R \otimes R'$ is well-defined and unique.

Exercise 9.37. Describe the algebra $\mathbb{C} \otimes_\mathbb{R} \mathbb{C}$.

Exercise 9.38. Describe the algebra $\mathbb{Q}[\sqrt{-1}] \otimes_\mathbb{Q} \mathbb{Q}[\sqrt{-1}]$.

and apply the problem

Exercise 9.39 (!). Let $P(t)$ and $Q(t)$ be polynomials over a field $k$. Denote $K_1 = k[t]/P(t)$ and $K_2 = k[t]/Q(t)$. Prove that $K_1 \otimes K_2 \cong K_1[t]/Q(t) \cong K_2[t]/P(t)$.

Exercise 9.40 (*). Let $R, R'$ be Artinian rings over $k, \text{char} k = 0$. Denote the natural bilinear forms $a, b \rightarrow \text{tr}(ab)$ on these rings by $g, g'$. Consider the tensor product $R \otimes R'$ with the natural structure of Artinian algebra. Consider the form $g \otimes g'$ on $R \otimes R'$. Prove that $g \otimes g'$ is equal to the form $a, b \rightarrow \text{tr}(ab)$.

Exercise 9.41 (*). Prove that the tensor product of semisimple Artinian rings over a field $k$ of characteristic 0 is semisimple.

Hint. Use the Problem 9.34.

Exercise 9.42 (*). Find two fields $K_1, K_2$, algebraic over but not equal to $\mathbb{Q}$, such that $K_1 \otimes_{\mathbb{Q}} K_2$ is also a field.
Exercise 9.43 (*). Let $P(t) \in \mathbb{Q}[t]$ be a polynomial that does not have rational roots but has exactly $r$ real and $2s$ complex roots (that are non-real). Prove that

$$(\mathbb{Q}[t]/P) \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_s \mathbb{C} \oplus \bigoplus_r \mathbb{R}.$$  

Exercise 9.44 (*). Let $P(t)$ be an irreducible polynomial over $\mathbb{Q}$ that does not have real roots and $v \in \mathbb{Q}[t]/P$ be an element that does not belong to $\mathbb{Q} \subset \mathbb{Q}[t]/P$. Prove that the minimal polynomial of $v$ does not have real roots.