Algebra 9: Artinian rings and idempotents

Definition 9.1. Consider a commutative algebra R with unity over a field k. One says that R is a **finitely generated Artinian ring over the field** k if R is finite-dimensional as a vector space.

Exercise 9.1. Consider a linear operator $A \in \operatorname{End} V$. Consider a subalgebra of $\operatorname{End} V$ generated by k and A. Prove that this in an Artinian ring over k.

Definition 9.2. An element $r \in R$ of an algebra (or ring) R is called **nilpotent** if $r^k = 0$ for some $k \in \mathbb{N}$.

Exercise 9.2. Let r, r' be nilpotent elements in an Artinian ring over a field. Prove that any linear combination r, r' is nilpotent.

Exercise 9.3. Let r, r' be nilpotent elements in the algebra $\operatorname{Mat}(V)$. Is $r + r'$ always nilpotent?

Remark. A nilpotent element in the matrix algebra is called a **nilpotent operator**.

Exercise 9.4. Let $A \in \operatorname{End} V$ be a nilpotent operator. Prove that there is a chain of subspaces $V \supset V_1 \supset V_2 \supset \cdots \supset V_k = 0$ in V such that $A(V_i) = V_{i+1}$.

Exercise 9.5 (!). Consider a nilpotent operator $A \in \operatorname{End} V$. Prove that in some basis A has the form:

$$\begin{pmatrix} 0 & * & * & \dots & * & * & * \\ 0 & 0 & * & \dots & * & * & * \\ 0 & 0 & 0 & \dots & * & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & * & * \\ 0 & 0 & 0 & \dots & 0 & 0 & * \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

(that is, an upper-triangular matrix with 0 on the diagonal). Prove that any matrix of this form is nilpotent.

Hint. Use the previous problem.

Exercise 9.6 (!). Let $A \in \operatorname{End} V$ be nilpotent operator. Prove that $\mathsf{tr}(A) = \det(A) = 0$ and $\mathsf{Chpoly}_A(t) = t^{\dim V}$.

Definition 9.3. Let R be a ring. A subset $\mathfrak{m} \subset R$ is called an **ideal** if the following it has the following properties:

(i) $\mathfrak{m}$ is closed under addition (that is, the sum of two elements from $\mathfrak{m}$ belongs to $\mathfrak{m}$)

(ii) For any $m \in \mathfrak{m}$, $a \in R$ the product am belongs to R.

Exercise 9.7. Consider a homomorphism of rings $R \longrightarrow R'$. Prove that the kernel of this homomorphism is an ideal.

Exercise 9.8. Consider a surjective homomorphism $f : R_1 \to R_2$ of algebras over a field k and let R_1 be a field. Prove that either $R_2 = 0$, or f is an isomorphism.

Exercise 9.9. Consider an ideal $\mathfrak{m} \subset R$. Consider the quotient $R/\mathfrak{m}$, that is the set of cosets of the form $r + \mathfrak{m}$. Define on $R/\mathfrak{m}$ the natural ring structure.

Definition 9.4. A ring $R/\mathfrak{m}$ is called a **quotient ring** of the ring R. An ideal is called **prime**, if the corresponding quotient ring is non-zero and has no zero divisors. An ideal is called maximal if, moreover, the quotient is a field.

Exercise 9.10. Prove that any prime ideal in an Artinian ring is maximal.

Exercise 9.11 (*). Describe all maximal ideals in the ring of polynomials $k[t]$.

Exercise 9.12. Consider the set of all nilpotent elements in the ring R. Prove that it is an ideal.

Definition 9.5. This ideal is called the **nilradical** of the ring R.

Exercise 9.13 (!). Consider the quotient ring $R/\mathfrak{n}$ of a ring by its nilradical. Prove that $R/\mathfrak{n}$ has no nilpotent elements.

Exercise 9.14. Consider an ideal in an Artinian ring that does not coincide with the whole ring. Prove that it is contained in a maximal one.

Exercise 9.15 (*). Consider an ideal in a ring (not necessary Artinian) that does not coincide with the whole ring. Prove that it is contained in a maximal one.

Hint. Use Zorn's lemma.

Definition 9.6. An Artinian ring R is called **semisimple**, if it does not have non-zero nilpotents.

Definition 9.7. Consider a direct sum $\oplus R_i$ with the natural (coordinate-wise) multiplication and addition. The resulting algebra is called the **direct sum of** R_i and is denoted $\oplus R_i$ too.

Exercise 9.16. Prove that the direct sum of semisimple Artinian rings in semisimple.

