

# Geometry 10: The fundamental group and the loop space

## Path-connectedness

**Definition 10.1.** Let  $M$  be a topological space. Recall that a **path** in  $M$  is a continuous mapping  $[a, b] \xrightarrow{\varphi} M$ . In this case one says that the path  $\varphi$  **connects the points**  $\varphi(a)$  and  $\varphi(b)$ .  $M$  is called **path-connected** when any two points of  $M$  can be connected by a path  $[a, b] \xrightarrow{\varphi} M$ .

**Exercise 10.1.** Let  $a, b, c$  are points in  $M$ , so that  $a$  can be connected (by a path) to  $b$ , and  $b$  can be connected to  $c$ . Show that  $a$  can be connected to  $c$ .

**Exercise 10.2.** From this derive that a union of path-connected subsets of  $M$  containing a point  $x \in M$  is path-connected.

**Definition 10.2.** The union of all the subsets of  $M$ , containing a fixed point  $x$  is called a **path-connected component** of  $M$

**Exercise 10.3.** Consider  $X \subset \mathbb{R}^2$  that is the union of the graph of the function  $\sin(1/t)$  and the segment  $[(0, 1), (0, -1)]$ . Show that  $X$  is locally compact, connected, but not path-connected. Find its path-connected components.

**Exercise 10.4 (\*).** Construct a compact, connected metrizable topological space with infinitely many path-connected components.

**Definition 10.3.** Let  $\{M_\alpha\}$  be a collection of topological spaces indexed by the set  $\mathfrak{A}$ . The disjoint union  $\coprod_{\alpha \in \mathfrak{A}} M_\alpha$  is a topological space whose points are pairs  $(\alpha, m) \mid \alpha \in \mathfrak{A}, m \in M_\alpha$ , and a base of the topology is given by the open sets in all  $M_\alpha$ .

**Exercise 10.5.** Show that the disjoint union of one-point spaces is discrete. Show that the natural projection  $\coprod_{\alpha \in \mathfrak{A}} M_\alpha \longrightarrow \mathfrak{A}$  on  $\mathfrak{A}$  with discrete topology is continuous.

**Definition 10.4.** A topological space  $M$  is called locally connected (respectively, locally path-connected), if any point  $x \in M$  is contained in a connected (respectively, path-connected) open set.

**Exercise 10.6.** Let  $M$  be a topological space. Show that  $M$  is locally connected (resp. locally path-connected) iff  $M$  is a disjoint union of its (path-)connected components.

**Exercise 10.7.** Show that a connected space is path-connected iff it is locally path-connected.

**Exercise 10.8.** Show that an open subset of  $\mathbb{R}^n$  is locally path-connected.

**Exercise 10.9 (\*\*).** Let  $\omega$  be the smallest uncountable ordinal, and  $\varphi : [0, 1] \longrightarrow \omega$  the corresponding bijection. Let  $X \subset [0, 1] \times [0, 1]$  be the subset of the square consisting of  $x, y$  satisfying  $\varphi(x) > \varphi(y)$ . Show that  $X$  is connected. Show that path-connected components of  $X$  are either points or segments of horizontal intervals.

**Hint.** Show that the intersection of  $X$  with any vertical segment is nowhere dense. Let  $V \subset [0, 1] \times [0, 1]$  be a connected closed subset of the square contained in  $X$ . Show that  $V$  intersects each vertical segment in no more than 1 point. Thus  $V$  is the graph of a continuous mapping  $\gamma : [a, b] \longrightarrow [0, 1]$ , satisfying  $\varphi(\gamma(a)) < \varphi(a)$ . Show that this mapping is constant.

## Geodesic connectedness

**Definition 10.5.** Let  $M$  be a complete locally compact metric space. Recall that a **geodesic** in  $M$  is a metric-preserving mapping  $[a, b] \rightarrow M$ . We say that  $M$  is **geodesically connected** if any two points can be connected by a geodesic. Obviously such a space is path-connected.

