GEOMETRY 12: fundamental group and homotopies

Homotopies

All topological spaces in this exercise sheet are assumed locally arcwise connected and Hausdorff, unless the contrary is stated.

Definition 12.1. Let $f_1, f_2 : X \longrightarrow Y$ be a continuous map of topological spaces. Recall that a **homotopy** between f_1 and f_2 is a continuous map $F : [0,1] \times X \longrightarrow Y$ such that $F\Big|_{\{0\}\times X}$ equals f_1, and $F\Big|_{\{1\}\times X}$ equals f_2.

Exercise 12.1. Prove that maps that are homotopic induce the same morphism $\pi_1(X) \longrightarrow \pi_1(Y)$.

Definition 12.2. Let $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ be continuous maps of topological spaces, moreover, $f \circ g$ $g \circ f$ are homotopic to identity maps from X to X and from Y to Y. Such maps are called **homotopy equivalences** and X and Y are then called **homotopy equivalent**.

Exercise 12.2. Prove that a composition of homotopy equivalence between maps is a homotopy equivalence. Prove that a homotopy equivalence of spaces is an equivalence relation.

Exercise 12.3 (!). Let $f : X \longrightarrow Y$ be a homotopy equivalence. Prove that f induces an isomorphism of fundamental groups.

Exercise 12.4. Let $X \subset Y$ be a retraction. Prove that X are Y homotopy equivalent.

Exercise 12.5 (!). Let X be a topological spaces. Prove that X is contractible if and only if it is homotopy equivalent to a point.

Exercise 12.6 (!). Consider the connected graph Γ which has n edges and n vertices. Prove that the associated topological space is homotopy equivalent to a circle.

Exercise 12.7 (!). Let M be a connected topological space and let $x, x', y, y' \in M$ be any points. Prove that the spaces of paths $\Omega(M, x, x')$ and $\Omega(M, y, y')$ are homotopy equivalent.

Hint. Consider a path γ_{xy} that connects x and y and let $\gamma_{x'y'}$ be the path that connects x' and y'. Let $\gamma_{xy}^{-1}(t) = \gamma_{xy}(1-t)$ and $\gamma_{x'y'}^{-1}(t) = \gamma_{x'y'}(1-t)$. Consider the map $f : \Omega(M, x, x') \longrightarrow \Omega(M, y, y')$ that maps any path $\gamma \in \Omega(M, x, x')$ into the composition $\gamma_{xy}^{-1}\gamma\gamma_{x'y'}$, and the analogous map $g : \Omega(M, y, y') \longrightarrow \Omega(M, x, x')$ that maps $\gamma \in \Omega(M, y, y')$ to $\gamma_{xy}\gamma\gamma_{x'y'}^{-1}$. Prove that fg is homotopic to the identity maps and that gf is homotopic to the identity map.

The space of paths on locally contractible spaces

Definition 12.3. Let M be a topological space. The space M is called **locally contractible** if every point has a contractible neighbourhood.

Exercise 12.8. Let M be a locally contractible topological space. Prove that M is locally arcwise connected.

Exercise 12.9 (*). Let M be a geodesically connected metric space such that for some $\delta > 0$ any two points that are at a distance $< \delta$ one from another are connected by a unique geodesic. Prove that M is locally contractible.

Exercise 12.10. Prove that any graph is locally contractible.

Definition 12.4. A topological space M is called a **manifold of dimension** n if any point has a neighbourhood that is homeomorphic to an open ball in $\mathbb{R}^n$.

Remark. Manifolds are easily seen to be locally contractible.

Exercise 12.11 (!). Prove that a sphere S^n is a manifold.

Hint. Use the stereographic projection.

Exercise 12.12. Let M be contractible, $x, y \in M$. Prove that all paths $\gamma \in \Omega(M, x, y)$ are homotopic.

Exercise 12.13 (!). Let $\gamma \in \Omega(M, x, y)$ be a path in a locally contractible space M and $\{U_\alpha\}$ be the set of contractible open sets in M. Choose a finite set in $\{U_\alpha\}$ such that it covers γ (this can be done since γ is compact). Let $V_1, \dots, V_n$ be the corresponding cover of $[0,1]$ with connected intervals where every V_i is a connected component of $\gamma^{-1}(U_i)$, and all U_i are contractible. Order V_i in such a way that V_i and V_{i+1} intersect at a point t_i, and let $a_i := \gamma(t_i)$. Prove that any path $\gamma' \in \Omega(M, x, y)$ such that $\gamma'(t_i) = a_i$, and $\gamma'([t_i, t_{i+1}]) \subset U_i$, is homotopic to γ.

Hint. Use the previous exercise.

Exercise 12.14 (!). Let M be a locally contractible topological space, and $\gamma \in \Omega(M, x, y)$ is a path. Prove that γ has a neighbourhood $\mathcal{U} \subset \Omega(M, x, y)$ such that all $\gamma' \in \mathcal{U}$ are homotopic.

Hint. Use the previous problem.

