GEOMETRY 4: Topology of metric spaces.

Definition 4.1. Let $M$ be a metric space and $X \subseteq M$. Then $X$ is called open when it contains, together with any point $x \in X$, some $\varepsilon$-ball with the center in $x$. A subset is called closed if its complement is open.

Exercise 4.1. Prove that $X$ is open iff for any sequence $\{a_i\}$ converging to $x \in X$ all but a finite number of $a_i$ belong to $X$.

Exercise 4.2. Prove that the union of any number of open sets is open. Prove that the intersection of a finite number of closed sets is closed.

Exercise 4.3. Prove that the closed ball $B_\varepsilon(x) = \{y \in X \mid d(x, y) \leq \varepsilon\}$ is a closed subset.

Exercise 4.4. Prove that a set is closed iff it contains all its accumulation points.

Definition 4.2. The closure of a set $A \subset M$ is the union of $A$ and the set of all the accumulation points of $A$.

Exercise 4.5. Consider a metric space, a closed ball $B_\varepsilon(x)$ and an open ball $B_\varepsilon(x)$. Is it always true that $\overline{B_\varepsilon(x)}$ is the closure of $B_\varepsilon(x)$? Prove that the closure of any subset is always closed.

Exercise 4.6. Let $A$ be a subset of $M$ which has no accumulation points (such a subset is called discrete). Prove that $M \setminus A$ is open.

Definition 4.3. Let $M$ be a metric space and $\varepsilon > 0$ be a number. Consider $R \subseteq M$ such that $M$ can be covered by a union of all $\varepsilon$-balls with center in $R$. Then $R$ is called an $\varepsilon$-net.

Exercise 4.7. Let any sequence in $M$ have an accumulation point. Prove that for any $\varepsilon > 0$ in $M$ there exists a finite $\varepsilon$-net.

Hint. Suppose that there is no such net, then for any finite set $R$ there exists a point $x$, whose distance to $R$ is more than $\varepsilon$. Add $x$ to $R$, and, using this operation as induction step, obtain an infinite discrete subset of $M$.

Definition 4.4. Let $X \subset M$ and $U_i \subset M$ be a collection of open sets. If $X \subset \bigcup U_i$ then it is said that $U_i$ is a cover of $X$. A collection of sets obtained from $\{U_i\}$ by throwing out some open sets in such a way that it remains a cover, is called a subcover.

Exercise 4.8. Let $M$ be a metric space, $S$ be an open cover of $M$. Let every subsequence of elements of $M$ have an accumulation point. Prove that there exists such an $\varepsilon > 0$, that any ball of radius $< \varepsilon$ is contained in one of the sets of the cover $S$.

Hint. Suppose that for any $\varepsilon$ there exists a point $x_\varepsilon$ such that a corresponding $\varepsilon$-ball is not contained entirely in any of the sets of the cover. Consider a sequence $\{\varepsilon_i\}$ which converges to zero and let $x$ be an accumulation point of $\{x_{\varepsilon_i}\}$. Prove that $x$ is not contained in any of the sets of $S$.

Exercise 4.9 (!). (Bolzano-Weierstrass lemma) Let $X \subset M$ be a subset of a metric space. Prove that the following conditions are equivalent
a. Every sequence of points from $X$ has an accumulation point in $X$.

b. Every open cover of $X$ has a finite subcover.

**Hint.** Use problem 4.6 to deduce (a) from (b). In order to deduce (b) from (a), take an arbitrary cover $S$, a number $\varepsilon$ from the problem 4.8 and a finite $\varepsilon$-net. Every ball of the $\varepsilon$-net is contained in some of the elements $U_i \in S$. Prove that $\{U_i\}$ is a finite subcover.

**Definition 4.5.** Let $M, M'$ be metric spaces, and $f : M \to M'$ be a function. Then $f$ is called **continuous**, if $f$ maps any sequence that converges to $x$ to a sequence that converges to $f(x)$, for all $x \in M$.

**Exercise 4.10 (!).** Let $X$ be any subset of $M$. Prove that a function $f : M \to \mathbb{R}$, $x \mapsto d(\{x\}, X)$ is continuous, where $d(\{x\}, X)$ (distance between $x$ and $X$) is defined as $d(\{x\}, X) := \inf_{x' \in X} d(x, x')$.

