GEOMETRY 4: Topology of metric spaces.

Definition 4.1. Let M be a metric space and $X \subseteq M$. Then X is called **open** when it contains, together with any point $x \in X$, some ε-ball with the center in x. A subset is called **closed** if its complement is open.

Exercise 4.1. Prove that X is open iff for any sequence $\{a_i\}$ converging to $x \in X$ all but a finite number of a_i belong to X.

Exercise 4.2. Prove that the union of any number of open sets is open. Prove that the intersection of a finite number of closed sets is closed.

Exercise 4.3. Prove that the closed ball

$$\overline{B}_\varepsilon(x) = \{y \in X \mid d(x, y) \leqslant \varepsilon\}$$

is a closed subset.

Exercise 4.4. Prove that a set is closed iff it contains all its accumulation points.

Definition 4.2. The **closure** of a set $A \subset M$ is the union of A and the set of all the accumulation points of A.

Exercise 4.5. Consider a metric space, a closed ball $\overline{B}_\varepsilon(x)$ and an open ball $B_\varepsilon(x)$. Is it always true that $\overline{B}_\varepsilon(x)$ is the closure of $B_\varepsilon(x)$? Prove that the closure of any subset is always closed.

Exercise 4.6. Let A be a subset of M which has no accumulation points (such a subset is called **discrete**). Prove that $M \backslash A$ is open.

Definition 4.3. Let M be a metric space and $\varepsilon > 0$ be a number. Consider $R \subseteq M$ such that M can be covered by a union of all ε-balls with center in R. Then R is called an **ε-net**.

Exercise 4.7. Let any sequence in M have an accumulation point. Prove that for any $\varepsilon > 0$ in M there exists a finite ε-net.

Hint. Suppose that there is no such net, then for any finite set R there exists a point x, whose distance to R is more than ε. Add x to R, and, using this operation as induction step, obtain an infinite discrete subset of M.

Definition 4.4. Let $X \subset M$ and $U_i \subset M$ be a collection of open sets. If $X \subset \cup U_i$ then it is said that U_i is a **cover of** X. A collection of sets obtained from $\{U_i\}$ by throwing out some open sets in such a way that it remains a cover, is called a **subcover**.

Exercise 4.8. Let M be a metric space, S be an open cover of M. Let every subsequence of elements of M have an accumulation point. Prove that there exists such an $\varepsilon > 0$, that any ball of radius $< \varepsilon$ is contained in one of the sets of the cover S.

Hint. Suppose that for any ε there exists a point x_ε such that a corresponding ε-ball is not contained entirely in any of the sets of the cover. Consider a sequence $\{\varepsilon_i\}$ which converges to zero and let x be an accumulation point of $\{x_{\varepsilon_i}\}$. Prove that x is not contained in any of the sets of S.

Exercise 4.9 (!). (Bolzano-Weierstrass lemma) Let $X \subset M$ be a subset of a metric space. Prove that the following conditions are equivalent

a. Every sequence of points from X has an accumulation point in X.

b. Every open cover of X has a finite subcover.

Hint. Use problem 4.6 to deduce (a) from (b). In order to deduce (b) from (a), take an arbitrary cover S, a number ε from the problem 4.8 and a finite ε-net. Every ball of the ε-net is contained in some of the elements $U_i \in S$. Prove that $\{U_i\}$ is a finite subcover.

Definition 4.5. Let M, M' be metric spaces, and $f: M \to M'$ be a function. Then f is called **continuous**, if f maps any sequence that converges to x to a sequence that converges to $f(x)$, for all $x \in M$.

Exercise 4.10 (!). Let X be any subset of M. Prove that a function $f: M \to \mathbb{R}, x \overset{f}{\mapsto} d(\{x\}, X)$ is continuous, where $d(\{x\}, X)$ (distance between x and X) is defined as $d(\{x\}, X) := \inf_{x' \in X} d(x, x')$.

