GEOMETRY 7: Set-theoretic topology: compactness

Definition 7.1. Consider a topological space M. We call any collection of open subsets $U_i \subset M$ (possibly infinite or even uncountable) such that $M = \bigcup U_i$ a **cover** of M . The topological space M is called **compact** (or **a compactum**) if it is possible to find a finite subcover of every open cover of M. A subset $Z \subset M$ of the topological space M is called compact if it is compact in the induced topology.

Exercise 7.1. Prove that the interval $[0, 1]$ is compact. In which case a set with discrete topology is compact? With codiscrete topology?

Exercise 7.2 (*). Consider the following topology on M: open sets are complements of finite sets (this topology is called **cofinite**). Find all compact subsets of M.

Exercise 7.3 (!). Consider a compact space Z and a closed subset $Z' \subset Z$. Prove that Z' is also compact. Does compactness of a set follow from its closedness?

Exercise 7.4. Consider a Hausdorff topological space M, an arbitrary subset Z of M and a point $x \notin Z$.

a. Prove that there is an open cover $\{U_i\}$ of Z such that the closure of every U_i does not contain x.

b. **(*)**Give an example of a non-Hausdorff T_1-space where this is not true.

Exercise 7.5 (!). Consider a Hausdorff space M. Prove that every compact subset of M is closed.

Hint. Use the previous problem.

Exercise 7.6. Consider two compact subsets of a Hausdorff space. Prove that there exist two non-intersecting open neighborhoods of these subsets.

Exercise 7.7 (!). Consider a compact Hausdorff topological space. Prove that it satisfies the T_4 separation axiom.

Definition 7.2. A topological space is called **locally compact** if every point has a neighborhood such that its closure is compact.

Exercise 7.8. Consider a locally compact Hausdorff topological space. Prove that is satisfies the T_3 separation axiom.

Exercise 7.9 (**). Does there exist a locally compact topological space which does not satisfy the first countability axiom?

Exercise 7.10 (**). Does there exist a countable topological space which is not locally compact?

Exercise 7.11. Consider a topological space X. Denote by $\widehat{X}$ the set $X \bigcup \{\infty\}$ (X with one point added, this point is denoted by ∞) with the following topology: $U \subset \widehat{X}$ is open if either $\infty \in U$ and the complement of U is compact as a subset of X, or if $\infty \notin X$ and U is open as a subset of X. Prove that this is indeed a topology and that the space $\widehat{X}$ is compact.

Definition 7.3. The space $\widehat{X}$ is called a **one-point compactification** of the space X.

Exercise 7.12 (*). Consider a Hausdorff space X. Is it true that $\widehat{X}$ is also Hausdorff?

Exercise 7.13. Consider the space $X = \mathbb{R}^n$ with the natural topology. Prove that $\widehat{X}$ is homeomorphic to the n-dimensional sphere.

Exercise 7.14. Consider a topological space M and a subset Z. Prove that the following are equivalent:

(i) Every point $z \in Z$ has a neighborhood $U \ni z$ that contains no other points from Z.

(ii) M induces the discrete topology on Z.

(iii) Z does not contain any of its accumulation points.

Definition 7.4. A closed subset $Z \subset M$ that satisfies one of the conditions from the statement of Problem 7.14 is called **discrete**.

Exercise 7.15. Consider a Hausdorff space M and suppose it has an infinite discrete subset $Z \subset M$. Prove that M is non-compact.

Consider a collection Z_i of subsets of a set M. We say that the collection is **incomplete**, if for every finite subcollection $Z_1, Z_2, \ldots, Z_k$ the intersection $Z_1 \cap Z_2 \cap \cdots \cap Z_k$ is non-empty. A **monotone collection** Z_i of subsets of the set M is a collection of subsets that is linearly ordered by inclusion (i.e. for all Z_i, Z_j from the collection either $Z_i \subset Z_j$, or $Z_j \subset Z_i$).