Remark. Notice that on all compact manifolds of dimension > 1 there are loops that are defined by a surjective map. Such loops can be constructed in the same way as the Peano curve.

Exercise 12.15 (!). Let M be a manifold (for instance, a sphere) of dimension greater than 1, and $\gamma \in \Omega(M, x)$ be a loop. Prove that γ is homotopic to a loop that is not surjective.

Hint. Use the previous exercise.

Exercise 12.16 (!). Let $n > 1$. Prove that n-dimensional sphere is simply connected.

Hint. Let γ be a loop on a sphere. Use the previous exercise and find a homotopy from γ to a loop that maps $[0,1]$ to $S^n \backslash \{x\}$ where x is some point. Prove that a sphere without a point is homeomorphic to $\mathbb{R}^n$, and in particular is contractible.

Exercise 12.17 (*). Let M be contractible and let $F: \ M \times [0,1] \longrightarrow M$ be a homotopy from the identity map to the constant map $M \to y \in M$. Consider the following map $M \longrightarrow \Omega(M, y, *)$, $t, m \longrightarrow F(m, t)$ ($t \in [0,1]$, $m \in M$). Prove that it is continuous.

Exercise 12.18. Let M be locally contractible, $x, y \in M$ be two points and $\gamma \in \Omega(M, x, y)$ be a path. Prove that γ has a neighbourhood $\mathcal{U} \in \Omega(M, x, *)$, such that all paths $\gamma' \in \mathcal{U}$ that connect x and a are homotopic in $\Omega(M, x, a)$.

Exercise 12.19 (*). Let M be a locally contractible topological space, and let $x \in M$ be a point, and $\Omega(M, x, *)$ be the set of all paths that start at the point x endowed with the compact-open topology. Consider the equivalence relation on $\Omega(M, x, *)$: $\gamma \sim \gamma'$ if γ and γ' connect x and y, and homotopic in $\Omega(M, x, y)$. Consider $\Omega(M, x, *)/\sim$ with the quotient topology. Consider a contractible neighbourhood $U_y \ni y$, and let $U_y \xrightarrow{F} \Omega(U_y, y, *)$ be a mapping that was constructed in the exercise 12.17. Let $\gamma \in \Omega(M, x, y)$ be a path and $U_y \xrightarrow{\Psi} \Omega(M, x, *)$ be a mapping that maps $a \in U_y$ to a path $\gamma F(a)$ (that is, to a path that is defined on $[0, 1/2]$ as $t \longrightarrow \gamma(2t)$, and on $[1/2, 1]$ as $F(a, 2t-1)$. Prove that (for sufficiently smallU_y) Ψ composed with $\Omega(M, x, *) \xrightarrow{\pi} \Omega(M, x, *)/\sim$ is a homeomorphism U_y on some open subset in $\Omega(M, x, *)/\sim$.

Hint. Continuity of $\Psi \circ \pi$ is obvious by construction and injectivity follows from the previous exercise. In order to show that $\Psi \circ \pi$ defines a homeomorphism U_y on $\Psi \circ \pi(U_y)$ we need to show that prove $\Psi \circ \pi$ maps open sets to open sets. This is clear from the fact that the natural map $\Omega(M, x, *)/\sim \longrightarrow M$, $\gamma' \longrightarrow \gamma'(1)$ is continuous and defines a homeomorphism U_y on its image.

Exercise 12.20 (*). Consider the mapping $\Omega(M, x, *)/\sim \longrightarrow M$ that maps a path $\gamma \in \Omega(M, x, y)$ to the point $y = \gamma(1)$. Prove that this is a covering.

Hint. Use the previous exercise.

Exercise 12.21 (!). Prove that $\Omega(M, x, *)$ is contractible.

Exercise 12.22 (*). Prove that γ is a path in $\Omega(M, x, *)/\sim$. Prove that γ is homotopic to an image of some path from $\Omega(M, x, *)$.

Hint. Prove that γ can be lifted to a path in $\Omega(M, x, *)$ locally and use the fact that for any point in $\Omega(M, x, *)/\sim$ its preimage in $\Omega(M, x, *)$ is connected.

Exercise 12.23 (*). Deduce that $\Omega(M, x, *)/\sim$ is simply connected.

Remark. Let (M, x) be a locally connected topological space with a marked point. The universal covering of M can be thus identified with the set of pairs ($y \in M$, homotopy class of a path $\gamma \in \Omega(M, x, y)$).

Free group and wedge sum

Definition 12.5. Let (M_1, x_1), (M_2, x_2), (M_3, x_3), ... be a collection (possibly infinite) of connected topological spaces with a marked point. Consider the quotient space of a disconnected sum of all (M_α, x_α) by the equivalence relation $\{x_1\} \sim \{x_2\} \sim \{x_3\} \sim \dots$ This quotient space is called a **wedge sum**, denoted by $\bigvee_\alpha (M_\alpha, x_\alpha)$. A wedge sum can also be denoted by $(M_1, x_1) \vee (M_2, x_2) \vee (M_3, x_3) \vee \dots$

Exercise 12.24. Assume that all M_α are connected (arcwise connected, Hausdorff). Prove that the wedge sum is connected (arcwise connected, Hausdorff).