Exercise 9.17. Let v be an element of a finite-dimensional algebra R over k. Consider a subspace R generated by $1, v, v^2, v^3, \dots$ (for all powers of v). Supposed this space has dimension n. Prove that $P(v) = 0$ for some polynomial $P = t^{n+1} + a_n t^n + \dots$ with coefficients in k. Prove that this polynomial is unique.

Definition 9.8. This polynomial is called the **minimal polynomial** of the element v and is denoted $\mathsf{Minpoly}(v)$.

Exercise 9.18. Let $v \in R$ be an element of an Artinian ring over k, and $P(t)$ be its minimal polynomial. R_v, v k. , R_v $k[t]/P$ P.

Definition 9.9. Let $v \in R$ be an element of an algebra R such that $v^2 = v$. Then v is called an **idempotent**.

Exercise 9.19. Let $e \in R$ be an idempotent in a ring. Prove that $1 - e$ is an idempotent too. Prove that a product of idempotents is and idempotent.

Exercise 9.20. Let $e \in R$ be an idempotent in a ring. Consider the space $eR \subset R$ (the image of the multiplication by e). Prove that eR is a subalgebra in R, that e is an identity in eR, and that $R = eR \oplus (1 - e)R$.

Exercise 9.21 (!). Let $R = k(t)/P$ where P is a polynomial that decomposes into a product of pairwise co-prime polynomials $P = P_1 P_2 \dots P_n$. Prove that R has m idempotents $e_1, \dots, e_n \subset R$, and that $e_i R \cong k[t]/P_i$.

Hint. Find polynomials $Q(t)$, $Q'(t)$ such that $QP_1 + Q'P_1P_3 \dots P_n = 1$. Let $e = Q'P_1P_3 \dots P_n$. Prove that $e^2 = e(\mod P)$, $eP_1(t) = 0(\mod P)$. Deduce that $k[z]/P_1(z) \cong eR$, and the isomorphism is given by $z \mapsto et$.

Exercise 9.22. Let R be a semisimple Artinian ring without non-identity idempotents. Prove that it is a field.

Hint. Let R be a field. Consider the subalgebra $k(x) \subset R$ generated by a non-invertible element $x \in R$, and apply the previous problem.

Definition 9.10. Two idempotents $e_1, e_2 \in R$ in a commutative algebra R are called **orthogonal** if $e_1e_2 = 0$.

Exercise 9.23. Let $e_1, e_2, e_3 \in R$ be idempotents in an Artinian ring R over a field k and let $e_1 = e_2 + e_3$, let e_2 and e_3 be orthogonal. Prove that $e_2, e_3 \in e_1R$ and $e_1R = e_2R \oplus e_3R$.

Exercise 9.24. Let $\operatorname{char} k \neq 2$. Suppose that e_1, e_2, e_3 be idepmotents in an Artinian ring R over a ring k and $e_1 = e_2 + e_3$. Prove that e_2 and e_3 are orthogonal.

Definition 9.11. Let R be an Artinian ring over a field k. An idempotent e in R is called **indecomposable** if there are no such non-zero orthogonal idempotents e_2, e_3 such that $e_1 = e_2 + e_3$.

Exercise 9.25 (!). Let R be a semisimple Artinian ring and e be an indecomposable idempotent. Prove that eR is a ring.

Exercise 9.26 (!). Let R be a semisimple Artinian ring over a field k. Prove that 1 decomposes into a sum of indecomposable orthogonal idempotents: $1 = \sum e_i$. Prove that this decomposition is unique.

Hint. For existence take some idempotent $e \in R$ and decompose $R = eR \oplus (1 - e)R$ then use induction. For uniqueness, consider the product of two possible decompositions of 1.

Exercise 9.27 (!). Let R be a semisimple Artinian ring over a ring k. Prove that R is isomorphic to a direct sum of fields.

Hint. Use the previous problem.

Exercise 9.28 (!). Let $R_1 \xrightarrow{\psi} R_2$ be a surjective homomorphism of Artinian rings, moreover, let R_1 be semisimple and thus decomposed into a direct sum of fields over some set of indices I, $R_1 = \oplus_{i \in I} K_i$. Prove that $R_2 = \oplus_{i \in I'} K_i$, where I' is some subset of I and ψ is the natural projection (that is, ψ acts identically on K_i, $i \in I'$ and is zero on K_i, $i \notin I'$).

Hint. Decompose $1 \in R_1$ into the sum of indecomposable idempotents e_i, $i \in I$, prove that $f : e_iR \to f(e_i)R_2$ is surjective for all $i \in I$, and apply Problem 9.8.