Definition 4.6. Let M be a metric space, $X \subset M$. It is said that X is a **compact set**, if any of the statements of the problem 4.9 holds. Note that these conditions do not depend on inclusion $X \hookrightarrow M$, but only on the metric on X.

Exercise 4.11 (!). Consider the completion of $\mathbb{Z}$ with respect to the norm ν_p defined above (it is called "a ring of integer p-adic numbers" and is denoted $\mathbb{Z}_p$). Prove that it is compact.

Hint. Prove that any p-adic number can be represented in the from $\sum a_i p^i$, where a_i are integers between 0 and $p-1$.

Exercise 4.12. Prove that a compact subset of M is always closed.

Hint. Prove that it contains all its accumulation points.

Exercise 4.13. Prove that a closed subspace of a compact set is always compact.

Exercise 4.14. Prove that a union of a compact sets is compact.

Exercise 4.15 (!). Let $f: X \to \mathbb{R}$ be a continuous function defined on a compact set. Prove that f achieves maximum on X.

Definition 4.7. Let X, Y be two subsets of a metric space. Denote the number $\inf_{x \in X, y \in Y}(d(x, y))$ by $d(X, Y)$.

Exercise 4.16 (!). Let X, Y be two compact subsets of a metric space. Prove that there exist points x, y in X, Y such that $d(x, y) = d(X, Y)$.

Definition 4.8. A subset $Z \subset M$ is called bounded if it is contained in a ball $B_r(x)$ for some $r \in \mathbb{R}, x \in M$.

Exercise 4.17. Let $Z \subset M$ be compact. Prove that it is bounded.

Definition 4.9. Let M be a metric space and $X \subset M$. The union of all open ε-balls with centers in all points of X is called the **ε-neighbourhood** of X.

Definition 4.10. Let M be a metric space and let X and Y be its bounded subsets. The **Hausdorff distance** $d_H(X, Y)$ is the infimum of all ε such that Y is contained in an ε-neighborhood of X and X is contained in an ε-neighborhood Y.

Exercise 4.18 (!). Prove that the Hausdorff distance defines a metric on the set $\mathcal{M}$ of all closed bounded subsets of M.

Exercise 4.19. Let X, Y be bounded subsets of M and $x \in X$. Prove that it is always the case that $d_H(X,Y) \geqslant d(x,Y)$.

Exercise 4.20 (!). Let M be a complete metric space. Prove that $\mathcal{M}$ is also complete.

Hint. Consider a Cauchy sequence $\{X_i\}$ of subsets of M. Let $\mathfrak{S}$ be the set of Cauchy sequences $\{x_i\}$ with $x_i \in X_i$. Let X be the set of accumulation points of sequences from $\mathfrak{S}$. Prove that $\{X_i\}$ converges to X.

Exercise 4.21 (*). Let $\{X_i\}$ be a Cauchy sequence of compact subsets of M and X be its limit. Prove that X is compact.

Hint. One can identify $\{X_i\}$ with its subsequence such that

$$d_H(X_i, X_j) < 2^{-\min(i,j)}. \tag{4.1}$$

Consider a sequence $\{x_i\}$ of points from X. For every X_j find a sequence $\{x_i(j) \in X_j\}$ such that $d(x_i(j), x_i) = d(x_i, X_j)$. Since X_j is compact, this sequence has an accumulation point. Choose an accumulation point $x(0)$ in $\{x_i(0)\}$ and replace $\{x_i\}$ with its subsequence such that $\{x_i(0)\}$ converges to $x(0)$. Then replace $\{x_i\}, i > 0$ with a subsequence such that $\{x_i(1)\}$ converges to $x(1)$. We replace $\{x_i\}, i > k$ with a subsequence on k-the step in such a way that $\{x_i(k)\}$ converges to $x(k)$. Prove that we will finally obtain a sequence $\{x_i\}$ such that $\{x_i(k)\}$ converges to $x(k)$ for all k. Prove that this operation can be carried out in such a way that $d(x_i(k), x(k)) < 2^{-i}$. Use (4.1) to prove that $d(x_i(k), x_i) < 2^{-\min(k,j)+2}$. Deduce that $\{x_i\}$ is a Cauchy sequence.

