GEOMETRY 1: real numbers.

You are supposed to know what field is, consult ALGEBRA-1 for the definition.

Cauchy sequences.

Real numbers are usually considered as something that can be approximated by rational numbers, for example, one can regard a real number a as infinite decimal fraction $a_0, a_1 a_2 \dots$, the finite fragments of that fraction $a_0, a_1 a_2 \dots a_n$ are then approximations of a. Some fractions are declared equivalent, for example, $1,00000\dots$ and $0,9999\dots$. It turns out that it is easier to rigorously define real numbers and operations on them when not only decimal fractions but just any sequences of rational numbers which approximate a given real number are considered. And again it should be taken into account that different sequences can be equivalent (when they approximate one and the same number). It appears quite logical to **define** a real number as a set of sequences of rational numbers that approximate it. This is Cauchy approach to real numbers definition.

Definition 1.1. We will say that something holds for **almost all** elements of a set if it holds for all elements except finite number of them. Let $\{a_i\} = a_0, a_1, a_2, \dots$ be a sequence of rational numbers. It is said that $\{a_i\}$ is a **Cauchy sequence** if for any rational number $\varepsilon > 0$ there exists an interval $[x, y]$ of length ε which contains almost all $\{a_i\}$.

Exercise 1.1. Let a be a rational number. Prove that a constant sequence $a, a, \dots$ is a Cauchy sequence.

We will denote such a sequence by $\{a\}$.

Exercise 1.2. Let $\{a_i\}$ be a Cauchy sequence. Let us permute arbitrarily its elements a_i. Prove that we obtain a Cauchy sequence then.

Exercise 1.3. Consider a sequence $\{a_i\}$ of rational numbers from an interval $I = [a, b]$, $a, b \in \mathbb{Q}$. Prove that one can select a subsequence out of $\{a_i\}$ which is a Cauchy subsequence.

Hint. Let us split the interval $I_0 = [a, b]$ into two equal parts. One of the halves (we will denote it by I_1) contains an infinite number of elements of the sequence. Let us delete from $\{a_i\}$ all elements that do not belong to I_1 except a_0. Let then divide I_1 into two equal parts and repeat the procedure over and over again. An interval I_k obtained on a k-th step contains all elements of the sequence starting from k-th and this interval is of length $\frac{b-a}{2^k}$.

Exercise 1.4 (!). Consider a monotonically increasing sequence $a_1 \leqslant a_2 \leqslant a_3 \leqslant \dots$. All a_i are bounded by some constant C: $a_i \leqslant C$. Prove that this is a Cauchy sequence.

Hint. Use the previous problem.

Definition 1.2. Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences. They are called **equivalent** if a sequence $a_0, b_0, a_1, b_1, a_2, b_2, \dots$ is a Cauchy sequence.

Exercise 1.5. Let a, b be two rational numbers. Prove that $\{a\}$ is equivalent to $\{b\}$ iff $a = b$.

Exercise 1.6. Prove that a Cauchy sequence is equivalent to any subsequence of it.

Exercise 1.7. Prove that if $\{a_i\}$ is equivalent to $\{b_i\}$ then $\{b_i\}$ is equivalent to $\{a_i\}$.

Exercise 1.8 (!). Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences. Prove that there exist two non-intersecting intervals I_1, I_2 such that almost all a_i belong to I_1 while almost all b_i belong to I_2.

Hint. Apply the definition of a Cauchy sequence with $\varepsilon = \frac{1}{2^n}$ for all n.

Exercise 1.9 (!). Prove that if a sequence $\{a_i\}$ is equivalent to a sequence $\{b_i\}$ and a sequence $\{b_i\}$ is equivalent to a sequence $\{c_i\}$ then $\{a_i\}$ is equivalent to $\{c_i\}$ (one says that "Cauchy sequences equivalence is transitive").

Definition 1.3. Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences. It is said that $\{a_i\} > \{b_i\}$ if $a_i > b_i$ for almost all i.