**Definition 10.6.** Let  $M$  be a complete locally compact metric space. We say that  $M$  is **Lipschitz connected**, with Lipschitz constant  $C \geq 1$ , if for any  $x, y \in M$  and any  $\varepsilon > 0$  there exists a sequence of points  $x = x_1, x_2, \dots, x_n = y$  such that  $d(x_i, x_{i+1}) < \varepsilon$ ,  $\sum_i d(x_i, x_{i+1}) \leq Cd(x, y)$ . In other words, one can place  $n$  points between  $x$  and  $y$  so that they are at distance at most  $\varepsilon$  from each other, whereas the length of the polygonal line they form is at most  $Cd(x, y)$ .

**Exercise 10.10 (\*).** Show that any geodesically connected metric space is Lipschitz connected with Lipschitz constant 1.

**Hint.** This is Hopf-Rinow Theorem.

**Exercise 10.11 (!).** Let  $(M, d)$  be a Lipschitz connected metric space, with constant  $C$ . Define a function  $d_h : M \times M \rightarrow \mathbb{R}$  as

$$\lim_{\varepsilon \rightarrow 0} \inf \left( \sum d(x_i, x_{i+1}) \right),$$

where inf is taken over such sequences  $x = x_1, x_2, \dots, x_n = y$  that  $d(x_i, x_{i+1}) < \varepsilon$ . Show that  $d(x, y) \leq d_h(x, y) \leq Cd(x, y)$  for any  $x, y \in M$ . Show that  $d_h$  is a metric and that  $(M, d)$  is homeomorphic to  $(M, d_h)$ .

**Exercise 10.12 (\*).** Show that  $(M, d_h)$  is Lipschitz connected, for any  $C > 1$ .

**Exercise 10.13 (\*).** Show that  $(M, d_h)$  satisfies Hopf-Rinow condition (and is therefore geodesically connected).

**Definition 10.7.** Recall that a mapping  $[a, b] \xrightarrow{\varphi} M$  **satisfies the Lipschitz condition, with constant  $C > 0$** , if  $d(\varphi(x), \varphi(y)) \leq C|x - y|$  for any  $x, y \in [a, b]$ . It is easy to see that a Lipschitz mapping is continuous.

**Exercise 10.14 (\*).** Let  $M$  be a complete locally compact metric space. Show that  $M$  is Lipschitz connected with constant  $C$  iff one can connect any two points by a Lipschitz path with the same (universal for  $M$ ) constant.

**Hint.** Use the previous problem and the inequality  $d(x, y) \leq d_h(x, y) \leq Cd(x, y)$ .

**Remark.** We have established that a Lipschitz connected metric space is path-connected.

**Exercise 10.15.** Consider the circle  $S$  on the plane with induced metric. Show that  $S$  is Lipschitz connected with constant  $\frac{\pi}{2}$ .

**Exercise 10.16 (\*).** Show that  $\frac{\pi}{2}$  is the smallest possible constant for which the circle with such a metric is Lipschitz connected.

**Exercise 10.17 (\*\*).** Consider the mapping  $]0, \infty[ \rightarrow \mathbb{R}^2$ , given in polar coordinates by the function  $\theta = 1/x, r = x$  (this is a spiral winding around 0 with the step  $\frac{1}{2\pi n}$ ). Let  $X$  be the closure of the graph of this function (that obviously consists of the graph itself and 0). Show that  $X$  is path-connected. Show that  $X$  is not Lipschitz connected, no matter what constant  $C$  we take.

**Exercise 10.18 (\*).** Let  $M$  be a locally compact complete metric space. Denote by  $S_\varepsilon(x)$  the sphere of radius  $\varepsilon$  with the centre in  $x$ . Show that the following conditions are equivalent.

- (i)  $M$  is Lipschitz connected, with constant  $C$
- (ii) for any  $x, y \in M$  and any  $r_1, r_2 > 0$  satisfying  $r_1 + r_2 \leq 1$ , the distance between  $S_{dr_1}(x)$  and  $S_{dr_2}(y)$  is not bigger than  $Cd(1 - r_1 - r_2)$ , where  $d = d(x, y)$ .

**Hint.** To derive (ii) from Lipschitz connectedness, consider a Lipschitz curve through  $x, y$ . Lipschitz connectedness follows immediately from (ii). The distance from  $x$  to  $S_{d(1-C^{-1}\varepsilon)}(y)$  is at most  $\varepsilon$ ; take as  $x_2$  the point of the sphere realizing this distance (that is possible, as the sphere is compact by Hopf-Rinow Theorem), and use induction.