Exercise 7.16. Prove that a topological space M is compact iff every incomplete collection of closed subsets $Z_i \subset M$ has a non-empty intersection $\cap_i Z_i$.

Exercise 7.17. Prove that if a topological space M is compact then every monotone collection of non-empty closed subsets $Z_i \subset M$ has a non-empty intersection $\cap_i Z_i$.

Exercise 7.18 (!). Consider a Hausdorff topological space M with a countable base. Prove that M is compact iff M does not have infinite discrete subsets.

Hint. If M has an infinite discrete subset then it follows from the Problem 7.17 that M is non-compact. Conversely, if M is non-compact then M has a countable cover S such that no finite subset of S covers M.

Exercise 7.19. Consider a Hausdorff topological space M with a countable base. Prove that M is compact iff every sequence of points from M has an accumulation point.

Exercise 7.20 (*). Consider a topological space M, not necessarily Hausdorff.

a. Is it possible that a compact subset of M contains an infinite discrete subset?

b. Is it possible that there is a non-compact subset of M that contains no infinite discrete subsets?

c. **(**)** Consider a Hausdorff space M. Does there exist a non-compact subset of M that does not contain infinite discrete subsets?

Exercise 7.21 (!). Consider a continuous mapping $f: M \longrightarrow N$ of topological spaces. Prove that for any compact subset $Z \subset M$, $f(Z)$ is compact.

Exercise 7.22. Consider a subset $Z \subset \mathbb{R}$.

a. Prove that Z is compact iff it is closed and bounded (i.e. contained in an interval $[a, b]$).

b. Prove that Z is compact iff every subset of it has an infimum and supremum in Z.

Exercise 7.23 (!). Consider a continuous mapping $f: M \longrightarrow \mathbb{R}$ of topological spaces. Prove that f reaches its maximum and minimum on any compact subset of M.

Exercise 7.24 (*). Consider a non-compact Hausdorff topological space with a countable base that satisfies the T_4 separation axiom. Construct a continuous function $f: M \longrightarrow \mathbb{R}$ that has no maximum.

Hint. Consider $\{x_i\}$, a countable discrete subset of M. Use the T_4 separation axiom to construct a collection of neighborhoods $U_i \supset x_i$ such that the closure of U_i does not intersect with the closure of $\bigcup_{j \neq i} U_j$. Now apply Urysohn lemma to closed sets $\{x_i\}$, $M \backslash U_i$ and sum up the Urysohn functions f_i obtained with the right coefficients.

Exercise 7.25. Consider a continuous mapping $f: M \longrightarrow N$ of topological spaces, where M is compact and N is Hausdorff. Prove that f maps closed sets to closed sets.

Exercise 7.26. Consider a continuous mapping $f: M \longrightarrow N$ of topological spaces, where M is compact and N is Hausdorff. Suppose that f is bijective. Prove that f is a homeomorphism.

Exercise 7.27. Give an example of a continuous mapping $f: M \longrightarrow N$, where M is compact, such that f is not a homeomorphism (N is not Hausdorff here).

Compact sets and products

Definition 7.5. A continuous mapping $f: X \to Y$ of topological spaces is called **proper** if for every compact $K \subset Y$ the preimage $f^{-1}(K) \subset X$ is compact.

Exercise 7.28 (!). Consider a Hausdorff space Y with a countable base. Prove that a proper mapping $f: X \to Y$ maps closed subsets of X to closed subsets of Y.

Hint. Take a closed set $Z \subset Y$ which has a non-closed image. Take a sequence of points $y_i \in f(Z)$ which converges to $y \in Y$ that does not belong to $f(Z)$.

Exercise 7.29 (*). Is the previous problem statement true if we do not require existence of a countable base?

Exercise 7.30 (*). Consider a continuous mapping $f: X \to Y$ that maps closed sets to closed sets and the preimage $f^{-1}(y)$ of any point $y \in Y$ is compact. Prove that the mapping f is proper.