Exercise 1.10. Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences. Prove that either $\{a_i\} < \{b_i\}$ or $\{b_i\} < \{a_i\}$.

Hint. Use the problem 1.8.

Exercise 1.11. Let $\{a_i\}$, $\{b_i\}$ be two non-equivalent Cauchy sequences and $\{a_i\} < \{b_i\}$. Prove that there exist two rational numbers c, d such that $\{a_i\} < \{c\} < \{d\} < \{b_i\}$.

Hint. Use the previous hint.

Exercise 1.12. Let $\{a_i\} < \{b_i\}$ and $\{b_i\}$ be equivalent to $\{c_i\}$. Prove that $\{a_i\} < \{c_i\}$.

Hint. Use the previous problem and the definition of Cauchy sequence for $\varepsilon < |c - d|$.

Exercise 1.13. Let $\{a_i\}$ be a Cauchy sequence and $c \in \mathbb{Q}$ be a rational number. Prove that the following properties are equivalent

a. $\{a_i\}$ is equivalent to a sequence $\{c\}$.

b. there are infinitely many elements of a sequence $\{a_i\}$ in any open interval $]x, y[$ containing c.

c. any open interval $]x, y[$ which contains c contains almost all elements of a sequence $\{a_i\}$ as well.

Definition 1.4. If any of these properties holds then it is said that $\{a_i\}$ converges to c.

Exercise 1.14. Let $\{a_i\}$, $\{b_i\}$ be a Cauchy sequence. Prove that $\{a_i + b_i\}$ and $\{a_i - b_i\}$ are Cauchy sequences.

Exercise 1.15. Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences and b_i converges to 0. Prove that $\{a_i\}$ is equivalent to $\{a_i + b_i\}$.

Exercise 1.16. Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences. Prove that $\{a_i b_i\}$ is a Cauchy sequence.

Exercise 1.17. Prove that if $\{b_i\}$ converges to 1 then $\{a_i b_i\}$ is equivalent to $\{a_i\}$.

Exercise 1.18. Let $\{a_i\}$ be a Cauchy sequence which does not contain zeros and which does not converge to 0. Prove that $\{a_i^{-1}\}$ is a Cauchy sequence.

Hint. Prove that there exists a closed interval $[x, y]$ which does not contain 0 such that almost all $\{a_i\}$ are contained in $[x, y]$. Let almost all $\{a_i\}$ belong to an interval $I \subset [x, y]$ of a length ε. Prove that all $\{a_i^{-1}\}$ except a finite number belong to an interval I^{-1} of a length $\varepsilon(\min(|x|, |y|)^{-1}$.

Definition 1.5. A set of all Cauchy sequences equivalent to a Cauchy sequence $\{a_i\}$ is called an **equivalence class** of a Cauchy sequence. The set of all equivalence classes is called a **set of real numbers** and is denoted by $\mathbb{R}$.

Exercise 1.19. Prove that to correspondence $c \mapsto \{c\}$ defines an injective mapping from a set $\mathbb{Q}$ of all rational numbers into $\mathbb{R}$.

Exercise 1.20 (!). Prove that four arithmetic operations that we have defined on $\mathbb{R}$ in the problems 1.14- 1.18 define on $\mathbb{R}$ the structure of a field.

Dedekind sections

The main disadvantage of defining real numbers using Cauchy sequences is that there are too many Cauchy sequences and the definition appears to be too implicit. This difficulty is rather psychological. Nevertheless, there exists a way to overcome it, it is to introduce more straightforward definition of real numbers that was proposed by Dedekind.

Definition 1.6. Let $R \subset \mathbb{Q}$ be a subset of a set of rational numbers which is non-empty and does not equal to the whole $\mathbb{Q}$. It is said that R is a **Dedekind section** if $a \in R$ and $b < a$ entails that $b \in R$. Dedekind section R is said to be **closed** if there exists a rational number a such that $b \in R$ as soon as $b \leqslant a$. Otherwise R is said to be **open**.

