GEOMETRY 3: Metric spaces and norm

You are supposed to know the definition of a linear space and dot product (i.e. a positive bilinear symmetric form). Consult ALGEBRA 3.

Metric spaces, convex sets, norm

Definition 3.1. A metric space is a set X equipped with a function $d: X \times X \to \mathbb{R}$ such that

a. $d(x,y) > 0$ for all $x \neq y \in X$; moreover, $d(x,x) = 0$.

b. Symmetry: $d(x,y) = d(y,x)$

c. "Triangle inequality": for all $x, y, z \in X$,

$$d(x,z) \leqslant d(x,y) + d(y,z).$$

A function d which satisfies these conditions is called **metric**. The number $d(x,y)$ is called "distance between x and y".

If $x \in X$ is a point and ε is a real number then the set

$$B_\varepsilon(x) = \{y \in X \ \mid \ d(x,y) < \varepsilon$$

is called an **(open) ball of radius** ε **with the center in** x. Such ball can be called as well an ε**-ball**. A closed ball is defined as follows

$$\overline{B}_\varepsilon(x) = \{y \in X \ \mid \ d(x,y) \leqslant \varepsilon.$$

Exercise 3.1. Consider any subset of a Euclidean plane $\mathbb{R}^2$ and the function d defined as $d(a,b) = |ab|$ where $|ab|$ is the length of a segment $[a,b]$ on the plane. Prove that this defines a metric space.

Exercise 3.2. Consider the function $d_\infty: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

$$(x,y),(x',y') \mapsto \max(|x-x'|,|y-y'|).$$

Prove that this is a metric. Describe a unit ball with the center in zero.

Exercise 3.3. Consider a function $d_1: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

$$(x,y),(x',y') \mapsto |x-x'|+|y-y'|.$$

Prove that this is a metric. Describe a unit ball with the center in zero.

Exercise 3.4 (*). A function $f: [0,\infty[\to [0,\infty[$ is said to be **upper convex** if $f(\frac{ax+by}{a+b}) \geqslant \frac{af(x)+bf(y)}{a+b}$, for any positive a, $b \in \mathbb{R}$. Let f be such a function and (X,d) be a metric space. Suppose that $f(\lambda) = 0$ iff $\lambda = 0$. Prove that the function $d_f(x,y) = f(d(x,y))$ defines a metric on X.

Exercise 3.5. Let V be a linear space with a positive bilinear symmetric form $g(x,y)$ (in what follows we will call that form a **dot product**). Define the "distance" $d_g: V \times V \to \mathbb{R}$ as $d_g(x,y) = \sqrt{g(x-y,x-y)}$. Prove that $d(x,y) \geqslant 0$ where equality holds iff $x = y$.

Definition 3.2. Let $x \in V$ be a vector from a vector space V. **Parallel transport along vector** x is a mapping $P_x : V \to V$, $y \mapsto y + x$.

Exercise 3.6. Prove that a function d_g is "invariant with respect to parallel transports", i.e. $d_g(a, b) = d_g(P_x(a), P_x(b))$.

Exercise 3.7. Prove that if $y \neq 0$, then d_g satisfies the triangle inequality:

$$\sqrt{g(x-y, x-y)} \leqslant \sqrt{g(x, x)} + \sqrt{g(y, y)}$$

Hint. Consider a two-dimensional subspace $V_0 \subset V$, generated by x and y. Prove that it is isomorphic (as a space with dot product) to the space $\mathbb{R}^2$ with dot product $g((x, y), (x', y')) = xx' + yy'$. Use the triangle inequality for $\mathbb{R}^2$.

Exercise 3.8 (!). Prove that d_g satisfies the triangle inequality.

Hint. Use invariance of parallel transports and reduce to the previous problem.

Definition 3.3. Consider a vector space V with a dot product g, and let d_g be the metric constructed above. This metric is called a **euclidean** metric.

Definition 3.4. Consider a vector space V, a parallel transport $P_x : V \to V$ and a one-dimensional subspace $V_1 \subset V$. Then the image $P_x(V_1)$ is called a **line** in V.

Exercise 3.9. Consider two different points in $x, y \in V$. Prove that there exists a unique line $V_{x,y}$ through x and y.

Definition 3.5. Consider a line l through points x and y, and a point a on l. We say that a lies **between** x and y, if $d(x, a) + d(b, y) = d(x, y)$. A **line segment between** x **and** y (denoted $[x, y]$) is the set of all points belonging to the line $V_{x,y}$, that "lie between" x and y.

Exercise 3.10. consider three different points on a line. Prove that one (and only one) of these points lies between two other points. Prove that the line segment $[x, y]$ is a set of all points z of the form $ax + (1 - a)y$, where $a \in [0, 1] \subset \mathbb{R}$.