**Remark.** Recall that in one version that Hopf-Rinow condition says that the distance between  $S_{dr_1}(x)$  and  $S_{dr_2}(y)$  equals  $d(1 - r_1 - r_2)$ .

## Loop space

**Definition 10.8.** Let  $(M, x)$  be a topological space with a marked point  $x$ . Consider the set  $\Omega(M, x)$  of paths  $[0, 1] \xrightarrow{\varphi} M$ ,  $\varphi(0) = \varphi(1) = x$ , with open-compact topology (the base of this topology consists of the set  $U(K, W)$  of mappings of a given compact  $K \subset [0, 1]$  into a given open set  $W \subset M$ ). The  $\Omega(M, x)$  is called **the loop space** for  $(M, x)$ .

**Exercise 10.19 (!).** Let  $M$  be metrizable. Show that  $\Omega(M, x)$  is metrizable too, with the metric

$$d(\gamma, \gamma') = \sup_{x \in [0, 1]} d(\gamma(x), \gamma'(x)).$$

**Exercise 10.20.** Let  $(M, x)$  be a space with a marked point  $x$ .  $M_0$  the connected component of  $x$ , and  $M_1$  the path-connected component of  $x$ . Show that  $\Omega(M, x) = \Omega(M_0, x) = \Omega(M_1, x)$ .

**Exercise 10.21.** Let  $X, Y$  be compacts,  $\mathcal{W}$  be the space of mappings from  $X$  to  $M$  endowed with open-compact topology. Construct a bijection between continuous mappings from  $Y$  to  $\mathcal{W}$  and continuous mappings  $X \times Y \rightarrow M$ .

**Exercise 10.22 (!).** Let  $\gamma, \gamma' \in \Omega(M, x)$ . Construct a bijection between the following sets:

- (i) Paths  $\Gamma : [0, 1] \rightarrow \Omega(M, x)$ , connecting  $\gamma$  and  $\gamma'$ .
- (ii) Continuous mappings  $\Psi$  from the square  $[0, 1] \times [0, 1]$  to  $M$  that map  $\{1\} \times [0, 1]$  to  $x$  and such that  $\Psi|_{[0, 1] \times \{0\}} = \gamma$ ,  $\Psi|_{[0, 1] \times \{1\}} = \gamma'$ .

**Definition 10.9.** Paths  $\gamma, \gamma' \in \Omega(M, x)$  for which the mappings  $\Psi : [0, 1] \times [0, 1] \rightarrow M$  exist, are called **homotopic**, and  $\Psi$ , that connects them, is called **homotopy**.

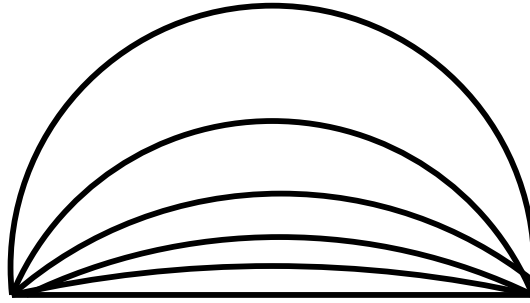
**Exercise 10.23.** Show that the set of loops homotopic to  $\gamma \in \Omega(M, x)$  is a path-connected component of  $\gamma \in \Omega(M, x)$ .

**Exercise 10.24.** Show that the homotopy of loops is an equivalence relation.

**Remark.** Loops homotopic to each other are also called **homotopy equivalent**.

**Definition 10.10.** Let  $(M, x)$  be path-connected. The set of homotopy equivalent classes of loops is denoted by  $\pi_1(M, x)$ .

**Exercise 10.25 (\*).** Let  $M \subset \mathbb{R}^2$  be the union of the closed segment  $[(0, 1), (0, -1)]$  and arcs of circles of diameters 3, 4, 5, ... that connect  $(0, 1)$  and  $(0, -1)$ .



Show that  $M$  is path-connected. Show that for any  $x \in M$  the space  $\Omega(M, x)$  is not locally path-connected.

**Exercise 10.26 (\*).** Let  $(M, d)$  be a geodesically connected locally compact metric space such that for a  $\delta > 0$  and any  $x, y \in M$ ,  $d(x, y) < \delta$ , the geodesic connecting  $x$  and  $y$  is unique. Let  $\Delta_\delta \subset M \times M$  be the set of pairs  $x, y \in M$ ,  $d(x, y) < \delta$ . Consider the mapping  $\Delta_\delta \rightarrow M$  of pairs to the middle points of the geodesics that connect pairs. Show that it is continuous.