**Definition 4.6.** Let $M$ be a metric space, $X \subset M$. It is said that $X$ is a **compact set**, if any of the statements of the problem 4.9 holds. Note that these conditions do not depend on inclusion $X \hookrightarrow M$, but only on the metric on $X$.

**Exercise 4.11 (!).** Consider the completion of $\mathbb{Z}$ with respect to the norm $\nu_p$ defined above (it is called “a ring of integer $p$-adic numbers” and is denoted $\mathbb{Z}_p$). Prove that it is compact.

**Hint.** Prove that any $p$-adic number can be represented in the from $\sum a_i p^i$, where $a_i$ are integers between 0 and $p - 1$.

**Exercise 4.12.** Prove that a compact subset of $M$ is always closed.

**Hint.** Prove that it contains all its accumulation points.

**Exercise 4.13.** Prove that a closed subspace of a compact set is always compact.

**Exercise 4.14.** Prove that a union of a compact sets is compact.

**Exercise 4.15 (!).** Let $f : X \to \mathbb{R}$ be a continuous function defined on a compact set. Prove that $f$ achieves maximum on $X$.

**Definition 4.7.** Let $X, Y$ be two subsets of a metric space. Denote the number $\inf_{x \in X, y \in Y} (d(x, y))$ by $d(X, Y)$.

**Exercise 4.16 (!).** Let $X, Y$ be two compact subsets of a metric space. Prove that there exist points $x, y$ in $X, Y$ such that $d(x, y) = d(X, Y)$.

**Definition 4.8.** A subset $Z \subset M$ is called bounded if it is contained in a ball $B_r(x)$ for some $r \in \mathbb{R}, x \in M$.

**Exercise 4.17.** Let $Z \subset M$ be compact. Prove that it is bounded.

**Definition 4.9.** Let $M$ be a metric space and $X \subset M$. The union of all open $\varepsilon$-balls with centers in all points of $X$ is called the $\varepsilon$-**neighbourhood** of $X$.

**Definition 4.10.** Let $M$ be a metric space and let $X$ and $Y$ be its bounded subsets. The **Hausdorff distance** $d_H(X, Y)$ is the infimum of all $\varepsilon$ such that $Y$ is contained in an $\varepsilon$-neighborhood of $X$ and $X$ is contained in an $\varepsilon$-neighborhood of $Y$. 
Exercise 4.18 (!). Prove that the Hausdorff distance defines a metric on the set \( \mathcal{M} \) of all closed bounded subsets of \( M \).

Exercise 4.19. Let \( X, Y \) be bounded subsets of \( M \) and \( x \in X \). Prove that it is always the case that \( d_H(X, Y) \geq d(x, Y) \).

Exercise 4.20 (!). Let \( M \) be a complete metric space. Prove that \( \mathcal{M} \) is also complete.

Hint. Consider a Cauchy sequence \( \{X_i\} \) of subsets of \( M \). Let \( \mathcal{G} \) be the set of Cauchy sequences \( \{x_i\} \) with \( x_i \in X_i \). Let \( X \) be the set of accumulation points of sequences from \( \mathcal{G} \). Prove that \( \{X_i\} \) converges to \( X \).

Exercise 4.21 (*). Let \( \{X_i\} \) be a Cauchy sequence of compact subsets of \( M \) and \( X \) be its limit. Prove that \( X \) is compact.

Hint. One can identify \( \{X_i\} \) with its subsequence such that

\[
d_H(X_i, X_j) < 2^{-\min(i,j)}.
\]

Consider a sequence \( \{x_i\} \) of points from \( X \). For every \( X_j \) find a sequence \( \{x_i(j) \in X_j\} \) such that \( d(x_i(j), x_i) = d(x_i, X_j) \). Since \( X_j \) is compact, this sequence has an accumulation point. Choose an accumulation point \( x(0) \) in \( \{x_i(0)\} \) and replace \( \{x_i\} \) with its subsequence such that \( \{x_i(0)\} \) converges to \( x(0) \). Then replace \( \{x_i\}, i > 0 \) with a subsequence such that \( \{x_i(1)\} \) converges to \( x(1) \). We replace \( \{x_i\}, i > k \) with a subsequence on \( k \)-the step in such a way that \( \{x_i(k)\} \) converges to \( x(k) \). Prove that we will finally obtain a sequence \( \{x_i\} \) such that \( \{x_i(k)\} \) converges to \( x(k) \) for all \( k \). Prove that this operation can be carried out in such a way that \( d(x_i(k), x(k)) < 2^{-i} \). Use (4.1) to prove that \( d(x_i(k), x_i) < 2^{-\min(k,i)+2} \). Deduce that \( \{x_i\} \) is a Cauchy sequence.