Exercise 12.25 (!). Assume that all M_α are connected and simply connected. Prove that their wedge sum is simply connected.

Exercise 12.26 (!). Let Γ be a connect graph that has n vertices and $n+k-1$ edges. Prove that its associated topological space M_Γ is homotopy equivalent to a wedge sum of k circles.

Hint. Assume Γ has an edge r that connects two distinct vertices v_1, v_2. Consdireth graph Γ' with $n-1$ vertices and $n+k-2$ edges that is obtained from Γ in the following way. Remove an edge r from Γ and glue vertices v_1 and v_2 together. Prove that M_Γ and $M_{\Gamma'}$ are homotopy equivalent.

Definition 12.6. Consider a set $\{a_1, a_2, \dots\}$ of cardinality N (N by either finite or infinite). An **N-ary tree** D_N is an infinite graph that is defined in the following way. The verticesof D_N are finite sequences of symbols a_i. The edges connect vertices that correspond to $A_1 A_2 \dots A_k$ and $A_1 A_2 \dots A_k A_{k+1}$ (all A_i belong to $\{a_1, a_2, \dots\}$).

Exercise 12.27. Prove that every vertex D_N has $N+1$ incoming edges.

Exercise 12.28 (!). Let M_N be a topological space of an N-ary tree, with the natural metric, defined in the beginning of this exercise sheet. Prove that M_N is star-shaped (any two points can be connected by a unique geodesic). Prove that it is contractible.

Exercise 12.29 (!). Consider an $2N-1$-ary tree. Colour its edges in N colours in such a way that any vertex has 2 incoming edges of each colour. Consider the wedge sum of N circles and colour each of the circles in a diffferent colour. Consider the mapping from M_{2N-1} to the wedge sum of N circles that maps the vertices of the graph to the vertices of the wedge sum and an edge of colour a_i to the circle of the same colour. Prove that this is a universal cover.

Exercise 12.30. Let $\{a_1, a_2, \dots\}$ be a set of cardinality N, and let $\mathcal{W}$ be the set of finite sequences ("words") of symbols a_i, a_j^{-1}, such that subsequences of the form $a_i a_i^{-1}$ and $a_i^{-1} a_i$ never occur. A sequence of length 0 is denoted e. We multiply words by juxtaposing them and striking out all $a_i a_i^{-1}$, $a_i^{-1} a_i$ that might occur. Prove that $\mathcal{W}$ forms a group.

Definition 12.7. This group is called the **free group generated by** $\{a_1, a_2, \dots\}$ and is denoted F_N.

Exercise 12.31. Prove that F_1 is isomorphic to $\mathbb{Z}$.

Exercise 12.32 (!). Let G be a group and $\{g_1, g_2, \dots\}$ be a collection of elements from G, labelled $\{a_1, a_2, \dots\}$. Prove that there exists a unique homomorphism $F_N \longrightarrow G$ that maps a_i to g_i.

Exercise 12.33 (!). Find a free action of F_N on the topological space M_{2N-1} of an $2N-1$-ary tree that is transitive on vertices.

Exercise 12.34 (!). Prove that M_{2N-1}/F_N is a wedge sum of N circles and that the fundamental group of the wedge sum is free.

Exercise 12.35 (!). Prove that any (possibly infinite) graph is homotopy equivalent to a wedge sum of circles.

Exercise 12.36 (!). Deduce that any subgroup of a free group is free.

Hint. Use the Galois theory of coverings.

Exercise 12.37 (*). Let $G_1, G_2, \dots$ be a set of groups. Consider the set $\mathcal{W}$ of finite sequences of non-identity elements from different G_i such that elements of the same group never occur next to each other. Given any sequence A of elements from G_i one can obtain an element $\mathcal{W}$ the following way. If A has to successive elements from G_i, we multiply them and replace these elements with their product. If A an identity element of one of the groups we strike it out. Repeat this procedure as many times as needed in order to get an element from $\mathcal{W}$. The elements of $\mathcal{W}$ can be multiplied by juxtaposing words and applying the procedure above. Prove that this defines a group.

Definition 12.8. This group is called the **free product of groups** G_1**,** G_2**,** $\dots$.

Exercise 12.38. Prove that the free group on N generators is the free product of N copies of $\mathbb{Z}$.

Exercise 12.39. Prove that a free product of free groups is free.

Exercise 12.40 (*). Let $(M_1, x_1), (M_2, x_2), (M_3, x_3), \dots$ be a collection of connected topological spaces with a marked point. Prove that $\pi_1(\bigvee_\alpha (M_\alpha, x_\alpha))$ is isomorphic to a free product of groups $\pi_1(M_1, x_1), \pi_1(M_2, x_2), \pi_1(M_3, x_3), \dots$.