GEOMETRY 11: Galois coverings

The subject of Galois covering that is covered in this exercise sheet is very similar to the Galois theory of field extensions. This is not a coincidence. In algebraic geometry methods from topology and differential geometry are applied to objects of algebro-geometric and number-theoretic nature. The version of Galois theory that is presented in ALGEBRA-11 goes back to A. Grothendieck. Grothendieck has given a definition of a fundamental group in such a way that Galois group and fundamental group of a topological space turned out to be particular cases of a more general construction. If one studies coverings and field extensions, it is very useful to keep in mind that these two things are similar.

All topological spaces in this exercise sheet are supposed to be Hausdorff.

Exercise 11.1. Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and let M_1 be a connected component of $\tilde{M}$. Prove that $\pi^{-1}(M_1)$ is a connected component of $\tilde{M}$.

Exercise 11.2 (!). Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and let $\tilde{M}$ and M be connected and non-empty, and π injective. Prove that π is a homeomorphism.

Definition 11.1. Let $\tilde{M} \xrightarrow{\pi} M$, $\tilde{M}' \xrightarrow{\pi'} M$ be coverings. A **morphism of coverings** is a continuous map $\varphi: \tilde{M} \longrightarrow \tilde{M}'$, that respects the projection to M, in other words, such that $\varphi \circ \pi' = \pi$. The set of all morphisms between coverings is denoted by $\operatorname{Mor}(\tilde{M}, \tilde{M}')$. An isomorphism of covering is a morphism that is invertible, and moreover $\varphi^{-1} \circ \varphi = \mathsf{Id}$, $\varphi \circ \varphi^{-1} = \mathsf{Id}$.

Exercise 11.3 (!). Let $\varphi: \tilde{M} \longrightarrow \tilde{M}'$ be a morphism of coverings. Prove that $\varphi: \tilde{M} \longrightarrow \tilde{M}'$ is a covering.

Exercise 11.4. Let M be connected and $\tilde{M} \xrightarrow{\pi} M$ be a covering. Prove that $\tilde{M}$ is locally connected.

Exercise 11.5. Let $M_1 \longrightarrow M_2$ and $M_2 \longrightarrow M_3$ be coverings.

** Is it true that the composition $M_1 \longrightarrow M_3$ is also a covering?

! Assume every point of $_3$ has a simply connected neighbourhood. Prove that $M_1 \longrightarrow M_3$ is a covering.

Exercise 11.6. Let $\tilde{M} \xrightarrow{\pi} M$, $\tilde{M}' \xrightarrow{\pi'} M$ be coverings and $\tilde{M}' \coprod \tilde{M}$ be their disjoint sum. Prove it is also a covering of M.

Exercise 11.7. Let M be connected and $\tilde{M} \xrightarrow{\pi} M$ be a covering. Prove that $\tilde{M} \cong \coprod_{\alpha \in I} \tilde{M}_\alpha$ where $\{\tilde{M}_\alpha\}$ is the set of connected components of $\tilde{M}$ regarded as coverings of M.

Definition 11.2. A **splitting** of a covering $\tilde{M} \xrightarrow{\pi} M$ is an isomorphism between $\tilde{M}$ and a covering of the form $\tilde{M} \cong V \times M$ where V is a set with discrete topologogy.

Exercise 11.8. Let $\tilde{M} \xrightarrow{\pi} M$ be a covering of a connected space M. Prove that π splits if and only if all connected components $\tilde{M}$ are isomorphic to M.

Galois coverings

Exercise 11.9 (!). Let $M_1 \xrightarrow{\pi_1} M$, $M_2 \xrightarrow{\pi_2} M$ be coverings. Consider the following subset in $M_1 \times M_2$

$$M_1 \times_M M_2 := \{(m_1, m_2) \in M_1 \times M_2 \ \mid \ \pi_1(m_1) = \pi_2(m_2)\}$$

We consider $M_1 \times_M M_2$ as a topological space (with the topology induced from $M_1 \times M_2$). Prove that the natural map $M_1 \times_M M_2 \longrightarrow M$ is a covering.

Definition 11.3. The space $M_1 \times_M M_2$ together with the natural map to M is called the **product of coverings M_1, M_2**. The product of arbitrary number of coverings is defined similarly.