Exercise 4.22 (!). Let \( M \) be compact and \( X \subset M \). Prove that for any \( \varepsilon > 0 \) there is a finite set \( R \subset M \) such that \( d_H(R, X) < \varepsilon \). (This statement can be rephrased as follows: “\( X \) allows approximation by finite sets with any prescribed accuracy”)

Hint. Find a finite \( \varepsilon \)-net in \( X \).

Exercise 4.23 (*). Let \( M \) be compact. Prove that \( \mathcal{M} \) is also compact.

Hint. Use the previous problem.

Definition 4.11. Let \( M \) be a metric space. It is said that \( M \) is **locally compact**, if for any point \( x \in M \) there exists a number \( \varepsilon > 0 \), such that the closed ball \( B_\varepsilon(x) \) is compact.

Exercise 4.24. Let \( M \) be a locally compact metric space and \( B_\varepsilon(x) \) be a closed compact ball. Prove that \( B_\varepsilon(x) \) is contained in an open set \( Z \) with compact closure.

Hint. Cover \( B_\varepsilon(x) \) with balls such that their closures are compact, and find a finite subcover.

Exercise 4.25 (!). Prove in the previous problem setting that for some \( \varepsilon' > 0 \) the ball \( B_{\varepsilon+\varepsilon'}(x) \) is also compact.

Hint. Take \( Z \) as in the previous problem. Take \( \varepsilon' \) to be \( d(M \setminus Z, B_\varepsilon(x)) \).
**Definition 4.12.** Let \((M, d)\) be a metric space. It is said that \(M\) satisfies Hopf-Rinow condition if for any two points \(x, y \in M\) and for any two numbers \(r_x, r_y > 0\) such that \(r_x + r_y < d(x, y)\)
\[
d(B_{r_x}(x), B_{r_y}(y)) = d(x, y) - r_x - r_y.
\]

**Exercise 4.26 (**) . If you know the definition of a Riemannian (or Finsler) manifold, prove that the Hopf-Rinow condition holds for the natural metric on such a manifold. Justify all the facts that you use in the proof.

**Exercise 4.27 (**) . Let \(M\) be a complete locally compact metric space which satisfies Hopf-Rinow condition, \(x \in M\) be a point and \(\varepsilon > 0\) be a number such that \(B_{\varepsilon'}(x)\) is compact for all \(\varepsilon' < \varepsilon\). Prove that the ball \(B_\varepsilon(x)\) is compact.

**Hint.** Let \(\{\varepsilon_i\}\), with \(\varepsilon_i < \varepsilon\), be a sequence that converges to \(\varepsilon\). Use the Hopf-Rinow condition to prove that \(\{B_{\varepsilon_i}(x)\}\) is a Cauchy sequence with respect to Hausdorff metric, \(\overline{B}_\varepsilon(x)\). Use the fact that the limit of such a sequence is compact (you have already proved it before).

**Exercise 4.28 (**) . (Hopf-Rinow theorem, I) Let \(M\) be a complete locally compact metric space which satisfies Hopf-Rinow condition. Prove that every closed ball \(\overline{B}_\varepsilon(x)\) in \(M\) is compact.

**Exercise 4.29.** Let \(M\) be a metric space such that every closed ball \(\overline{B}_\varepsilon(x)\) in \(M\) is compact. Prove that \(M\) is complete.

**Exercise 4.30 (**) . Let \(M\) be a locally compact complete metric space which satisfies Hopf-Rinow condition, \(x, y \in M\). Prove that there is a point \(z \in M\) such that \(d(x, z) = d(y, z) = \frac{1}{2}d(x, y)\).

**Exercise 4.31 (**) . Let \(S\) be a set of all rational numbers of the form \(\frac{n}{2^k}\), \(n \in \mathbb{Z}\) which belong to the interval \([0, 1]\). Prove in the previous problem setting that there exists a mapping \(S \xrightarrow{\xi} M\) such that \(d(\xi(a), \xi(b)) = |a - b|d(x, y)\) and \(\xi(0) = x\) and \(\xi(1) = y\).