**Hint.** Let  $\{(x_i, y_i)\}$  is a sequence of pairs converging to  $(x, y)$ , and  $\{z_i\}$  the sequence of middle points of corresponding geodesics. Due to local compactness,  $\{z_i\}$  has limit points and does not contain infinite discrete subsets. Any limit point of  $\{z_i\}$  will be the middle of geodesic connecting  $x$  and  $y$ . Thus  $\{z_i\}$  has unique limit point.

**Exercise 10.27 (\*).** Consider the mapping  $\Delta_\delta \otimes [0, 1] \xrightarrow{\Psi} M$ , of pairs  $x, y \in M$ ,  $d(x, y) = d$ ,  $t \in [0, 1]$  to points  $\gamma_{x,y}(\frac{t}{d})$ , where  $\gamma_{x,y}$  is a geodesic connecting  $x$  and  $y$  (when  $x = y$  set  $\Psi(x, y, t) = x$ ). Show that this mapping is continuous.

**Hint.** Use the previous problem and the construction of a geodesic as the limit of middle points of segments used in the proof of Hopf-Rinow Theorem.

**Definition 10.11.** Let  $M$  be a metric space. A path  $\gamma : [0, 1] \rightarrow M$  is called **piecewise geodesic** if  $[0, 1]$  is subdivided into  $[0, a_1], [a_1, a_2], \dots, [a_n, 1]$ , and on each of these closed intervals  $\gamma$  satisfies  $d(\gamma(x), \gamma(y)) = \lambda_i|x - y|$ , for some constant  $\lambda_i$

**Remark.** If  $M$  is an open set in  $\mathbb{R}^n$  with the natural metric then, as shown in Sheet 4, geodesics are segments. Thus piecewise geodesics are piecewise linear. Such mappings are also called **piecewise linear**.

**Exercise 10.28 (\*).** In the conditions of Exercise 10.26, consider  $\Omega(M, x)$  as a metric space (with sup-metric). Show that any loop  $\gamma \in \Omega(M, x)$  is homotopic to a piecewise geodesic, so that the homotopy can be chosen in any  $\varepsilon$ -neighbourhood  $B_\varepsilon(\gamma) \subset \Omega(M, x)$ .

**Exercise 10.29 (\*).** Derive from this that  $\Omega(M, x)$  is locally path-connected.

**Remark.** In such a situation  $\pi_1(M, x)$  is the set of connected components of  $\Omega(M, x)$ .

**Exercise 10.30.** Let  $M$  be an open set in  $\mathbb{R}^n$ . Show that  $\Omega(M, x)$  is locally path-connected.

**Hint.** Show that any loop can be homotopically deformed, in a sufficiently small  $\varepsilon$ -neighbourhood, into a piecewise linear.

## Fundamental group

**Exercise 10.31.** Given loops  $\gamma_1, \gamma_2 \in \Omega(M, x)$ , consider the loop  $\gamma_1\gamma_2 \in \Omega(M, x)$ , defined as follows:

$$\gamma_1\gamma_2(\lambda) = \begin{cases} \gamma_1(2\lambda) & \lambda \in [0, 1/2], \\ \gamma_2(2\lambda - 1) & \lambda \in [1/2, 1]. \end{cases}$$

Show that the class of the homotopy  $\gamma_1\gamma_2$  depends only on classes of homotopies  $\gamma_1, \gamma_2$ : if  $\gamma_1 \sim \gamma'_1$ ,  $\gamma_2 \sim \gamma'_2$  then  $\gamma_1\gamma_2 \sim \gamma'_1\gamma'_2$ .

**Exercise 10.32.** Show that  $(\gamma_1\gamma_2)\gamma_3$  is homotopy equivalent to  $\gamma_1(\gamma_2\gamma_3)$ .

**Exercise 10.33.** Given a loop  $\gamma \in \Omega(M, x)$ , denote by  $\gamma^{-1}$  the loop  $\gamma^{-1}(x) = \gamma(1 - x)$ . Show that the loops  $\gamma\gamma^{-1}$  and  $\gamma^{-1}\gamma$  are homotopic to the trivial loop  $[0, 1] \rightarrow x$ .