Remark. If one uses the analogy between field extensions and coverings then disjoint unions of coverings correspond to a direct sums of semisimple Artinian rings, and products of coverings correspond to tensor products.

Exercise 11.10. Let M_1, M_2, M_3 be coverings of M. Proe that morphisms from M_3 to $M_1 \times M_2$ are in bijective correspondence with pairs of morphisms $\varphi_1 : \ M_3 \longrightarrow M_1$, $\varphi_2 : \ M_2 \longrightarrow M_1$.

Exercise 11.11. Consider $\mathbb{R}$ as a covering of S^1. How many connected components does $\mathbb{R} \times_{S^1} \mathbb{R}$ have?

Definition 11.4. Let $M_1 \xrightarrow{\varphi} M_2$ be a morphism between two coverings of . Define the **graph of φ** as a subset in $M_1 \times_M M_2$ that consists of pairs of the form $(m, \varphi(m))$ for all $m \in M_1$.

Exercise 11.12 (!). Let $M_1 \xrightarrow{\varphi} M_2$ be a morphism between two coverings and let Γ_φ be its graph. Prove that Γ_φ is both open and closed in $M_1 \xrightarrow{\varphi} M_2$.

Exercise 11.13. Let $[\tilde{M} : M]$ be a covering, and moreover let M and $\tilde{M}$ be connected (such a covering is called **connected**). Let $X \subset \tilde{M} \times_M \tilde{M}$ be a connected component. Prove that X is the graph of an automorphism $\nu : \ \tilde{M} \longrightarrow \tilde{M}$ if and only if the projection on the first components is an isomorphism $X \cong \tilde{M}$.

Exercise 11.14 (!). Let $[\tilde{M} : M]$ be a connected covering. Consider the projection on the first argument $\tilde{M} \times_M \tilde{M} \longrightarrow \tilde{M}$ as a covering of $\tilde{M}$. Construct a bijective correspondence between $\operatorname{Mor}_{\tilde{M}}(\tilde{M}, \tilde{M} \times_M)$ and the set of automorphisms of $\tilde{M}$ over M.

Hint. Use the previous problem.

Definition 11.5. Let $[\tilde{M} : M]$ be a covering and assume M and $\tilde{M}$ are connected. Then $[\tilde{M} : M]$ is called a **Galois covering**, if the covering $\tilde{M} \times_M \tilde{M} \longrightarrow \tilde{M}$ is split. In this situation the automorphism group of $\tilde{M}$ over M is called the **Galois group of the covering** $[\tilde{M} : M]$ (denoeted $\operatorname{Gal}([\tilde{M} : M])$). Sometimes the Galois group is called **monodromy group**, or **deck transform group**.

Exercise 11.15 (!). Let M be connected and let $[\tilde{M} : M]$ be such a Galois covering that every point of M has exactly n preimages (such a covering is called n-sheet covering). Prove that the Galois group $[\tilde{M} : M]$ has exactly n elements.

Hint. Prove thta $[\tilde{M} \times_M \tilde{M} : \tilde{M}]$ is n-sheet covering too, and use the previous problem.

Definition 11.6. Let a group G act on a set S. The action is called **free** if $s \neq gs$ for any $g \in G$, $s \in S$, if $g \neq 1$. The action is called **transitive** if for any two points s_1, $s_2 \in S$ there exists $g \in G$ such that $g(s_1) = s_2$.

Exercise 11.16. Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and $G = \operatorname{Aut}_M(\tilde{M})$ be its automorphism group. Assume that M is connected. Prove that for any $x \in M$ the group G acts freely on $\pi^{-1}(x)$.

Exercise 11.17 (!). Let $\tilde{M} \xrightarrow{\pi} M$ be a Galois covering and let $x \in M$ be any point. Prove that $\operatorname{Gal}([\tilde{M} : M])$ acts on $\pi^{-1}(x)$ freely and transitively.

Hint. Find a bijective correspondence between $\pi^{-1}(X)$ and the set of connected componentns $\tilde{M} \times_M \tilde{M}$, and apply Exercise 11.14.

Exercise 11.18 (!). Let $\tilde{M} \xrightarrow{\pi} M$ be a covering and let $x \in M$ be any point. Prove that $\operatorname{Aut}_M(\tilde{M})$ acts transitively on $\pi^{-1}(X)$ if and only if $[\tilde{M} : M]$ is a Galois covering.

