GEOMETRY 5: Set-theoretic topology.

Definition 5.1. Consider a set M and a collection of distinguished sets $S \subset \wp(M)$ called **open subsets**. A pair (M, S) (and, by abuse of notation, M itself) is called a **topological space**, if the following conditions are met:

1. An empty set and M are open;
2. The union of any number of open sets is open;
3. The intersection of a finite number of open sets is open.

A mapping $\varphi : M \longrightarrow M'$ of topological spaces is called **continuous**, if the preimage of every open set is open. Continuous mappings are also called **morphisms** of topological spaces. An **isomorphism** of topological spaces is a morphism $\varphi : M \longrightarrow M'$ such that there is an inverse morphism $\psi : M' \longrightarrow M$ (i.e. $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity morphisms). An isomorphism of topological spaces is called **homeomorphism**.

A subset $Z \subset M$ is called **closed**, if its complement is open. A **neighborhood** of a point $x \in M$ is an open subset of M which contains x. A **neighborhood** of a subset $Z \subset M$ is an open subset of M that contains Z.

Exercise 5.1. Prove that a composition of continuous mappings is continuous.

Exercise 5.2 (!). Consider a set M and let S be a set of all subsets of M. Prove that S defines a topology on M. This topology is called **discrete**. Describe a set of all continuous mappings from M to a given topological space.

Exercise 5.3 (!). Consider a set M and let S be the set containing an empty set and M itself. Prove that S defines a topology on M. This topology is called **codiscrete**. Describe a set of all continuous mappings from M to a space with discrete topology.

Exercise 5.4. Give an example of a continuous bijection between topological spaces that is not a homeomorphism.

Exercise 5.5. Consider a subset Z of a topological space M. Open subsets of Z are defined to be intersections of the form $Z \cap U$, where U is open in M.

a. Prove that this defines a topology on Z. Prove that a natural embedding $Z \hookrightarrow M$ is continuous.

b. **(*)** Can all the continuous embeddings be obtained in this way?

Definition 5.2. Such a topology on $Z \subset M$ is said to be **induced by** M. We will consider any subset of any topological space as a topological space with induced topology.

Definition 5.3. Consider a topological space M, and let S_0 be such a collection of open sets such that any open set can be represented as a union of sets from S_0. Then S_0 is called a **base** of M.

Exercise 5.6. Describe all bases of a space M with discrete topology; of a space M with codiscrete topology.

Definition 5.4. Consider a metric space M. Recall that a subset $U \subset M$ is called **open**, if for every point $u \in U$, U contains a ball of radius $\varepsilon > 0$ with the center u.

Exercise 5.7. Prove that this definition defines a topology on a metric space.

Definition 5.5. A topological space is called **metrizable** if it can be obtained from a metric space as described above.

Exercise 5.8. Prove that a discrete space is metrizable and a codiscrete space is not.

Exercise 5.9. Prove that open balls in a metric space M are open. Prove that open balls define a base of topology on M.

Exercise 5.10 (!). Consider a topological space M and two topologies S, S' on M. Suppose that for every point $m \in M$ and every neighborhood $U' \ni m$ which is open in the topology S' there is a neighborhood $U \ni m$, $U \subset U'$, which is open in the topology S. Prove that the identity mapping $(M, S) \xrightarrow{i} (M, S')$ is continuous. Give an example where i is not a homeomorphism.

Remark. It is said in this case that the topology defined by S' is **stronger** than the topology defined by S.

Exercise 5.11. Consider the space $\mathbb{R}^n$ with a norm ν (see GEOMETRY 3). This norm defines a metric and hence a topology on $\mathbb{R}^n$. Denote this topology by S_ν. Let ν, ν' be two norms satisfying $C^{-1}\nu'(x) < \nu(x) < C\nu'(x)$ for a fixed $C \in \mathbb{R}$. Prove that the identity mapping on $\mathbb{R}^n$ defines a homeomorphism $(\mathbb{R}^n, S_\nu) \longrightarrow (\mathbb{R}^n, S_{\nu'})$.

Hint. Use the previous problem.