**Remark.** Loops that are homotopic to the trivial one are called **null-homotopic**.

**Exercise 10.34 (!).** Show that the operation  $\gamma_1, \gamma_2 \rightarrow \gamma_1\gamma_2$  makes  $\pi_1(M, x)$  into a group.

**Definition 10.12.** This group is called **the fundamental group** of  $M$ .

**Exercise 10.35.** Let  $X \xrightarrow{f} Y$  be a continuous mapping of path-connected spaces, and  $x \in X$ . Consider the corresponding mapping

$$\Omega(X, x) \xrightarrow{\check{f}} \Omega(Y, f(y)), \quad \gamma \mapsto \gamma \circ f.$$

Show that  $\check{f}$  maps homotopic paths to homotopic and induces a homomorphism of fundamental groups.

**Exercise 10.36.** Let  $M$  be a path-connected topological space, and  $x, y \in M$ . Consider the space  $\Omega(M, x, y)$  of paths  $[0, 1] \rightarrow M$  connecting  $x$  and  $y$  with open-compact topology. As above, paths are called homotopic (homotopy equivalent) if they lie in the same path-connected component of  $\Omega(M, x, y)$ . Define an operation  $\Omega(M, x, y) \times \Omega(M, y, z) \rightarrow \Omega(M, x, z)$ ,  $\gamma_1, \gamma_2 \mapsto \gamma_1\gamma_2$  using the same formula as in Exercise 10.31. Show that this mapping is continuous and maps homotopic paths to homotopic.

**Exercise 10.37 (!).** Let  $x, y \in M$ , and  $\gamma_{xy}[0, 1] \rightarrow M$  be a path connecting  $x$  and  $y$ . Define  $\gamma_{xy}^{-1}$  using  $\gamma_{xy}^{-1}(\lambda) = \gamma_{xy}(1 - \lambda)$ . Consider the mapping  $\Omega(M, x) \rightarrow \Omega(M, y)$ ,  $\gamma \mapsto \gamma_{xy}^{-1}\gamma\gamma_{xy}$  and  $\Omega(M, y) \rightarrow \Omega(M, x)$ ,  $\gamma \mapsto \gamma_{xy}\gamma\gamma_{xy}^{-1}$ . Show that these mappings map homotopic paths to homotopic. Let  $f, g$  be corresponding maps on fundamental groups. Show that  $f, g$  are inverses of each other and induce an isomorphism of groups  $\pi_1(M, x) \xrightarrow{\varphi_{\gamma_{xy}}} \pi_1(M, y)$ .

**Remark.** As can be seen from the preceding problem, if  $\pi_1(M)$  is not abelian then the isomorphism  $\pi_1(M, x) \cong \pi_1(M, y)$  obtained there nontrivially depends upon the choice of the path  $\gamma_{xy}$ . Nevertheless, when the dependence upon the marked point is not important, the fundamental group  $M$  is denoted simply by  $\pi_1(M)$ . This notation is not quite correct.

**Exercise 10.38 (!).** In conditions of the preceding problem, let  $x = y$ , and  $\gamma_{xx}$  a path. Show that the isomorphism  $\pi_1(M, x) \xrightarrow{\varphi_{\gamma_{xx}}} \pi_1(M, x)$  obtained above can be expressed via  $\gamma_{xx}$  as follows:  $\gamma \longrightarrow \gamma_{xx}\gamma\gamma_{xx}^{-1}$ .

## Simply connected spaces

**Definition 10.13.** Let  $M$  be a path-connected topological space. We say that  $M$  is **simply connected** when all the loops on  $M$  are contractible, i.e. when  $\pi_1(M) = \{1\}$ .

**Exercise 10.39.** Show that  $\mathbb{R}^n$  is simply connected.

**Definition 10.14.** Let  $(M, x)$  be a topological space with a marked point,  $M \times [0, 1] \xrightarrow{\varphi} M$  be a continuous mapping such that  $\varphi(M \times \{1\}) = \{x\}$  and  $\varphi|_{M \times \{0\}}$  the identity mapping from  $M = M \times \{0\}$  to  $M$ . Then  $(M, x)$  is called **contractible**. In such a situation one says that  $\varphi$  defines a **homotopy between the identity mapping and the projection**  $M \longrightarrow \{x\}$ .

