

GEOMETRY 2: real numbers, part 2.

Roots of polynomials of an odd degree.

Exercise 2.1 (!). Consider a polynomial over \mathbb{Q} of an odd degree, $P = t^{2n+1} + a_{2n}t^{2n} + a_{2n-1}t^{2n-1} + \dots + a_0$. Let R_P be the set of all $x \in \mathbb{Q}$ such that $P(t) < 0$ on an interval $] -\infty, x[$. Prove that R_P is not empty.

Hint. Prove that R_P contains $-\max(1, \sum |a_i|)$.

Exercise 2.2 (!). Prove that R_P is not the set of all real numbers.

Hint. Prove that the complement $\mathbb{Q} \setminus R_P$ contains $\max(1, \sum |a_i|)$.

Exercise 2.3 (!). Prove that R_P is a Dedekind section.

Exercise 2.4 (!). Prove that P satisfies **Lipschitz property**: for any interval I there exists a constant $C > 0$ such that $|P(a) - P(b)| < C|a - b|$ for any $a, b \in I$.

Exercise 2.5 (!). Consider a Dedekind section R_P as a real number. Prove that $P(R_P) = 0$. It follows that any polynomial over \mathbb{R} of an odd degree has a root.

Hint. First prove that $P(R_P) \leq 0$. Then prove that $P(R_P) < 0$ contradicts the problem 2.4.

Limits.

Definition 2.1. Let $A \subset \mathbb{R}$ be a set of real numbers and c be a real number. Then c is called **accumulation point (limit point)** of a set A if every open interval $I =]x, y[$ containing c contains infinitely many elements of A .

Definition 2.2. Let $\{a_i\}$ be a sequence of real numbers and c be a real number. Let any open interval $I =]x, y[$ containing c contain all elements of $\{a_i\}$ except a finite number of them. Then c is called the **limit of the sequence** $\{a_i\}$ (denoted by $c = \lim_{i \rightarrow \infty} a_i$). It is said that a sequence a_i **converges to** c .

Exercise 2.6. Let c be an accumulation point of a sequence $\{a_i\}$. Prove that there exists a subsequence of $\{a_i\}$ that converges to c .

Exercise 2.7 (*). Consider a sequence $\{a_i\}$ of points from an interval $[x, y]$. Prove the existence of accumulation points of that sequence.

Definition 2.3. A set $A \subset \mathbb{R}$ is called **discrete** if it has no accumulation points.

Exercise 2.8 (*). Let $\{a_i\}$ be a sequence. Denote a set of all a_i by A . Prove that $\{a_i\}$ converges if and only if A has no infinite discrete subsets and has a unique accumulation point.

Exercise 2.9. Consider a sequence $0, 1, 2, 3, 4, \dots$. Prove that this sequence has no limit.

Exercise 2.10. Consider a sequence $0, 1, 1/2, 1/3, 1/4, \dots$. Prove that this sequence converges to 0.

Exercise 2.11. Consider an increasing sequence $a_1 \leq a_2 \leq a_3 \leq \dots, a_i \in \mathbb{R}$. Let all a_i be bounded above by C : $a_i \leq C$. Prove that this sequence has a limit. Use the definition of real numbers as Dedekind sections.

Hint. Prove that $\lim_{i \rightarrow \infty} a_i = \sup\{a_i\}$, and use the fact that the supremum exists.

Definition 2.4. Let $\{a_i\} = a_0, a_1, a_2, \dots$ be a sequence of real numbers. $\{a_i\}$ is called a **Cauchy sequence** if for any $\varepsilon > 0$ there exists an interval $[x, y]$ of length ε which contains all members $\{a_i\}$ except a finite number of them.

Remark. This is the same definition as the definition of Cauchy sequences of rational numbers.

Exercise 2.12. Let a sequence $\{a_i\}$ converge to a real number c . Prove that this is a Cauchy sequence.

Exercise 2.13. Let a Cauchy sequence $\{a_i\}$ have a subsequence that converges to $x \in \mathbb{R}$. Prove that $\{a_i\}$ converges to x .

Exercise 2.14. Let $\{a_i\}$ be a Cauchy sequence. Consider the sequence $\{b_i\}$, $b_i = \inf_{i \geq k} a_i$. Prove that this infimum is correctly defined and that the sequence b_i increases.

Exercise 2.15. Consider the previous problem and prove that if the sequence $\{b_i\}$ has a limit then $\lim_{i \rightarrow \infty} a_i = \lim_{i \rightarrow \infty} b_i$.

