GEOMETRY 6: Set-theoretic topology: product of spaces

Definition 6.1. Consider a topological space M and a collection B of open subsets of M. The collection B is called a **prebase** of the topology on M, if every open set can be obtained as a union (potentially infinite) of finite intersections of open subsets from B.

Exercise 6.1. Consider $\mathbb{R}$ with discrete topology. Prove that it does not have a countable prebase.

Exercise 6.2 (!). Consider a topological space M with a countable prebase. Prove that M has a countable base.

Exercise 6.3 (*). Consider a finite set M, $|M| = 2^n$ with discrete topology and a prebase B of M. Prove that $|B| \geqslant 2n$. Find a prebase that has $2n$ elements.

Exercise 6.4. Consider $\mathbb{R}$ with natural topology and let B be the set of all intervals such that their end-points are finite binary fractions. Prove that B is a base of topology of $\mathbb{R}$.

Exercise 6.5. Consider a collection B of subsets of a set M such that $\cup B = M$. Consider all subsets of M that are finite intersections and arbitrary unions of elements of B, as well as M and $\emptyset$. Prove that these sets define a topology on M.

Definition 6.2. This topology is called the **topology defined by the prebase** B.

Definition 6.3. Consider topological spaces M_1 and M_2. Consider a topology S on $M_1 \times M_2$ defined by the prebase of subsets of the form $U_1 \times M_2$, $M_1 \times U_2$ where U_1, U_2 are open in M_1, M_2 respectively. Then $(M_1 \times M_2, S)$ is called the **product of M_1 and M_2**.

Exercise 6.6. Prove that the natural projection $M_1 \times M_2 \longrightarrow M_1$ is continuous. Prove that sets of the form $U_1 \times U_2$ define a base of the topology of $M_1 \times M_2$.

Exercise 6.7. Consider mappings of topological space $X \xrightarrow{\gamma_1} M_1$, $X \xrightarrow{\gamma_2} M_2$. Prove that they are continuous iff the product

$$X \xrightarrow{\gamma_1 \times \gamma_2} M_1 \times M_2$$

is continuous.

Exercise 6.8. Consider topological spaces M_1, M_2 that have one of the properties from the list below. Prove that $M_1 \times M_2$ has the same property.

a. Separation axiom T_1.

b. **(!)** Hausdorff separation axiom (T_2).

c. Separation axiom T_3.

d. Being separable.

e. **(!)** Having a countable neighborhood base for every point.

f. Having a countable base.

Exercise 6.9 (**). Does this hold for the separation axiom T_4? What about $T_4 + T_1$?

Definition 6.4. The mapping $x \xrightarrow{\Delta} (x, x) \in X \times X$ is called the **diagonal embedding** and its image is called the **diagonal** in $X \times X$.

Exercise 6.10. Prove that the diagonal embedding is a homeomorphism onto its image (supposing that the topology on $\Delta \subset X \times X$ is induced from $X \times X$).

Hint. Use the Problem 6.7.

Exercise 6.11. Prove that X satisfies the T_1 separation axiom iff the diagonal is the intersection of all open sets that contain it.

Exercise 6.12 (!). Prove that X is Hausdorff iff the diagonal is closed in $X \times X$.

Exercise 6.13 (*). Suppose that the graph $\Gamma \subset X \times Y$ of a mapping of topological spaces $X \xrightarrow{\gamma} Y$ is closed. Is it true that γ is continuous?

Exercise 6.14 (!). Consider a morphism of topological spaces $X \xrightarrow{\gamma} Y$ and suppose X is Hausdorff. Prove that the graph of γ is closed.

Exercise 6.15. Consider metric spaces M_1, M_2 and their product $M = M_1 \times M_2$, and let d be one of the functions defined on $M \times M$ listed below. Prove that d defines a metric on M.

a. $d((m_1, m_2), (m_1', m_2')) = d(m_1, m_2) + d(m_1', m_2')$

b. $d((m_1, m_2), (m_1', m_2')) = \max(d(m_1, m_2), d(m_1', m_2'))$

c. **(!)** $d((m_1, m_2), (m_1', m_2')) = \sqrt{d(m_1, m_2)^2 + d(m_1', m_2')^2}$

Exercise 6.16 (!). Prove that all the three metric structures from the previous problem define the same topology on $M_1 \times M_2$. Prove that this topology is equivalent to the topology of the product $M_1 \times M_2$ considered as a product of topological spaces.

Tychonoff cube and Hilbert cube

Definition 6.5. Consider a (possibly uncountable) index set I and the set $M = X^I$ of all mappings from I to a fixed topological space X. One can regard X^I as a set of sequences of points of X indexed by I or as an infinite product of X with itself. Denote by $W(i, U) \subset X^I$ the set of all mappings $I \longrightarrow X$ that map a fixed index i to an element from a subset $U \subset X$. Define a prebase B of topology on X^I in the following way: let $V \in B$ if $V = W(i, U)$ for some index element $i \in I$ and some open subset $U \subset X$. This topology is called **weak**.

Exercise 6.17 (!). Consider a sequence of points $\alpha_1, \alpha_2, \ldots$ in X^I. Prove that it converges iff the sequence $\alpha_k(i)$ converges for every index $i \in I$.

Remark. The previous problem statement is often expressed as follows: "a space X^I with weak topology is the set of mappings from I to X with the pointwise convergence topology".

Definition 6.6. Consder an index set I. The space $[0,1]^I$ with the weak topology is called a **Tychonoff cube**.