Exercise 5.12 (*). Consider two norms ν, ν' on $\mathbb{R}^n$ such that the identity mapping on $\mathbb{R}^n$ defines a homeomorphism $(\mathbb{R}^n, S_\nu) \longrightarrow (\mathbb{R}^n, S_{\nu'})$. Prove that there exists a constant C such that $C^{-1}\nu'(x) < \nu(x) < C\nu'(x)$.

Exercise 5.13 (*). Consider a finite-dimensional vector space V endowed with a symmetric positive bilinear form g. We will consider V as a metric space with the metric d_g, constructed in GEOMETRY 3. Denote by S_g the topology defined by d_g. Prove that the corresponding topology on V does not depend upon g, i.e. for any (symmetric positive bilinear) g, g', the identity map on V is a homeomorphism $(V, S_g) \longrightarrow (V, S_{g'})$.

Exercise 5.14 (**). Consider a finite-dimensional vector space V with norm ν. Prove that the topology S_ν does not depend on norm ν: the identity map on $\mathbb{R}^n$ is a homeomorphism $(\mathbb{R}^n, S_\nu) \longrightarrow (\mathbb{R}^n, S_{\nu'})$. Is it true for a infinite-dimensional V?

Definition 5.6. Consider a metric d on $\mathbb{R}^n$, defined by the norm

$$|(\alpha_1, \ldots, \alpha_n)| = \sqrt{\sum_i \alpha_i^2}.$$

The topology on $\mathbb{R}^n$, defined by d is called the **natural** topology. The **natural topology** on subsets of $\mathbb{R}^n$ is the topology induced by the natural $\mathbb{R}^n$-topology.

Exercise 5.15. Consider $\mathbb{R}$ with the natural topology. Consider a space M with discrete topology and a space M' with a codiscrete topology. Find the set of all continuous maps

a. from $\mathbb{R}$ to M

b. from M to $\mathbb{R}$

c. from M' to $\mathbb{R}$

d. from $\mathbb{R}$ to M'.

Exercise 5.16. Consider a mapping $\varphi : M \longrightarrow M'$, where M, M' are topological spaces. Is it true that the continuity of φ implies that the preimage of any closed set is closed? Is it true that if a preimage of any closed set is closed then φ is continuous?

Exercise 5.17. Give an example of a continuous mapping of topological spaces such that the image of an open set is not open. Give an example of a continuous mapping of topological spaces such that the image of a closed set is closed.

Definition 5.7. Consider a topological space M and arbitrary $Z \subset M$. The intersection of the closed sets of M containing Z is denoted by $\overline{Z}$ and is called the **closure** of Z.

Exercise 5.18. Prove that $\overline{Z}$ is closed.

Definition 5.8. Consider a topological space M. The following conditions T0-T4 are called **separation axioms.**

T0. Let $x \neq y \in M$. Then at least one of the points x, y has a neighborhood containing the other point.

T1. Every point in M is closed.

T2. For any $x \neq y \in M$ there are non-intersecting neighborhoods U_x, U_y.

T3. For any point $y \in M$, every $M \supseteq U \ni y$ contains an open neighborhood $U' \ni y$ such that U contains the closure of U'.

T4. For any closed subset $Z \in M$, any neighborhood $U \supset Z$ contains an open neighborhood $U' \supset Z$ such that U contains the closure of U'.

The condition T_2 is widely known as the **Hausdorff axiom**. A topological space that satisfies the T_2 condition is called a **Hausdorff**.

Exercise 5.19. Prove that the condition T_1 is equivalent to the following one: for any two distinct points $x, y \in M$, there exists a neighborhood of y, which does not contain x.

Exercise 5.20. Prove that the condition T_4 is equivalent to the following one: any two distinct closed sets $X, Y \subset M$ have two non-intersecting neighborhoods.

Exercise 5.21. Let M be a topological space. Consider an equivalence relation on M defined the following way: x is equivalent to y iff $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$. Denote the set of equivalence classes as M'.

a. Verify that this is indeed an equivalence relation . Prove that M satisfies the T_0 iff $M = M'$.

b. Define $U \subset M'$ to be open iff its preimage w.r.t. the mapping $M \to M'$ is open. Prove that this defines a topology on M'. Does it satisfy the T_0 condition?

c. Prove that the open subsets of M are exactly the preimages of the open subsets of M'.

d. Suppose that M has the codiscrete topology. What is M'?