Exercise 2.16 (!). Let $\{a_i\}$ be a Cauchy sequence. Prove that $\{a_i\}$ converges. Use the definition of real numbers as Dedekind sections.

Hint. Use the previous problem.

Exercise 2.17 (!). Let $\{a_i\}$ be a Cauchy sequence. Prove that $\{a_i\}$ converges. Use the definition of real numbers as Cauchy sequences.

Hint. Let a real number $\{a_i\}$ be represented by a Cauchy sequence of rational numbers $a_i(0), a_i(1), a_i(2), \dots$. Passing to a suitable subsequence one can suppose that all a_i ($i > n$) are contained in an interval of length 2^{-n} and that all $a_i(j)$ ($j > m$) are contained in an interval of length 2^{-m} . Prove that the sequence $\{a_i(i)\}$ is a Cauchy sequence and that the sequence $\{a_i\}$ converges to the real number represented by it.

Exercise 2.18 (!). Let $\{a_i\}, \{b_i\}, \{c_i\}$ be converging sequences of real numbers and $a_i \leq b_i \leq c_i$ for all i . Suppose that $\lim_{i \rightarrow \infty} a_i = \lim_{i \rightarrow \infty} c_i = x$. Prove that $\lim_{i \rightarrow \infty} b_i = x$.

Exercise 2.19 (*). Let a sequence $\{a_i\}$ converge to x . Prove that $b_j = \frac{1}{j} \sum_{i=0}^j a_i$ converges to x . Give an example, when $\{b_j\}$ converges but $\{a_i\}$ does not.

Series.

Definition 2.5. Let $\{a_i\}$ be a sequence of real numbers. Consider a sequence of partial sums $\sum_{i=0}^n a_i$. If this sequence converges it is said that **series** $\sum_{i=0}^{\infty} a_i$ **converge**. It is denoted by $\sum_{i=0}^{\infty} a_i = x$ where

$$x = \lim_{i \rightarrow \infty} \sum_{i=0}^n a_i.$$

It is often denoted by $\sum a_i = x$.

Definition 2.6. A series $\sum a_i$ **converges absolutely**, if series $\sum |a_i|$ converges.

Exercise 2.20 (!). Consider a series $\sum a_i$ which converges absolutely. Prove that these series converges.

Exercise 2.21. Consider a series $\sum a_i$ which converges absolutely. Let b_i be a sequence of non-negative numbers such that $a_i \geq b_i$. Prove that the series $\sum b_i$ converges absolutely.

Exercise 2.22 ().** Let a_i, b_i be sequences of integer numbers such that the series $\sum a_i^2, \sum b_i^2$ converge. Prove that the series $\sum a_i b_i$ converge.

Exercise 2.23 (*). Let a_i be a sequence of positive real numbers. Limit of the sequence of products

$$\lim_{n \rightarrow \infty} \prod_{i=0}^n (1 + a_i)$$

is denoted by $\prod_{i=0}^{\infty} (1 + a_i)$. If this limit exists it is said that an infinite product $\prod_{i=0}^{\infty} (1 + a_i)$ converges. Let the product $\prod_{i=0}^{\infty} (1 + a_i)$ converge. Prove that the series $\sum_{i=0}^{\infty} a_i$ converges.

Exercise 2.24 (*). Prove that the infinite product $\prod_{i=0}^{\infty} (1 + \frac{1}{3^n})$ converges.

Exercise 2.25 ().** Let a series $\sum a_i$ converge. Prove that $\prod_{i=0}^{\infty} (1 + a_i)$ converges, as well.

Exercise 2.26 (!). Let $a_0 \geq a_1 \geq a_2 \geq \dots$ be a decreasing sequence of positive real numbers converging to 0. Consider the series $\sum_{i=0}^{\infty} (-1)^i a_i$. Prove that these series converges. Such series are called **sign-alternating**.

Exercise 2.27. Prove that the series $\sum \frac{1}{n(n+1)}$ converges.

Hint. Consider the value $\frac{1}{n} - \frac{1}{(n+1)}$.

Exercise 2.28. Prove that the series $\sum \frac{1}{n^2}$ converges.

Exercise 2.29. Prove that the series $\sum \frac{1}{n!}$ converges.

Exercise 2.30 (!). Prove that the series $\sum \frac{1}{2^n}$ converges. Calculate the value it converges to.

Exercise 2.31 (*). Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.

Exercise 2.32 ().** Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ in a complete ordered field A . Does this series converge for all $x \in A$?