Exercise 11.19. Consider the covering $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$. Prove that it is a Galois covering.

Exercise 11.20. Take $n \in \mathbb{Z}$ and consider an n-sheeted covering $S^1 \longrightarrow S^1$, $t \mapsto nt$. Prove that this is a Galois covering.

Definition 11.7. Let M be a topological space and let G be a group that acts on M by continuous transformations. Consider the space of G-orbits M/G. Recall (GEOMETRY-10) that the topology on M/G is introduced a follows: a subset of M/G is open if and only if its preimage in M is open. The set M/G with this topology is called a **quotient of** M by the action of G.

Exercise 11.21 (!). Let $[\tilde{M} : M]$ be a covering and assume $G \subset \operatorname{Aut}_M(\tilde{M})$ acts on $[\tilde{M} : M]$ by automorphisms. Prove that this action is free and that the quotient $\tilde{M}/G$ is Hausdorff and is a covering of M.

Remark. Taking a quotient by the action of G plays the same role in the Galois coverings theory as taking G-invariant in the Galois theory of field extensions.

Exercise 11.22 (!). Let $[\tilde{M} : M]$ be a covering and let G be its automorphism group. Prove that $\tilde{M}/G$ is isomorphic to M if and only if $[\tilde{M} : M]$ is a Galois covering.

Hint. Use Exercise 11.18.

Remark. In the several exercises that follow the statement and the proof mimic almost verbatim the corresponding exercises about Galois field extensions.

Exercise 11.23. Let $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3$ be a sequence of coverings, and moreover φ_i are surjective and their composition is split. Prove that φ_i's split.

By analogy with Galois theory of field extension the coverings of the form $\tilde{M} \xrightarrow{\pi} M$ will further be denoted $[\tilde{M} : M]$.

Exercise 11.24 (!). Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings, and assume all M_i are connected and $[M_1 : M_3]$ is a Galos covering. Prove that $M_1 \times_{M_3} M_2$ splits as a covering of M_1.

Hint. Use Exercise 11.23, apply it to the sequence of coverings

$$M_1 \times_{M_3} M_1 \longrightarrow M_1 \times_{M_3} M_2 \longrightarrow M_1 \times_{M_3} M_3.$$

Exercise 11.25 (!). Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings and assume that $[M_1 : M_3]$ is a Galois covering. Prove that $[M_1 : M_2]$ is a Galois covering.

Hint. Use Exercise 11.23.

Exercise 11.26. Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings. Prove that

$$M_1 \times_{M_3} M_1 \cong M_1 \times_{M_2} (M_2 \times_{M_3} M_2) \times_{M_2} M_1.$$

Exercise 11.27. Deduce the following statement from the previous exercise: if $M_1 \longrightarrow M_2 \longrightarrow M_3$ is a sequence of coverings and if $[M_1 : M_2]$ and $[M_2 : M_3]$ is a Galois covering then $[M_1 : M_3]$ is also a Galois covering.

Exercise 11.28. Let $[\tilde{M} : M]$ be a covering and let G be its Galois gorup and $G' \subset G$ be its subgroup. Consider the quotient $\tilde{M}/G'$. Prove that $[\tilde{M} : \tilde{M}/G']$ is a Galois covering with Galois group G'.

Definition 11.8. Let $\tilde{M} \longrightarrow M$ be a covering. A **quotient covering** $[\tilde{M} : M]$ is a covering $\tilde{M}' \longrightarrow M$ together with a sequence of coverings $\tilde{M} \longrightarrow \tilde{M}' \longrightarrow M$.

Exercise 11.29 (!). (fundamental theorem of Galois theory) Let $[\tilde{M} : M]$ be a Galois covering with Galois group G. Consider the correspondence that to a subgroup $G' \subset G$ associates a quotient covering $[\tilde{M}/G' : M]$. Prove that this correspondence defines a bijection between the set of subgroups of G and the set of isomorphism classed of quotient coverings.

Exercise 11.30. Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of coverings and assume that $[M_1 : M_3]$ is a Galois covering. Consider the natural projection

$$M_1 \times_{M_3} M_1 \xrightarrow{\Psi} M_2 \times_{M_3} M_2.$$

Let $g \in \operatorname{Gal}([M_1 : M_3])$ and $e_g \subset M_1 \times_{M_3} M_1$ be the graph $\{(m, g(m))\}$ of the action of g in $M_1 \times_{M_3} M_1$. Prove that $g \in \operatorname{Gal}([M_1 : M_2]) \subset \operatorname{Gal}([M_1 : M_3])$ if and only if e_g projects to the diagonal componont in $M_2 \times_{M_3} M_2$.