Definition 3.6. Consider a vector space V, and let $B \subset V$ be its subset. We say that B is **convex** if B contains all points of the line segment $[x, y]$ for any $x, y \in V$.

Definition 3.7. Let V be a vector space over $\mathbb{R}$. A **norm** on V is a function $\rho : V \to \mathbb{R}$, such that the following hold:

a. For any $v \in V$ one has $\rho(v) \geqslant 0$. Moreover, $\rho(v) > 0$ for all nonzero v.

b. $\rho(\lambda v) = |\lambda| \rho(v)$

c. For any $v_1, v_2 \in V$ one has $\rho(v_1 + v_2) \leqslant \rho(v_1) + \rho(v_2)$.

Exercise 3.11. Consider a vector space V over $\mathbb{R}$, and let $\rho : V \to \mathbb{R}$ be a norm on V. Consider the function $d_\rho : V \times V \to \mathbb{R}$, $d_\rho(x, y) = \rho(x - y)$. Prove that this is a metric on V.

Exercise 3.12 (*). Let $d : V \times V \to \mathbb{R}$ be a metric on V, invariant w.r.t. the parallel transports. Suppose that d satisfies

$$d(\lambda x, \lambda y) = |\lambda| d(x, y)$$

for all $\lambda \in \mathbb{R}$. Prove that d can be obtained from the norm $\rho : V \to \mathbb{R}$ by using the formula $d(x, y) = \rho(x - y)$.

Exercise 3.13. Let V be a linear space over $\mathbb{R}$ and $\rho: V \to \mathbb{R}$ be a norm on V. Consider the set $B_1(0)$ of all points with the norm $\leqslant 1$. Prove that this set is convex.

Definition 3.8. Consider a vector space V over $\mathbb{R}$ and let v be a nonzero vector. Then the set of all vectors of the form $\{\lambda v \mid \lambda > 0\}$ is called a **half-line** (or a **ray**) in V.

Definition 3.9. A **central symmetry** in V is the mapping $x \mapsto -x$.

Exercise 3.14 (*). Consider a central symmetric convex set $B \subset V$ that does not contain any half-lines and has an intersection with any half-line $\{\lambda v \mid \lambda > 0\}$. Consider the function ρ

$$v \overset{\rho}{\mapsto} \inf\{\lambda \in \mathbb{R}^{>0} \quad | \quad \lambda v \notin B\}$$

Prove that this is a norm on V. Prove that all the norms can be obtained that way.

Exercise 3.15. Consider an abelian group G and a function $\nu: G \to \mathbb{R}$ satisfying $\nu(g) \geqslant 0$ for all $g \in G$, and $\nu(g) > 0$ whenever $g \neq 0$. Suppose that $\nu(a+b) \leqslant \nu(a) + \nu(b)$, $\nu(0) = 0$ and that $\nu(g) = \nu(-g)$ for all $g \in G$. Prove that the function $d_\nu: G \times G \to \mathbb{R}$, $d_\nu(x, y) = \nu(x - y)$ is a metric on G.

Exercise 3.16. A metric d on an abelian group G is called an **invariant** metric if $d(x+g, y+g) = d(x, y)$ for all $x, y, g \in G$. Prove that any invariant metric d is obtained from a function $\nu: G \to \mathbb{R}$ by setting $d(x, y) = \nu(x - y)$.

Definition 3.10. Fix a prime number $p \in \mathbb{Z}$. The function $\nu_p: \mathbb{Z} \to \mathbb{R}$, which given a number $n = p^k r$ (r is not divisible by p) yields p^{-k}, and satisfies $\nu_p(0) = 0$, is called the **p-adic norm on $\mathbb{Z}$**.

Exercise 3.17. Prove that the function $d_p(m, n) = \nu_p(n - m)$ defines a metric on $\mathbb{Z}$. This metric is called **p-adic metric on $\mathbb{Z}$**.

Hint. Check that $\nu_p(a + b) \leqslant \nu(a) + \nu(b)$ holds and use the previous problem.

Definition 3.11. Let R be a ring and $\nu: R \to \mathbb{R}$ be a function that is positive and yields strictly positive values for all nonzero r. Suppose that $\nu(r_1 r_2) = \nu(r_1)\nu(r_2)$ and $\nu(r_1 + r_2) \leqslant \nu(r_1) + \nu(r_2)$. Then ν is called a **norm** on R. A ring endowed with a norm is called a **normed ring**.

Remark. It follows from the problems above that a norm on a ring R defines an invariant metric on R. In what follows any normed ring will be regarded as a metric space.

Exercise 3.18. Prove that ν_p is a norm on a ring $\mathbb{Z}$. Define a norm on $\mathbb{Q}$ that extends ν_p.

Complete metric spaces.