**Exercise 10.40 (!).** Let  $(M, x)$  be path-connected and contractible. Show that for any point  $y \in M$  the space  $(M, y)$  is contractible.

**Hint.** Let  $M \times [0, 1] \xrightarrow{\varphi} M$  be a homotopy between the identity mapping and the projection onto  $\{x\}$ , and  $[1, 0] \xrightarrow{\gamma} M$  be a path connecting  $x$  and  $y$ . Take  $M \times [0, 1] \xrightarrow{\varphi_1} M$ , mapping  $(m, t)$  to  $\varphi(m, 2t)$  for  $t \in [0, 1/2]$  and  $(m, t)$  in  $\gamma(2t - 1)$  for  $t \in [1/2, 1]$ .

**Exercise 10.41.** Show that a contractible topological space is path-connected.

**Remark.** Two preceding problems immediately imply that the contractibility of  $(M, x)$  does not depend upon the choice of  $x$ . Thus we say in the remainder simply “ $M$  is contractible”.

**Exercise 10.42.** Show that a contractible space is simply connected.

**Exercise 10.43 (!).** Let  $V \subset \mathbb{R}^n$  be a **star subset** of  $(\mathbb{R}^n, x)$ , that is, it satisfies the property that any line through  $x \in \mathbb{R}^n$  intersects  $V$  in a connected set, and  $x \in V$ . Show that  $V$  is contractible.

**Exercise 10.44.** Let  $V \subset \mathbb{R}^n$  be a convex set. Show that it is contractible.

**Definition 10.15.** Let  $N$  be a subset of a topological space  $M$ . The **deformation retract** (or simply **retract**) of  $M$  to  $N$  is a continuous mapping  $M \times [0, 1] \xrightarrow{\varphi} M$ , such that  $\varphi(M \times \{1\}) \subset N$ , its restriction onto  $N$  the identity, and  $\varphi|_{M \times \{0\}}$  an identity mapping. In this case  $N$  is called a **retract** of  $M$ .

**Exercise 10.45 (!).** Let  $N$  be a retract of  $M$ , and  $n \in N$ . Show that the natural mapping  $\pi_1(N, n) \longrightarrow \pi_1(M, n)$  is an isomorphism.

**Definition 10.16.** Let  $M$  be a topological space, and  $\sim$  be an equivalence relation. As always, the set of equivalence classes is denoted by  $M/\sim$ . We introduce on  $M/\sim$  the **quotient topology**: the open subsets of  $M/\sim$  are those, whose preimages in  $M$  are open. In particular, if  $G$  is a group acting on  $M$ , there is the natural (orbit) equivalence relation on  $M$ :  $x \sim y$  if there exists  $g \in G$  satisfying  $g \cdot x = y$ . The quotient of  $M$  w.r.t. this equivalence relation is called the quotient space w.r.t. the  $G$ -action and denoted by  $M/G$ . The corresponding equivalence classes are called  **$G$ -orbits** of  $M$ .

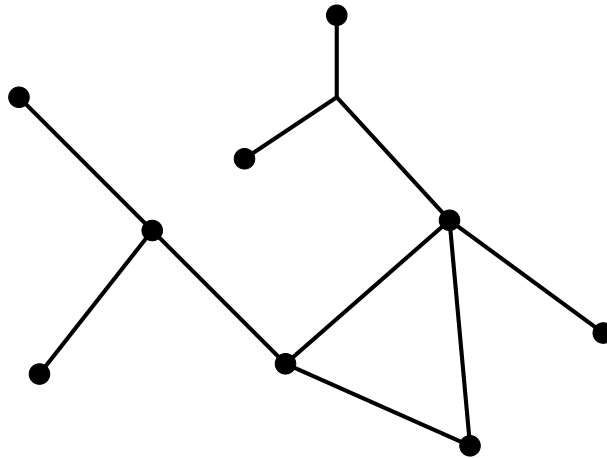
**Exercise 10.46.** Let  $M$  be a Hausdorff topological space and  $\{x_1, \dots, x_n\} \subset M$  and  $\{y_1, \dots, y_m\} \subset M$  two disjoint finite subsets. Show that for these subsets there exist non-intersecting neighbourhoods.