**Exercise 4.32 (**) . (Hopf-Rinow theorem, II) Let \(M\) be a locally compact complete metric space which satisfies Hopf-Rinow condition, \(x, y \in M\). Prove that the mapping \(\xi\) can be naturally extended to the completion of \(S\) with respect to the standard metric, so that the resulting mapping \([0, 1] \xrightarrow{\overline{\xi}} M\) satisfies \(\overline{\xi}(0) = x, \overline{\xi}(1) = y\) and \(d((\overline{\xi}(a), \overline{\xi}(b)) = |a - b|d(x, y)\) for any two reals \(a, b \in [0, 1]\).

**Remark.** Such a mapping \(\overline{\xi}\) is called geodesic. The Hopf-Rinow theorem can be restated as follows: for any two points in a complete metric locally compact space which satisfies Hopf-Rinow condition there is a geodesic that connects them.

**Definition 4.13.** Such a space is called geodesically connected.

**Exercise 4.33 (**) . Give an example of a metric space, which is not locally compact but geodesically connected.

**Exercise 4.34.** Let \(V = \mathbb{R}^n\) be the metric space with the standard (Euclidean) metric. Prove that geodesics in \(V\) are intervals (sets of the form \(ax + (1 - a)y\), where \(a\) belongs to \([0, 1] \subset \mathbb{R}\), and \(x, y \in V\)).

**Exercise 4.35.** Let \(V\) be a finite dimensional vector space with a norm that defines a metric \(d\) and \(d_0\) be the Euclidean metric on \(V\). Prove that the identity mapping \((V, d) \rightarrow (V, d_0)\) is continuous iff a unit ball in \((V, d)\) contains a ball from \((V, d_0)\). Prove that the inverse mapping is continuous provided that a unit ball in \((V, d)\) is contained in a ball from \((V, d_0)\).
Exercise 4.36. In the previous problem settings, consider a function $D(x) := d(0, x)$ on a unit sphere $S^{n-1} \subset V$

$$S^{n-1} = \{ x \in V \mid d_0(0, x) = 1 \}$$

Let $D$ be a continuous function on $S^{n-1}$. Prove that the mapping $(V, d) \to (V, d_0)$ is continuous and the inverse mapping is continuous.

Hint. Use the fact that a continuous function on a compact set achieves its minimum and maximum values.

Exercise 4.37 (**). Prove that $D$ is a continuous function.

Exercise 4.38. Let $V$ be a finite dimensional vector space with a norm that defines the metric $d$. Suppose that the identity mapping $(V, d) \to (V, d_0)$ is continuous and the inverse mapping is also continuous. Prove that $(V, d)$ is complete and locally compact.

Exercise 4.39 (*). Let $d$ be the metric on $\mathbb{R}^n$ associated with the norm $(x_1, x_2, \ldots) \mapsto \max |x_i|$. Prove that it satisfies the Hopf-Rinow condition. Prove that $\mathbb{R}^n$ with such a metric is geodesically connected. Describe how the geodesics look like.

Exercise 4.40 (*). Is it true that the metric $d$ defined by a norm always satisfies the Hopf-Rinow condition?

Definition 4.14. Let $X$ be a metric space and $0 < k < 1$ be a real number. A mapping $f : X \to X$ is called contraction mapping with a contraction coefficient $k$ if $kd(x, y) \geq d(f(x), f(y))$.

Exercise 4.41 (!). Let $X$ be a metric space and $f : X \to X$ be a contraction mapping. Prove that for any $x \in X$ the sequence $\{a_i\}$, $a_0 := x, a_1 := f(x), a_2 := f(f(x)), a_3 := f(f(f(x))), \ldots$ is Cauchy sequence.

Hint. Use the fact that $d(a_i, a_{i+1}) = k^id(x, f(x))$, and deduce that the series $\sum d(a_i, a_{i+1})$ converges.

Exercise 4.42 (!). (The Contraction Mapping Theorem) Let $X$ be a complete metric space and $f : X \to X$ be a contraction mapping. Prove that $f$ has a fixed point.

Hint. Find the limit of the sequence $x, f(x), f(f(x)), f(f(f(x))), \ldots$. 