Definition 3.12. Let (X, d) be a metric space and $\{a_i\}$ be a sequence of point from X. A sequence $\{a_i\}$ is called a **Cauchy sequence**, if for every $\varepsilon > 0$ there exists an ε-ball in X which contains all but a finite number of a_i.

Exercise 3.19. Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences in X. Prove that $\{d(a_i, b_i)\}$ is a Cauchy sequence in $\mathbb{R}$.

Definition 3.13. Let (X, d) be a metric space and $\{a_i\}$, $\{b_i\}$ be Cauchy sequences in X. Sequences $\{a_i\}$ and $\{b_i\}$ are called **equivalent**, if the sequence $a_0, b_0, a_1, b_1, \ldots$ is a Cauchy sequence.

Exercise 3.20. Let $\{a_i\}$, $\{b_i\}$ be Cauchy sequences in X. Prove that $\{a_i\}$, $\{b_i\}$ are equivalent iff $\lim\limits_{i\to\infty} d(a_i, b_i) = 0$.

Exercise 3.21. Let $\{a_i\}$, $\{b_i\}$ be equivalent Cauchy sequences in X, and $\{c_i\}$ be another Cauchy sequence. Prove that

$$\lim_{i\to\infty} d(a_i, c_i) = \lim_{i\to\infty} d(b_i, c_i)$$

Exercise 3.22 (!). Let (X, d) be a metric space and $\overline{X}$ be the set of equivalence classes of Cauchy sequences. Prove that the function

$$\{a_i\}, \{b_i\} \mapsto \lim_{i\to\infty} d(a_i, b_i)$$

defines a metric on $\overline{X}$.

Definition 3.14. In that case, $\overline{X}$ is called the **completion of** X.

Exercise 3.23. Consider a natural mapping $X \to \overline{X}$, $x \mapsto \{x, x, x, x, ...\}$. Prove that it is an injection which preserves the metric.

Definition 3.15. Let A be a subset of X. An element $c \in X$ is called an **accumulation point (limit point)** of a set A if any open ball containing c contains an infinite number of elements of A.

Exercise 3.24. Prove that a Cauchy sequence cannot have more than one accumulation point.

Definition 3.16. Let $\{a_i\}$ be a Cauchy sequence. It is said that $\{a_i\}$ **converges to** $x \in X$, or that $\{a_i\}$ **has the limit** x (denoted as $\lim\limits_{i\to\infty} a_i = x$), if x is an accumulation point of $\{a_i\}$

Definition 3.17. A metric space (X, d) is called **complete** if any Cauchy sequence in X has a limit.

Exercise 3.25 (!). Prove that the completion of a metric space is complete.

Definition 3.18. A subset $A \subset X$ of a metric space is called **dense** if any open ball in X contains an element from A.

Exercise 3.26. Prove that X is dense in its completion $\overline{X}$.

Exercise 3.27 (*). Let X be a metric space and consider a metric preserving mapping $j : X \to Z$ from X into a complete metric space Z. Prove that j can be uniquely extended to $\overline{j} : \overline{X} \to Z$.

Remark. This problem can be used as a definition of $\overline{X}$. The definition 3.14 then becomes a theorem.

Exercise 3.28 (!). Let R be a ring endowed with a norm ν. Define addition and multiplication on the completion of R with respect to the metric corresponding to ν. Prove that $\overline{R}$ has a norm that extends the norm ν on R.

Definition 3.19. The normed ring $\overline{R}$ is called the **completion of R with respect to the norm ν.**

Exercise 3.29 (*). Let R be a normed ring and $\overline{R}$ be its completion. Suppose that R is a field. Prove that $\overline{R}$ is also a field.

Exercise 3.30 (*). Let R be a ring without zero divisors (i.e. it satisfies the following property: if r_1, r_2 are nonzero elements, then $r_1 r_2$ is also non-zero). Consider a function $\nu : R \to \mathbb{R}$ which maps all non-zero elements of R to unity and maps zero to zero. Prove that ν is a norm. What is $\overline{R}$?

Exercise 3.31. Prove that $\mathbb{R}$ can be obtained as the completion of $\mathbb{Q}$ with respect to the norm $q \mapsto |q|$. Can this statement be used as a definition of $\mathbb{R}$?

Definition 3.20. The completion of $\mathbb{Z}$ with respect to the norm ν_p is called the **ring of integer p-adic numbers**. This ring is denoted by $\mathbb{Z}_p$.

Exercise 3.32. Let (X, d) be a metric space and $\{a_i\}$ be a sequence of points in X. Suppose that the series $\sum d(a_i, a_{i-1})$ converges. Prove that $\{a_i\}$ is a Cauchy sequence. Is the converse true?

Exercise 3.33 (!). Prove that for any sequence of integer numbers a_k the series $\sum a_k p^k$ converges in Z_p.