Let $\{a_i\}$ be a Cauchy sequence. Let us denote the set of all rational numbers b such that $\{b\} < \{a_i\}$ by $R_{\{a_i\}}$.

Exercise 1.21. Prove that $R_{\{a_i\}}$ is a Dedekind section (i.e. if $b \in R_{\{a_i\}}$ and $c < b$ then $c \in R_{\{a_i\}}$). Prove that this section is open.

Exercise 1.22. Let $\{a_i\}$ and $\{b_i\}$ be equivalent Cauchy sequences. Prove that $R_{\{a_i\}} = R_{\{b_i\}}$.

Exercise 1.23. Let $\{a_i\}$ and $\{b_i\}$ be non-equivalent Cauchy sequences and $\{a_i\} < \{b_i\}$. Prove that $R_{\{a_i\}} \subset R_{\{b_i\}}$ but those two sets do not coincide.

Hint. Consider the points of an interval $[c, d]$ from the problem 1.11; which of the sets $R_{\{a_i\}}$, $R_{\{b_i\}}$ do they belong?

Exercise 1.24 (*). Let $\{a_i\}$, $\{b_i\}$ be two Cauchy sequences. Prove that they are equivalent if and only if $R_{\{a_i\}} = R_{\{b_i\}}$.

Hint. Use the problem 1.10 (as well as the preceding problems).

Exercise 1.25 (*). Let $R \subset \mathbb{Q}$ be an open Dedekind section. Prove that $R = R_{\{a_i\}}$ holds for some Cauchy sequence $\{a_i\}$.

Hint. Consider an interval $I_0 = [a, b]$ such that a belongs to R and b does not. Split it into two equal parts, select the half I_1 with the same property. Repeat this process and select any point of I_i as a_i.

We observe that the set of equivalence classes of Cauchy sequences is the same thing as the set of open Dedekind sections. That is why the real numbers can be defined as Dedekind sections. In what follows you can use the definition that suits you best.

Exercise 1.26 (**). Define arithmetic operations on $\mathbb{R}$ explicitly on Dedekind sections without using Cauchy sequences. Check that the axioms of a field hold.

Hint. To define multiplication define first the operations "multiplication by a positive real number a" and "multiplication by -1", then prove distributivity for each of them separately.

Supremum and infimum

Definition 1.7. Let $A \subset \mathbb{R}$ be some subset of $\mathbb{R}$. A set A is said to be **bounded above** if all elements of A are greater that some constant $C \in \mathbb{R}$. A set A is said to be **bounded below** if all elements of A are less than some constant $C \in \mathbb{R}$. A set A is called **bounded** if it bounded above and bounded below.

Definition 1.8. Let $A \subset \mathbb{R}$ be some subset of $\mathbb{R}$. Infimum of A (notation: $\inf A$) is by definition a number $c \in \mathbb{R}$ such that $c \leqslant a$ for all $a \in A$ and in any open interval $]x, y[$ containing c there are elements of a. Supremum of A (notation: $\sup A$) is by definition a number $c \in \mathbb{R}$ such that $c \geqslant a$ for all $a \in A$ and in any open interval $]x, y[$ containing c there are elements of a.

Exercise 1.27. Prove that $\inf A$ and $\sup A$ are unique (if they exist).

Exercise 1.28 (!). Let A be a set bounded above. Prove that $\sup A$ exists.

Hint. Consider every $a \in A$ as Dedekind sections, i.e. subsets of $\mathbb{Q}$. Consider their union R; since every $a \leqslant C$ this will be Dedekind section too. Prove that $\inf A = R$.

Exercise 1.29 (!). Let $A \subset \mathbb{R}$ a set bounded below. Prove that $\inf A$ exists.

Remark. Let $A \subset \mathbb{R}$ is not bounded above (below). It is denoted by $\inf A = -\infty$ ($\sup A = \infty$).