Exercise 11.31. Let $M_1 \longrightarrow M_2 \longrightarrow M_3$ be a sequence of Galois coverings. Prove that the natural projection

$$M_1 \times_{M_3} M_1 \xrightarrow{\Psi} M_2 \times_{M_3} M_2.$$

defines a surjective homomorphism $\operatorname{Gal}([M_1 : M_3]) \xrightarrow{\psi} \operatorname{Gal}([M_2 : M_3])$. Prove that $\ker\psi = \operatorname{Gal}([M_1 : M_2])$.

Hint. Use the fact the Galois $\operatorname{Gal}([M_i : M_3])$ is identified with the set of connected components of $M_i \times_{M_3} M_i$ and use the previous problem.

Exercise 11.32 (!). Let $\tilde{M} \longrightarrow M$ be a Galois covering and $G' \longrightarrow \tilde{M}/G'$ be the bijective correspondence between quotient coverings and subgroups of the Galois gorup defined above. Prove that G' is a normal subgroup if and only if $[\tilde{M}/G' : M]$ is a Galois covering.

Coverings of linearly connected spaces

Definition 11.9. Let M be a metric space. Recall that a **geodesic** in M is a path $[a, b] \xrightarrow{\gamma} M$ such that $d(\gamma(x), \gamma(y)) = |x - y|$. The **length** of the geodesic is the distance between its ends. A path is called **piece-wise geodesic** if it can be decomposed into a union of a finite number of geodesic segments. The **length** of a piece-wise geodesic path is defined to be the sum of lengths of its geodesic pieces. We denote the length of a path γ by $|\gamma|$.

Definition 11.10. Let Γ be a graph and M_Γ be its topological space. We say that Γ is **connected** if its topological space is connected.

Exercise 11.33 (!). Prove that a graph is connected if and only if any two vertices are connected by a finite sequence of edges. Prove that a connected graph is linearly connected.

Exercise 11.34 (!). Let Γ be a connected graph. By construction, on each edge $r_\alpha \subset M_\gamma$ of the graph are defined the coordinates that identify the edge with $[0,1]$. Let γ be a piece-wise linear path in Γ_M, that is, a path that consists of a finite number of intervals of the form $[a_i, b_i] \xrightarrow{\varphi_i} [\lambda_i, \mu_i] \subset r_\alpha$, where φ_i is linear. Define $|\gamma| := \sum |\lambda_i, \mu_i|$ as the sum of lengths of all intervals that contain this path. Define $d(x,y) := \inf |\gamma|$ where γ runs throuh all piece-wise linear paths from x to y. Prove that $d(x,y)$ defines a metric and M_Γ is geodesically connected.

Definition 11.11. This metric is called the **standard metric on the topological space of a graph**.

Definition 11.12. Geodesically connected manifold M is called **star-shaped** if any two points of M are connnected by a unique geodesic.

Exercise 11.35. Prove that any convex subset in $\mathbb{R}^n$ (with the standard metric) is star-shaped.

Exercise 11.36 (*). Find a metric on $M = \mathbb{R}^2$ such that is geodesically connected and there are infinitely many geodesics connecting arbitrary two fixed points.

Exercise 11.37 (*). Let Γ be a tree, that is, a connected finite graph that has n vertices and $n-1$ edges. Prove that M_Γ with the standard metric is start-shaped.

Exercise 11.38 (*). Let Γ be a finite graph such that Γ_M is star-shaped. Prove that Γ is a tree.

Exercise 11.39 (!). Let M be a geodesically connected manifold, $\tilde{M} \xrightarrow{\pi} M$ be a covering, and x and y be two points in $\tilde{M}$. Consider the set $S_{x,y}$ of all paths on $\tilde{M}$ that connect x and y, such that their projection to M is piece-wise geodesic. Consider the following function on $\tilde{M} \times \tilde{M}$

$$\tilde{d}(x,y) = \inf_{\gamma \in S_{x,y}} |\pi(\gamma)|$$

Prove that it is a metric. Prove that $\tilde{d}(x,y) \geqslant d(\pi(x), \pi(y))$.