Exercise 9.29 (*). Let $R = k[t]/P$ and the polynomial P has multiple roots over the algebraic closure $\overline{k}$. Can R be semisimple? Analyse the cases $\operatorname{char} k = 0$, $\operatorname{char} k \neq 0$.

Exercise 9.30 (*). Let R be a semisimple Artinian ring over a field k, and $1 = e_1 + \dots + e_n$ be the decomposition of 1 into the sum of indecomposable orthogonal idempotents. Prove that R has exactly n prime ideals. Describe these ideals in terms of e_i.

Exercise 9.31 (*). Let R be an Artinian ring over a field k (of any characteristic). Prove that the intersection of all simple ideals R is the nilradical of R.

Definition 9.12. Let R be an algebra over a field k, and g be a bilinear form on R. The form g is called **invariant**, if $g(x, yz) = g(xy, z)$ for any x, y, z.

Exercise 9.32. Let R be an Artinian ring endowed with a bilinear invariant form, and $\mathfrak{m}$ be an ideal in R. Prove that $\mathfrak{m}^\perp$ is an ideal too.

Exercise 9.33 (*). Find an Artinian ring that does not admit a non-degenerate invariant bilinear form.

Exercise 9.34 (!). Let R be an Artinian ring over a field k. Consider a the bilinear form $a, b \longrightarrow \mathsf{tr}(ab)$, where $\mathsf{tr}(ab)$ is the trace of the endomorphism $L_{ab} \in \operatorname{End} R$, $x \overset{L_{ab}}{\mapsto} abx$. Prove that if this form is non-degenerate then R is semisimple. Prove that if R is semisimple and $\mathsf{char}\, k = 0$ then the form is non-degenerate.

Hint. One direction can be proved using the Problem 9.6. For the other direction consider first the case when R is a field.

Exercise 9.35. Let V, V' be vector spaces over k endowed with bilinear forms g, g'. Define on $V \otimes V'$ the bilinear form $g \otimes g'$ that would satisfy

$$g \otimes g'(v \otimes v', w \otimes w') = g(v, w) g'(v', w')$$

Prove that the bilinear form on $V \otimes V'$ is well-defined and unique.

Exercise 9.36. Let R, R' be commutative algebras over k. Consider a tensor product $R \otimes R'$. Endow $R \otimes R'$ with a multiplicative structure such that $v \otimes v' \cdot w \otimes w = vw \otimes v'w'$. Prove that the ring structure on $R \otimes R'$ is well-defined and unique.

Exercise 9.37. Describe the algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise 9.38. Describe the algebra $\mathbb{Q}[\sqrt{-1}\,] \otimes_{\mathbb{Q}} \mathbb{Q}[\sqrt{-1}\,]$.

and apply the problem

Exercise 9.39 (!). Let $P(t)$ and $Q(t)$ be polynomials over a field k. Denote $K_1 = k[t]/P(t)$ and $K_2 = k[t]/Q(t)$. Prove that $K_1 \otimes K_2 \cong K_1[t]/Q(t) \cong K_2[t]/P(t)$.

Exercise 9.40 (*). Let R, R' be Artinian rings over k, $\mathsf{char}\, k = 0$. Denote the natural bilinear forms $a, b \longrightarrow \mathsf{tr}(ab)$ on these rings by g, g'. Consider the tensor product $R \otimes R'$ with the natural structure of Artinian algebra. Consider the form $g \otimes g'$ on $R \otimes R'$. Prove that $g \otimes g'$ is equal to the form $a, b \longrightarrow \mathsf{tr}(ab)$.

Exercise 9.41 (*). Prove that the tensor product of semisimple Artinian rings over a field k of characteristic 0 is semisimple.

Hint. Use the Problem 9.34.

Exercise 9.42 (*). Find two fields K_1, K_2, algebraic over but not equal to $\mathbb{Q}$, such that $K_1 \otimes_{\mathbb{Q}} K_2$ is also a field.

Exercise 9.43 (*). Let $P(t) \in \mathbb{Q}[t]$ be a polynomial that does not have rational roots but has exactly r real and $2s$ complex roots (that are non-real). Prove that

$$(\mathbb{Q}[t]/P) \otimes_{\mathbb{Q}} \mathbb{R} = \bigoplus_s \mathbb{C} \oplus \bigoplus_r \mathbb{R}.$$

Exercise 9.44 (*). Let $P(t)$ be an irreducible polynomial over $\mathbb{Q}$ that does not have real roots and $v \in \mathbb{Q}[t]/P$ be an element that does not belong to $Q \subset \mathbb{Q}[t]/P$. Prove that the minimal polynomial of v does not have real roots.