Exercise 5.22. Are $T_0 - T_4$ conditions satisfied by a space with the discrete topology? With the codiscrete topology?

Exercise 5.23. Prove that $T_0 - T_4$ are satisfied by $\mathbb{R}$.

Exercise 5.24. Prove that T_1 implies T_0 and that T_2 implies T_1.

Exercise 5.25. Give an example of a space that does not satisfy the T_1 condition. Give an example of a non-Hausdorff space such that all the singleton sets are closed in it.

Exercise 5.26 (*). Give an example of a space that satisfies the T_1 condition such that any two non-empty open sets have an non-empty intersection.

Exercise 5.27 (*). Prove that T_2 follows from T_1 and T_3.

Exercise 5.28 (*). Give an example of a space that satisfies T_4 but does not satisfy T_1.

Exercise 5.29. Consider a metrizable topological space. Prove that it satisfies conditions T_1, T_2, T_3.

Exercise 5.30 (*). Consider a metrizable topological space. Prove that it satisfies the condition T_4.

Exercise 5.31 (*). Let M be a finite set.

a. Find all topologies on M that satisfy the T_1 condition.

b. Are there any topologies on M that do not satisfy T_1?

c. Are there any topologies on M that do not satisfy T_1, but satisfy T_0?

Definition 5.9. A set M is said to be **partially ordered**, if there is a binary relation $x \leq y$ ("x less than or equal y") defined on it such that:

1. If $x \leq y$ and $y \leq z$, then $x \leq z$.
2. If $x \leq y$ and $y \leq x$, then $x = y$.

Exercise 5.32 (*). a. Consider a partially ordered set M; say that $S \subset M$ is open if together with any $x \in S$ it contains all $y \in M$ satisfying $y \leq x$. Prove that this defines a topology on M. When does this topology satisfy the T_0 condition? The T_1 condition?

b. Consider a finite set M and a topology on M that satisfies the T_0 condition. Prove that it is induced by a partial order on M.

Definition 5.10. Let $Z \subset M$ be a subset of a topological space. A subset Z is called **dense**, if Z has a non-empty intersection with every open subset of M.

Exercise 5.33 (!). Prove that Z is dense iff the closure $\overline{Z}$ is the entire M.

Exercise 5.34. Find all dense subsets in a topological space with the discrete topology; with the codiscrete topology.

Exercise 5.35. Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Exercise 5.36 (!). A subset Z in a topological space M is called **nowhere dense**, if for every open $U \subset M$ the subset $Z \cap U$ is not dense in U. Prove that Z is nowhere dense iff $M \backslash \overline{Z}$ is dense in M.

Exercise 5.37 (*). Construct a nowhere dense subset of the interval $[0, 1]$ (endowed with the natural topology) of the continuum cardinality.

Exercise 5.38. Find all nowhere dense subsets in a space with discrete topology; with codiscrete topology.

Definition 5.11. Let M be a topological space and $x \in M$ be an arbitrary point. A neighborhood base of x is a collection B of neighborhoods of x such that any neighborhood $U \ni x$ contains some neighborhood from B.

Exercise 5.39. Consider a collection B of open subsets of a topological space M such that for any $x \in M$ the collection of all $U \in B$ containing x is a neighborhood base of x. Prove that B is a base of the topology of M.

Definition 5.12. Consider a topological space M. One can impose two countability conditions on M. If every point of M has a countable neighborhood base, then it is said that M satisfies **the first countability axiom**. If M has a countable base of open sets, then it is said that M satisfies **the second countability axiom**, or that M is a **space with a countable base**. If there exists a countable dense subset of M then it is said that M is **separable**.

Exercise 5.40. Consider a space M with discrete topology. Prove that M satisfies the first countability axiom.

Exercise 5.41. Consider a topological space M with a countable base. Prove that it is separable.

Exercise 5.42 (*). Consider a separable topological space M. Prove that M has a countable base.