Exercise 11.40 (*). In the previous problem setting prove that $\tilde{M}$ is geodesically connected.

Exercise 11.41. Let M be a geodesicall connected metric space and $\tilde{M} \longrightarrow M$ be its covering. Prove that the connected component of the preimage of a geodesic is a geodesic in $(\tilde{M}, \tilde{d})$.

Hint. Prove that the preimage of a geodesic is a geodesic in a neighbourhood of every point. Then use the inequality $\tilde{d}(x,y) \geqslant d(\pi(x), \pi(y))$.

Exercise 11.42 (!). Let (M,d) be a star-shaped metric space and $\tilde{M} \xrightarrow{\pi} M$ be its connected covering. Let moreover $x \in \tilde{M}$ be any point and U_x be the set of points $y \in M$ that can be connected with x by a geodesic. Prove that U_x is open and closed in $\tilde{M}$ and that $(U_x, \tilde{d})$ is star-shaped. Deduce that the natural projection $\tilde{M} \xrightarrow{\pi} M$ is an isometry and a homeomorphism.

Hint. Use the previous exercise.

Exercise 11.43. Let $M = [0,1] \times [0,1]$ be a square and $\tilde{M} \longrightarrow M$ be its connected covering. Prove that it is a homeomorphism.

Exercise 11.44. Let M be a linearly connected and simply connected space, and $\tilde{M} \xrightarrow{\pi} M$ be a connected covering. Prove that it is a homeomorphism

Hint. Prove that $\tilde{M}$ is linearly connected. Let $x, y \in \pi^{-1}(x_0)$ be two points and $\tilde{\gamma}$ be a path that connects them. Then $\gamma := \pi(\tilde{\gamma})$ is a loop. Since M is simply connected, γ can be extended to a map from the square to $X \subset M$ (prove it). Consider the preimage of this square in $\tilde{M}$ and let $\tilde{X}$ be the component of the preimage that contains $\tilde{\gamma}$. Use the previous exercise to prove that $\tilde{X} \xrightarrow{\pi} X$ is a homeomorphism and deduces that $x = y$.

Exercise 11.45. In the previous problem setting prove that any covering M splits.

Definition 11.13. Let M be a any (not necessarily linearly connected) connected topological space. The space M is called **simply connected** if any covering of M is split.

Remark. Thanks to the previous exercise this definition is consistent with the definition of a simply connected linearly connected topological spaces given in GEOMETRY 10.

Definition 11.14. Let M be connected. A covering $\tilde{M} \longrightarrow M$ is called **universal** if it is simply connected.

Exercise 11.46 (!). Prove a universal covering is a Galois covering.

Exercise 11.47 (!). Prove that universal covering is unique up to isomorphism.

Hint. Let $\tilde{M}$, $\tilde{M}'$ be two universally coverings of M. Since $\tilde{M} \times_M \tilde{M}'$ is a covering of $\tilde{M}$, $\tilde{M}'$, it splits over $\tilde{M}$, $\tilde{M}'$. This means that any connected component $\tilde{M} \times_M \tilde{M}'$ projects isomorphically to $\tilde{M}$, $\tilde{M}'$.

Existence of the universal covering

Exercise 11.48. Let M be linearly connected, $\tilde{M} \xrightarrow{\pi} M$ be a connected covering and $x \in M$ be any point. Prove that the cardinality of the set $\pi^{-1}(x)$ is not greater than the cardinality of $\pi_1(M)$.

Exercise 11.49. Prove that the cardinality $\pi^{-1}(x)$ is not greater than the cardinality of the set $M^{[0,1]}$ of maps from $[0,1]$ to M.

Exercise 11.50 (*). Let $\tilde{M} \xrightarrow{\pi} M$ be a connected covering of a connected M and $x \in M$ be any point. Prove that the cardinality of $\pi^{-1}(x)$ is not greater than $|2^{2S}|$, where $|2^{2S}|$ is the cardinality of the set of subsets of $S \times S$.