**Exercise 10.47 (!).** Let  $M$  be a Hausdorff topological space and  $G$  a finite group of  $M$ -homeomorphisms. Show that  $M/G$  is Hausdorff.

**Hint.** Let  $x, y$  be two points in distinct  $G$ -orbits. Find non-intersecting  $G$ -invariant neighbourhoods of  $x$  and  $y$ . For this, apply Exercise 10.46 to the orbits  $Gx, Gy$ , obtain neighbourhoods  $U, U'$ , and pick  $\bigcap_{g \in G} gU, \bigcap_{g \in G} gU'$ .

**Exercise 10.48 (\*).** Give an example of a Hausdorff space  $M$  and non-Hausdorff space  $M/G$  (here the group  $G$  will be infinite).

**Definition 10.17.** Let  $\Gamma$  be a graph, that is, a data collection consisting of “vertex set”  $\{\mathcal{V}\}$  and “edge set”  $\{\mathcal{R}\}$ , and information on which vertices are endpoints of which edges.



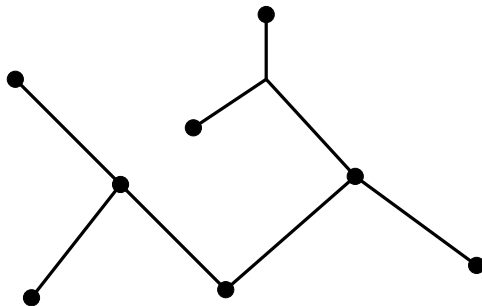
More precisely, one may define  $\Gamma$  as a pair of sets  $\mathcal{V}, \mathcal{R}$  and a surjection  $\{\mathcal{R}\} \times \{l, \infty\} \xrightarrow{\bar{\cdot}} \{\mathcal{V}\}$ . Introduce on  $\{\mathcal{R}\} \times [l, \infty]$  the equivalence relation generated by the following: endpoints of two edges are equivalent if they are incident to the same vertex. This relation glues together endpoints of edges through the same vertex. The quotient  $\{\mathcal{R}\} \times [l, \infty]$  w.r.t. this equivalence relation is called the **topological space of the graph**.

**Exercise 10.49.** Show that the topological space of any graph is Hausdorff.

**Exercise 10.50.** A graph is called connected if any vertex is connected to any other vertex by a sequence of edges. Show that the topological space of a connected graph is path-connected.

**Exercise 10.51 (\*\*).** Let  $\Gamma$  be a graph with infinite vertex set. Show that  $\Gamma$  contains either an infinite clique (i.e. the set of pairwise connected by edges vertices), or an infinite coclique (i.e. the set of vertices such that none of them are connected by an edge).

**Exercise 10.52 (!).** Let  $\Gamma$  be a connected graph with  $n$  vertices and  $n - 1$  edges (such a graph is called a **tree**).



Show that the topological space  $M_\Gamma$  of  $\Gamma$  is contractible.

**Exercise 10.53 (\*)**. Let  $\Gamma$  be an infinite graph so that each of its connected finite subgraphs is a tree. Show that  $\pi_1(M_\Gamma) = \{1\}$ .

**Exercise 10.54 (\*)**. Let  $S^n$  be an  $n$ -dimensional sphere ( $n > 1$ ). Show that  $S^n$  is simply connected.

**Hint**. Use geodesic connectedness.

## Coverings

**Definition 10.18**. Let  $\tilde{M} \xrightarrow{\pi} M$  be a continuous mapping of topological spaces;  $\pi$  is called a **covering** when any point has a neighbourhood  $U$  such that  $\pi^{-1}(U)$  is the product of  $U$  and a discrete topological space  $K$ , so that the natural mapping  $\pi^{-1}(U) \xrightarrow{\pi} U$  coincided the projection  $\pi^{-1}(U) = U \times K \rightarrow U$ . In this case one also says that  $\tilde{M}$  **covers**  $M$ .

We consider the circle  $S^1$  as the quotient  $S^1 = \mathbb{R}/\mathbb{Z}$ . This gives a natural group structure on  $S^1$ .

**Exercise 10.55**. Let  $n \neq 0$  be an integer. Consider a natural mapping  $S^1 \rightarrow S^1, t \rightarrow nt$ . Show that it is a covering.