Hint. Use the previous problem.

Exercise 3.34. Prove that $(1-p)(\sum_{k=0}^{\infty} p^k) = 1$ in Z_p.

Exercise 3.35 (*). Prove that any integer number which is not divisible by p is invertible in Z_p.

Definition 3.21. The completion of $\mathbb{Q}$ with respect to the norm obtained by extension of ν_p, is denoted by $\mathbb{Q}_p$ and is called the **field of p-adic numbers**.

Exercise 3.36 (*). Take $x \in \mathbb{Q}_p$. Prove that $x = \frac{x'}{p^k}$, where $x' \in \mathbb{Z}_p$.

Exercise 3.37 (*). Prove that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Definition 3.22. A norm ν on a ring R is called **non-Archimedean**, if $\nu(x+y) \leqslant \max(\nu(x), \nu(y))$ for all x, y. Otherwise the norm is called **Archimedean**.

Exercise 3.38 (*). Let ν be a norm on $\mathbb{Q}$. Prove that ν is non-Archimedean iff $\mathbb{Z}$ is contained in the unit ball.

Hint. Use the following equality: $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. Find an estimate of $\sqrt[n]{((\nu(x+y)^n)}$ for big n using the estimate of binomial coefficients: $\nu(\binom{k}{n}) \leqslant 1$.

Exercise 3.39 (*). Let ν be a non-Archimedean norm on $\mathbb{Q}$. Consider $\mathfrak{m} \subset \mathbb{Z}$ consisting of all integers n such that $\nu(n) < 1$. Prove that $\mathfrak{m}$ is an *ideal* in $\mathbb{Z}$ (ideal in a ring R is a subset which is closed under addition and multiplication by elements of R). Prove that the ideal

$$\mathfrak{m} = \{n \in \mathbb{Z} \mid \nu(n) < 1\}$$

is *prime* (prime ideal is an ideal such that $xy \notin \mathfrak{m}$ for all $x, y \notin \mathfrak{m}$).

Exercise 3.40 (*). Prove that any ideal in $\mathbb{Z}$ is of the form $\{0, \pm 1m, \pm 2m, \pm 3m, ...\}$ for some $m \in \mathbb{Z}$. Prove that any prime ideal $\mathfrak{m}$ in $\mathbb{Z}$ is of the form $\{0, \pm p, \pm 2p, \pm 3p, ...\}$, where $p = 0$ or p prime.

Hint. Use the Euclid's Algorithm.

Exercise 3.41 (*). Let ν be a non-Archimedean norm $\mathbb{Q}$ and $\mathfrak{m} = \{p, 2p, 3p, 4p, ...\}$ be an ideal constructed above. Prove that there exists a real number $\lambda > 1$ such that $\nu(n) = \lambda^{-k}$ for any $n = p^k r$, $r \not| p$.

Exercise 3.42 (*). Let ν be a norm on $\mathbb{Q}$ such that $\nu(2) \leqslant 1$. Prove that $\nu(a) < \log_2(a) + 1$ for any integer $a > 0$.

Hint. Use the binary representation of a number.

Exercise 3.43 (*). Let ν be a norm on $\mathbb{Q}$ such that $\nu(2) < 1$. Prove that $\nu(a) \leqslant 1$ for any integer $a > 0$ (i.e. ν is non-Archimedean).

Hint. Deduce from $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$ that $\lim\limits_{n\to\infty} \frac{\log n}{n} = 0$. Prove $\lim\limits_{N\to\infty} \nu(a^N) \leqslant 1$, using the previous problem.

Exercise 3.44 (*). Let a_i be a Cauchy sequence of rational numbers of the form $\frac{x}{2^n}$ ("Cauchy sequence" here means the same thing as Cauchy sequence of real numbers). Suppose that a norm ν on $\mathbb{Q}$ is Archimedean. Prove that $\nu(a_i)$ is a Cauchy sequence.

Hint. Write down x in the binary system and prove that

$$\nu(x/2^n) \leqslant \nu(2)^{\log_2(x)+1}/\nu(2)^n \leqslant \nu(2)^{\log_2|x+1/2^n|}.$$

Exercise 3.45 (*). Deduce that ν can be extended to a continuous function on $\mathbb{R}$, which satisfies $\nu(xy) = \nu(x)\nu(y)$. Prove that ν can be obtained as $x \mapsto |x|^\lambda$ for some constant $\lambda > 0$. Express λ in terms of $\nu(2)$.

Exercise 3.46 (*). For which $\lambda > 0$ the function $x \mapsto |x|^\lambda$ defines a norm on $\mathbb{Q}$?

We have obtained a complete classification of norms on $\mathbb{Q}$: any norm can be obtained as a power of either a p-adic norm or the absolute value norm. This classification is called **Ostrovsky theorem**.