Hint. Choose $x_1, x_2 \in \pi^{-1}(x)$. Prove that there exists a collection of such connected open subsets $\{\tilde{U}_\alpha\} \in \pi^{-1}(S)$ that $\tilde{U}_{\alpha_0}$ has non-empty intersection with the union of all $\tilde{U}_\alpha$ that are not equal to U_{α_0}, and moreover

$$\{x_1, x_2\} = \pi^{-1}(x) \cap (\bigcup \tilde{U}_\alpha)$$

Decreasing the base S if necessary one can assume that π splits over $\pi(U_\alpha)$ for all α. Prove that x_2 is determined uniquely if x_1, $\{\pi(U_\alpha)\}$ is given, and if it is known which U_α have non-empty intersection.

Exercise 11.51. Let M be connected and let V be a set of a cardinality defined below. Denote by $\mathcal{R}$ the set of all topologies defined on some subset $X \subset M \times V$ in such a way that the only natural projection $X \longrightarrow M$ is a covering. Prove that any connected covering M is isomorphic to some element of $\mathcal{R}$ if

a. M is linearly connected and the cardinality of V is $|M^{[0,1]}|$

b. **(*)** The cardinality of V is $|2^{2^S}|$ where S is the base of topology on M.

Remark. This exercise allows one to speak of "the set of isomorphism classes of coverings". Recall that not all mathematical objects are sets; thus, the class of all sets is not a set. In order to prove that a class is a set, one has to restrict its cardinality.

Definition 11.15. Let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be a collection of maps onto M indexed b I (possibly infinite, and even uncountable). Consider the set of all $(m_{\alpha_1}, m_{\alpha_1}, \dots) \in \prod M_\alpha$ such that $\pi_\alpha(m_\alpha) = m$ for some $m \in M$. This set is called the **fibre product of** $\{M_\alpha\}$ and is denoted by $\prod_M M_\alpha$.

Exercise 11.52. Let M be a topological space and $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be a collection of its coverings. Introduce on $\prod_M M_\alpha$ a topology in the following way. Let $U \subset M$ be open and let $\{U_\alpha \subset M_\alpha\}$ be the collection of open sets that cover U. Prove that the sets of the form $\prod_U U_\alpha \subset \prod_M M_\alpha$ define the base of topology on $\prod_M M_\alpha$. Prove thta $\prod_M M_\alpha$ is Hausdorff.

* Is it true that the natural projection $\prod_M M_\alpha \longrightarrow M$ is a covering?

! Suppose that every point of has a simply connected neighbourhood. Prove that the natural projection $\prod_M M_\alpha \longrightarrow M$ is a covering.

Definition 11.16. In this situation $\prod_M M_\alpha$ is called the **fibred product of** M_α **over** M or just a product of coverings $M_\alpha \xrightarrow{\pi_\alpha} M$.

Exercise 11.53. Assume all coverings $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ are split. Prove that $\prod_M M_\alpha$ split too.

Exercise 11.54 (!). Let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be a Galois covering. Prove thta any connected component of their product over M is a Galois covering too.

Hint. Use the Exercise 11.53.

Exercise 11.55. Let $\tilde{M}$ be a covering of M. Constructi the natural bijection between $\operatorname{Mor}(\prod_M M_\alpha, \tilde{M})$ and $\prod \operatorname{Mor}(M_\alpha, \tilde{M})$

Exercise 11.56 (*). Let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be the set of all coverings $S^1 \longrightarrow S^1$, $t \longrightarrow nt$, indexed by $n \in \mathbb{Z}$. Prove that any connected component of $\prod_M M_\alpha$ is isomorphic to $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} = S^1$.

Exercise 11.57. Let $\tilde{M} \xrightarrow{\pi} M$ be a covering, and assume that $\tilde{M}$ and M are connected, $x \in M$, $x_1, x_2 \in \pi^{-1}(x)$, W is the connected component of $\tilde{M} \times_M \tilde{M}$ that contains $x_1 \times x_2$, and W_1 is the connected component of $\tilde{M} \times_M \tilde{M} \times_M \tilde{M}$, that contains $x_1 \times x_2 \times x_2$. Prove that the natural projection $W_1 \longrightarrow W$ (forgetting the third argument) is an isomorphism.

Exercise 11.58. In the same situation, let $\{x_\alpha\}$ be a set of points in $\pi^{-1}(x)$, indexed by $\alpha \in I$, and let W be the corresponding component in the fibre product $\prod_{M,I} \tilde{M}$ of I copies of $\tilde{M}$, and W_1 be a component in $\left(\prod_{M,I} \tilde{M}\right) \times_M \tilde{M}$, that contains $\{x_\alpha\}$ and x_0, and moreover $x_0 \in \{x_\alpha\}$. Prove that the natural projection $W_1 \longrightarrow W$ is an isomorphism.