Exercise 6.18. Consider a set of continuous functions $\alpha_i : M \longrightarrow [0,1]$ indexed by a set I. Prove that the mapping of the form

$$\prod \alpha_i : m \longrightarrow \prod_{i \in I} \alpha_i(m)$$

from M to Tychonoff cube $[0,1]^I$ is continuous.

Exercise 6.19. Prove that any point of a Tychonoff cube is closed.

Exercise 6.20 (*). Prove that a Tychonoff cube satisfies T_2 and T_3 separation axioms.

Exercise 6.21 (!). Consider a Tychonoff cube $[0,1]^I$ where I is countable. Prove that it has a countable base.

Hint. Prove that the collection of all $U = W(i,]a, b[)$ with a, b rational numbers defines a countable prebase in $[0,1]^I$ and use the Problem 6.2.

Exercise 6.22 (**). Prove that if the index set I has the cardinality greater than or equal to continuum then the Tychonoff cube $[0,1]^I$ is non-separable.

Hint. Consider a countable subset W of a Hausdorff space. Prove that the cardinality of the closure of W is not greater than continuum.

Exercise 6.23 (!). Consider a set $M = [0,1]^{\mathbb{N}}$ of sequences of real numbers in $[0,1]$ indexed by $\mathbb{N}$. Consider the function $d: M \times M \longrightarrow \mathbb{R}$,

$$d(\{\alpha_i\}, \{\beta_i\}) = \sqrt{\sum_i i^{-2}|\alpha_i - \beta_i|^2}.$$

Prove that this function is well-defined and defines a metric on $[0,1]^{\mathbb{N}}$.

Definition 6.7. A metric space $[0,1]^{\mathbb{N}}$ with the metric defined as above is called **Hilbert cube**.

Exercise 6.24 (!). Consider a sequence $\{\alpha_i(n)\}$ of points of $[0,1]^{\mathbb{N}}$. Prove that it converges in the Tychonoff topology iff it converges in the topology of the Hilbert cube.

Exercise 6.25 (*). Deduce that the identity mapping is a homeomorphism of the Tychonoff cube and the Hilbert cube.

Remark. We actually proved that if the index set I is countable then the Tychonoff cube $[0,1]^I$ is metrizable.

Exercise 6.26 (*). Consider a uncountable index set I. Is the Tychonoff cube $[0,1]^I$ metrizable in that case?

Urysohn lemma and metrization of topological spaces

Definition 6.8. Consider two non-intersecting closed subsets $A, B \subset M$ of a topological space M. A continuous function $f: M \longrightarrow [0,1]$ is called an **Urysohn function** if $f(A) = 0, f(B) = 1$.

Exercise 6.27. Suppose that Urysohn function exists for any two non-intersecting closed subsets $A, B \subset M$. Prove that M satisfies the separation axiom T_4.

Exercise 6.28. Prove in the previous problem setting that it is possible that M does not satisfy T_1 separation axiom.

Exercise 6.29 (*). Suppose M satisfies T_4 separation axiom and $A, B \subset M$ are non-intersecting and closed. Prove that there exists a sequence of neighborhoods $U_{p/2^q} \supset A$ indexed by rational numbers of the form $0 < p/2^q < 1$ that satisfies the following conditions:

(i) for all p, q, B does not intersect $U_{p/2^q}$.

(ii) if $p_1/2^{q_1} < p_2/2^{q_2}$ then the closure of $U_{p_1/2^{q_1}}$ is contained in $U_{p_2/2^{q_2}}$.

Hint. Use an inductive argument.

Exercise 6.30 (*). In the previous problem setting define a function $f: M \longrightarrow [0,1]$ to be

$$f(m) = \sup \left\{ p_2/2^{q_2} \;\; | \;\; m \notin U_{p_1/2^{q_1}} \right\}$$

outside A and equal to zero on A. Prove that f is continuous and that f is an Urysohn function.

Hint. Prove that the intervals of the form $]p_1/2^{q_1}, p_2/2^{q_2}[$ form a prebase of the topology on $[0,1]$. Prove that

$$f^{-1}(]p_1/2^{q_1}, p_2/2^{q_2}[) = U_{p_2/2^{q_2}} \backslash \overline{U_{p_1/2^{q_1}}}.$$

Deduce that f is continuous.

Remark. We have proven the following "Urysohn lemma": if M satisfies the T_4 condition, then for any two non-intersecting closed subsets of M there is an Urysohn function.

Exercise 6.31 (*). Consider a Hausdorff space M with a countable base B, which satisfies the T_4 condition and let I be a set of all pairs $U_1, U_2 \in B$ such that the closures of U_1, U_2 do not intersect, F_{U_1,U_2} are respective Urysohn functions and $F: M \longrightarrow [0,1]^I$ is a mapping to Tychonoff cube define as $F(m) = \prod F_{U_1,U_2}$. Prove that F is continuous and injective.

Exercise 6.32 (*). In the previous problem setting denote the inverse mapping of F as $G: F(M) \longrightarrow M$. Consider a sequence of points $\{x_i\}$ such that $F_{U_1,U_2}(x_i)$ converges for every pair (U_1, U_2) from I. Deduce that the sequence $\{x_i\}$ converges. Prove that G is continuous.

Exercise 6.33 (*). Prove that any Hausdorff space M with a countable base which satisfies the T_4 condition (such space is called a **Polish** space) is a subspace of a Hilbert cube.

Remark. We have proved the following **metrization theorem**: every Polish space is metrizable.

Exercise 6.34. Prove that any subset of a Hilbert cube is Polish.

Exercise 6.35 (*). Is it true that every metrizable space is a Polish space?