Exercise 4.22 (!). Let M be compact and $X \subset M$. Prove that for any $\varepsilon > 0$ there is a finite set $R \subset M$ such that $d_H(R, X) < \varepsilon$. (This statement can be rephrased as follows: "X allows approximation by finite sets with any prescribed accuracy")

Hint. Find a finite ε-net in X.

Exercise 4.23 (*). Let M be compact. Prove that $\mathcal{M}$ is also compact.

Hint. Use the previous problem.

Definition 4.11. Let M be a metric space. It is said that M is **locally compact**, if for any point $x \in M$ there exists a number $\varepsilon > 0$, such that the closed ball $\overline{B}_\varepsilon(x)$ is compact.

Exercise 4.24. Let M be a locally compact metric space and $\overline{B}_\varepsilon(x)$ be a closed compact ball. Prove that $\overline{B}_\varepsilon(x)$ is contained in an open set Z with compact closure.

Hint. Cover $\overline{B}_\varepsilon(x)$ with balls such that their closures are compact, and find a finite subcover.

Exercise 4.25 (!). Prove in the previous problem setting that for some $\varepsilon' > 0$ the ball $\overline{B}_{\varepsilon+\varepsilon'}(x)$ is also compact.

Hint. Take Z as in the previous problem. Take ε' to be $d(M \backslash Z, \overline{B}_\varepsilon(x))$.

Definition 4.12. Let (M, d) be a metric space. It is said that M **satisfies Hopf-Rinow condition** if for any two points $x, y \in M$ and for any two numbers $r_x, r_y > 0$ such that $r_x + r_y < d(x, y)$

$$d(B_{r_x}(x), B_{r_y}(y)) = d(x, y) - r_x - r_y.$$

Exercise 4.26 (**). If you know the definition of a Riemannian (or Finsler) manifold, prove that the Hopf-Rinow condition holds for the natural metric on such a manifold. Justify all the facts that you use in the proof.

Exercise 4.27 (*). Let M be a complete locally compact metric space which satisfies Hopf-Rinow condition, $x \in M$ be a point and $\varepsilon > 0$ be a number such that $\overline{B}_{\varepsilon'}(x)$ is compact for all $\varepsilon' < \varepsilon$. Prove that the ball $\overline{B}_\varepsilon(x)$ is compact.

Hint. Let $\{\varepsilon_i\}$, with $\varepsilon_i < \varepsilon$, be a sequence that converges to ε. Use the Hopf-Rinow condition to prove that $\{\overline{B}_{\varepsilon_i}(x)\}$ is a Cauchy sequence with respect to Hausdorff metric, $\overline{B}_\varepsilon(x)$. Use the fact that the limit of such a sequence is compact (you have already proved it before).

Exercise 4.28 (*). (Hopf-Rinow theorem, I) Let M be a complete locally compact metric space which satisfies Hopf-Rinow condition. Prove that every closed ball $\overline{B}_\varepsilon(x)$ in M is compact.

Exercise 4.29. Let M be a metric space such that every closed ball $\overline{B}_\varepsilon(x)$ in M is compact. Prove that M is complete.

Exercise 4.30 (*). Let M be a locally compact complete metric space which satisfies Hopf-Rinow condition, $x, y \in M$. Prove that there is a point $z \in M$ such that $d(x, z) = d(y, z) = \frac{1}{2}d(x, y)$.

Exercise 4.31 (*). Let S be a set of all rational numbers of the form $\frac{n}{2^k}$, $n \in \mathbb{Z}$ which belong to the interval $[0, 1]$. Prove in the previous problem setting that there exists a mapping $S \xrightarrow{\xi} M$ such that $d(\xi(a), \xi(b)) = |a - b|d(x, y)$ and $\xi(0) = x$ and $\xi(1) = y$.