**Exercise 10.56**. Show that the natural projection  $\mathbb{R} \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$  is a covering.

**Exercise 10.57**. Show that the natural projection  $\mathbb{R}^n \rightarrow (S^1)^n$  is a covering.

**Exercise 10.58**. Consider the quotient  $S^n \rightarrow S^n/\{\pm 1\} = \mathbb{R}P^n$  of the sphere w.r.t. the central symmetry, with the natural topology. Show that it is a covering.

**Exercise 10.59**. Let  $\tilde{M} \xrightarrow{\pi} M$  be a covering, and  $\tilde{M}' \subset \tilde{M}$  a subspace that covers  $M$ , too. Show that  $\tilde{M}'$  is clopen in  $\tilde{M}$ .

**Exercise 10.60**. Let  $\tilde{M} \xrightarrow{\pi} M$  be a covering, and  $M$  path-connected. Show that  $\tilde{M}$  is locally path-connected. Show that any path-connected component of  $\tilde{M}$  covers  $M$ .

**Exercise 10.61 (!)**. Let  $\tilde{M} \xrightarrow{\pi} M$  be a covering, and  $M$  path-connected. Show that  $\tilde{M}$  is connected iff it is path-connected.

**Definition 10.19**. Let  $\gamma : [a, b] \rightarrow M$  be a path, and  $\tilde{M} \xrightarrow{\pi} M$  a covering of  $M$ . A mapping  $\tilde{\gamma} : [a, b] \rightarrow \tilde{M}$  is called a **lifting of  $\gamma$**  if  $\tilde{\gamma} \circ \pi = \gamma$ .



**Exercise 10.62 (!).** Let  $\tilde{M} \xrightarrow{\pi} M$  be a covering, and  $\gamma : [a, b] \rightarrow M$  a path joining  $x$  and  $y$ . Show that for any  $\tilde{x} \in \pi^{-1}(\{x\})$  the lifting  $\tilde{\gamma}$ , mapping  $a$  to  $\tilde{x}$ , exists, and is unique.

**Exercise 10.63 (!).** Show that homotopic paths are lifted to homotopic paths, and that  $\tilde{\gamma}(y) \in \pi^{-1}(\{y\})$  is uniquely determined by the class of the homotopy  $\gamma$  in  $\Omega(M, x, y)$  and the point  $\tilde{x}$ .

**Remark.** Denote by  $\pi_1(M, x, y)$  the set of classes of homotopic paths from  $x$  to  $y$ . We have a mapping

$$\pi^{-1}(\{x\}) \times \pi_1(M, x, y) \xrightarrow{\Psi} \pi^{-1}(\{y\})$$

**Definition 10.20.** Let  $\tilde{M} \xrightarrow{\pi} M$  be a cover, and  $M$  path-connected. The space  $\tilde{M}$  is called a **universal cover** if it is connected and simply connected.

**Remark.** Simple connectedness was defined for path-connected spaces only. But this does not present an obstacle, as it follows from the Exercise 10.61 that  $\tilde{M}$  is path-connected.

**Exercise 10.64 (!).** Let  $\tilde{M} \xrightarrow{\pi} M$  be a universal cover. Fix  $x \in M$  and  $\tilde{x} \in \pi^{-1}(\{x\})$ . Consider the mapping  $\pi_1(M, x) \xrightarrow{\psi} \pi^{-1}(\{x\})$ , constructed in Exercise 10.63, and  $\psi(\gamma) = \Psi(\tilde{x}, \gamma)$ . Show that it is a bijection.

**Exercise 10.65.** Show that  $\pi_1(S^1) = \mathbb{Z}$ .

**Exercise 10.66.** Show that  $\pi_1((S^1)^n) = \mathbb{Z}^n$ .

**Exercise 10.67 (\*).** Show that for  $(n > 1)$  one has  $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ .

**Exercise 10.68.** Find the fundamental groups of all the letters of Greek alphabet, except  $\Phi$  and  $B$ . (More precisely, graphs modelled by these letters.)

**Exercise 10.69 (\*).** Given a finite connected graph with  $n$  edges and  $n$  vertices, consider its topological space  $M$ . Show that  $\pi_1(M) = \mathbb{Z}$ .