Exercise 5.43 (!). Consider a metrizable topological space. Prove that it has a countable neighborhood base for every point.

Exercise 5.44. Construct a non-separable metrizable topological space.

Exercise 5.45 (**). Give an example of a countable Hausdorff space without a countable base.

5.1 Topology and convergence

Topological space were invented as a language to speak about continuous functions. In GEOMETRY 4 we defined a continuous function as a function that preserves limits of convergent sequences. One can consider topology from the axiomatic viewpoint as above, or from the point of view of geometric intuition, by giving a class of convergent sequences on a space to define its topology and considering a mapping continuous if it preserves limits.

The second approach (despite all its obvious advantages) encounters set-theoretical problems: if the space does not have a countable base, then one has to use well-founded uncountable sequences. We are going to work mostly with spaces which have a countable neighborhood base and it is convenient to define topology and continuity via limits of sequences.

Definition 5.13. Let M be a topological space and $Z \subset M$ be an infinite subset. A point $x \in M$ is called an **accumulation point** of Z, if every neighborhood of x contains some point $z \in Z$. A **limit** of a sequence $\{x_i\}$ is defined to be a point x such that every neighborhood of x contains almost all x_i's. A sequence is called **convergent** if it has a limit.

Exercise 5.46. Find all convergent subsequences in a space with discrete topology; in a space with codiscrete topology.

Exercise 5.47. Consider a Hausdorff space M. Prove that every sequence has at most one limit.

Exercise 5.48 (*). Is the converse true (i.e. does it follow from the uniqueness of a limit that the space is Hausdorff)? What if M has a countable neighborhood base of its point?

Exercise 5.49. Consider a space M where any sequence has at most one limit. Prove that M satisfies the separation axiom T_1.

Exercise 5.50. Consider a continuous mapping $f: M \longrightarrow M'$ and a subset $Z \subset M$. Prove that f maps accumulation points of Z to accumulation point of $f(Z)$. Prove that f maps limits to limits.

Exercise 5.51 (!). Consider a mapping that maps accumulation points to accumulation points. Prove that it is continuous.

Exercise 5.52. Consider a space M with a countable neighborhood base for every point, and an arbitrary $Z \subset M$. Prove that the closure of Z is the set of limits of all sequences from Z.

Exercise 5.53 (!). Consider topological spaces M, M' with a countable neighborhood base for every point and a mapping $f: M \longrightarrow M'$ that preserves limits of sequences. Prove that f is continuous.

Hint. Use the previous problem.

Exercise 5.54 (*). What happens if we do not require in the previous problem that neighborhood bases are countable in M? In M'?

Exercise 5.55 (*). Consider a set M and let some sequences of elements of M be declared to **converge** to points from M (it is denoted like this: $x \in \lim x_i$; note that there can be more than one limit of a sequence[1]). Let the following conditions hold for the notion of convergence:

(i) The limit of a sequence $x, x, x, x, x, \ldots$ contains x.

(ii) If $x \in \lim x_i$ then the limit of any subsequence $\{x_{i_\ell}\}$ is nonempty and contains x.

(iii) Consider an infinite number of elements of a sequence $\{x_i\}$. Let us permute them and denote the result by $\{y_i\}$. If $x \in \lim x_i$ then $x \in \lim y_i$.

(iv) If $x \in \lim x_i$ and $x \in \lim y_i$ then the sequence $x_1, y_1, x_2, y_2, \ldots$ converges to x.

a. Define closed subsets of M as these $Z \subset M$ that contain the limits of all sequences $\{x_i\} \subseteq Z$. Define open sets as complements of closed sets. Prove that this defines a topology on M.

[1] Thus we talk here about limits of sequences as *sets* of points. (DP)

b. Consider a topology S on M with a countable neighborhood base for every point. Define the limits of sequences with respect to this topology. Prove that conditions (i)-(iv) hold for this notion of convergence. Let S' be a topology obtained from limits with the help of construction in (a). Prove that the topologies S' and S coincide.

c. Take a uncountable set with the following topology: open sets are complements of finite sets (this topology is called cofinite). Consider a topology S' defined by limits as above. Describe S'. Prove that S' does not satisfy the first countability axiom.