Exercise 4.32 (*). (Hopf-Rinow theorem, II) Let M be a locally compact complete metric space which satisfies Hopf-Rinow condition, $x, y \in M$. Prove that the mapping ξ can be naturally extended to the completion of S with respect to the standard metric, so that the resulting mapping $[0, 1] \xrightarrow{\overline{\xi}} M$ satisfies $\overline{\xi}(0) = x$, $\overline{\xi}(1) = y$ and $d((\overline{\xi}(a), \overline{\xi}(b)) = |a - b|d(x, y)$ for any two reals $a, b \in [0, 1]$.

Remark. Such a mapping $\overline{\xi}$ is called **geodesic**. The Hopf-Rinow theorem can be restated as follows: for any two points in a complete metric locally compact space which satisfies Hopf-Rinow condition there is a geodesic that connects them.

Definition 4.13. Such a space is called **geodesically connected**.

Exercise 4.33 (*). Give an example of a metric space, which is not locally compact but geodesically connected.

Exercise 4.34. Let $V = \mathbb{R}^n$ be the metric space with the standard (Euclidean) metric. Prove that geodesics in V are intervals (sets of the form $ax + (1 - a)y$, where a belongs to $[0, 1] \subset \mathbb{R}$, and $x, y \in V$).

Exercise 4.35. Let V be a finite dimensional vector space with a norm that defines a metric d and d_0 be the Euclidean metric on V. Prove that the identity mapping $(V, d) \to (V, d_0)$ is continuous iff a unit ball in (V, d) contains a ball from (V, d_0). Prove that the inverse mapping is continuous provided that a unit ball in (V, d) is contained in a ball from (V, d_0).

Exercise 4.36. In the previous problem settings, consider a function $D(x) := d(0, x)$ on a unit sphere $S^{n-1} \subset V$

$$S^{n-1} = \{x \in V \mid d_0(0, x) = 1\}$$

Let D be a continuous function on S^{n-1}. Prove that the mapping $(V, d) \to (V, d_0)$ is continuous and the inverse mapping is continuous.

Hint. Use the fact that a continuous function on a compact set achieves its minimum and maximum values.

Exercise 4.37 (**). Prove that D is a continuous function.

Exercise 4.38. Let V be a finite dimensional vector space with a norm that defines the metric d. Suppose that the identity mapping $(V, d) \to (V, d_0)$ is continuous and the inverse mapping is also continuous. Prove that (V, d) is complete and locally compact.

Exercise 4.39 (*). Let d be the metric on $\mathbb{R}^n$ associated with the norm $(x_1, x_2, ...) \mapsto \max |x_i|$. Prove that it satisfies the Hopf-Rinow condition. Prove that $\mathbb{R}^n$ with such a metric is geodesically connected. Describe how the geodesics look like.

Exercise 4.40 (*). Is it true that the metric d defined by a norm always satisfies the Hopf-Rinow condition?

Definition 4.14. Let X be a metric space and $0 < k < 1$ be a real number. A mapping $f: X \to X$ is called **contraction mapping with a contraction coefficient** k if $kd(x, y) \geqslant d(f(x), f(y))$.

Exercise 4.41 (!). Let X be a meric space and $f: X \to X$ be a contraction mapping. Prove that for any $x \in X$ the sequence $\{a_i\}$, $a_0 := x, a_1 := f(x), a_2 := f(f(x)), a_3 := f(f(f(x))), \dots$ is Cauchy sequence.

Hint. Use the fact that $d(a_i, a_{i+1}) = k^i d(x, f(x))$, and deduce that the series $\sum d(a_i, a_{i+1})$ converges.

Exercise 4.42 (!). (The Contraction Mapping Theorem) Let X be a complete metric space and $f: X \to X$ be a contraction mapping. Prove that f has a fixed point.

Hint. Find the limit of the sequence $x, f(x), f(f(x)), f(f(f(x))), \dots$.