Exercise 11.59 (!). Let $\tilde{M} \xrightarrow{\pi} M$ be a connected covering and $x \in M$. Consider the product $\prod_{M,\{\pi^{-1}(x)\}} \tilde{M}$ of $\tilde{M}$ with itself indexed by the set $\pi^{-1}(x)$, and let $\tilde{M}_G$ be the connected component in $\prod_{M,\{\pi^{-1}(x)\}} \tilde{M}$ containing the product of all $x_\alpha \in \{\pi^{-1}(x)\}$. Prove that $\tilde{M}_G \times_M \tilde{M}$ splits over $\tilde{M}_G$. Prove that $\tilde{M}_G \longrightarrow M$ is a Galois covering.

Remark. We have proved that any covering is a quotient covering of a Galois covering.

Exercise 11.60. Let M be a connected topological space, $\mathcal{R}$ be the set of all isomorphism classes of connected coverings of M, and let $\{M_\alpha \xrightarrow{\pi_\alpha} M\}$ be the corresponding set of coverings, and $\tilde{M} \subset \prod_M M_\alpha$ be the connected component of their product. Prove that for any connected covering $\tilde{M}' \longrightarrow M$ there exists a surjective morphism of coverings $\tilde{M} \longrightarrow \tilde{M}'$.

Hint. Use the previous exercise.

Exercise 11.61. In the previous problem setting prove that $\tilde{M}$ is a Galois covering.

Exercise 11.62 (!). Deduce tht for any $\tilde{M} \longrightarrow M$ the covering $\tilde{M} \times_M \tilde{M}' \longrightarrow \tilde{M}$ splits.

Hint. Use the Exercise 11.24.

Exercise 11.63 (!). Let M any connected topological space, $\tilde{M} \longrightarrow M$ be a Galois covering constructed above. Prove that $\tilde{M}$ is simply connected.

Remark. We have obtained that any connected topological space has a universal covering. As was shown above, the universal cover is unique.

Exercise 11.64 (!). Let M be linearly connected, and $\tilde{M}$ be its universal covering, and $\operatorname{Gal}([\tilde{M} : M])$ be the corresponding Galois group. Prove that $\operatorname{Gal}([\tilde{M} : M])$ is not isomorphic to the Galoi group of M.

Definition 11.17. The **fundamental group** of a topological space is the group $\pi_1(M) := \operatorname{Gal}([\tilde{M} : M])$, where $\tilde{M}$ is the universal covering.

Definition 11.18. Subgroups $G_1, G_2 \subset G$ are called **conjugated** if there exists $g \in G$ such that G_1 is mapped to G_2 by the automorphism $x \longrightarrow x^g$.

Exercise 11.65 (*). Let $M_1 \longrightarrow M$ be a covering, and let $\tilde{M} \longrightarrow M_1 \longrightarrow M$ be the universal covering. Consider the subgroup $G_1 \subset \operatorname{Gal}([\tilde{M} : M]) = \pi_1(M)$, obtained as a result of the fundamental theory of the Galois theory. Prove that this correspondence defines a bijection between isomorphism classes of coverings of M and conjugacy classes of subgroups of $\pi_1(M)$.

Exercise 11.66 (!). Find all coverings of a circle up to isomorphism. Construct them explicitly.

Exercise 11.67 (*). Let M be a connected topological space such that all linear connected components of it are simply connected. Can it have a non-trivial fundamental group?

Exercise 11.68 (*). Let B be the set of polynomials $P(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \cdots + a_0$ over $\mathbb{C}$ that have distinct roots and let B_1 be the set of all tuplese $(x_1, \ldots, x_n) \in \mathbb{C}^n$ of pairwise distinct numbers $x_i \in \mathbb{C}$. Introduce on B and B_1 the natural topology of a subset of $\mathbb{C}^n$. Consider the map $B_1 \xrightarrow{\pi} B$, $(x_1, \ldots, x_n) \longrightarrow \prod(t - x_i)$. Prove that π is a Galois covering. Find its Galois group.

Exercise 11.69 (*). Construct a connected covering that is not a Galois cover.