Hint. Use the compactness criterion from the Problem 7.16.

Definition 7.6. A continuous mapping $f: X \to Y$ is called **closed** if the image of any any closed subset is closed. The mapping is called **universally closed** if for any continuous mapping $g: Z \to Y$ the induced mapping $X \times_Y Z \to Z$ is closed ($X \times_Y Z$ is a subset of $X \times Z$ that contains all pairs $\langle x, z \rangle$ such that $f(x) = g(z)$).

Exercise 7.31 (*). Consider a continuous mapping $F : X \to Y$ which is universally closed. Prove that it is a proper mapping.

Hint. Use the Problem 7.30 to justify that only the case when Y is a one point space can be considered. Then use the Problem 7.18: if X contains an infinite discrete subset M then take $Z = \widehat{M}$, i.e. a one-point compactification of M and deduce the contradiction.

Exercise 7.32 (!). Consider compact topological spaces X, Y. Prove that the product $X \times Y$ is compact.

Hint. Use the fact that sets of the form $U \times V$, where U is open in X and V is open in Y, form a base of the topology on $X \times Y$ and prove that it suffices to consider covers of $X \times Y$ that contain only sets of this form. Then for every point $y \in Y$ choose a finite subcover of the subset $X \times \{y\} \subset X \times Y$ that contains sets of the form $U_i \times V_i$, and notice that sets $V_y = \cap V_i$ form an open cover of Y.

Thus every projection $X \times Y \to Y$ for any Y and compact X is a proper mapping.

Exercise 7.33. Consider a subset $X \subset \mathbb{R}^n$. Prove that the following are equivalent:

(i) X is compact

(ii) X is closed and bounded (i.e. lies within a ball).

Tychonoff's theorem

Exercise 7.34. Consider a sequence $a_i(n)$ of mappings from $\mathbb{N}$ to $[0, 1]$. Prove that one can select a subsequence $a_{i_1}, a_{i_2}, a_{i_3}, \dots$ such that $\{a_{i_k}(n)\}$ converges for any n.

Exercise 7.35 (!). Deduce that the Tychonoff cube $[0, 1]^{\mathbb{N}}$ is compact.

Exercise 7.36 (*). Consider a topological space M. Consider a (possibly uncountable) collection $\{V_\alpha\}$ of covers of M, such that every V_α either contains $V_{\alpha'}$ or is contained in it (in other words, in $\{V_\alpha\}$ every cover can be obtained from any other cover by adding or removing some elements). Suppose every V_α does not have a finite subcover. Prove that the union of all V_α does not have a finite subcover either.

Exercise 7.37 (*). Use the Zorn's lemma to prove that every non-compact subset $X \subset M$ has a cover $\{V_\alpha\}$ that does not have a finite subcover, but if one adds to $\{V_\alpha\}$ any open set that does not belong to it, then the cover obtained has a finite subcover.

Hint. Use the previous problem.

We will call such covers **maximal**.

Exercise 7.38 (*). Consider a maximal cover $\{V_\alpha\}$ of a non-compact topological space M. Prove that if open sets U_1, U_2 do not belong to $\{V_\alpha\}$ and they have a non-empty intersection then the intersection does not belong to $\{V_\alpha\}$ either. Prove that any non-empty finite intersection of open sets that do not belong to $\{V_\alpha\}$, does not belong to $\{V_\alpha\}$ either.

Hint. Use the previous problem.

Exercise 7.39 (*). Consider a topological space M with a given prebase of topology R. Consider then a non-compact subset $X \subset M$ and a maximal cover $\{V_\alpha\}$. Prove that $\{V_\alpha\}$ has a subcover whose elements belong to R.

Hint. Use the previous problem.

Remark. We have proved the following theorem (Alexander's theorem about prebase). Consider a topological space M with a given prebase S. Then a subset $X \subset M$ is compact iff every cover of X whose elements are from S has a finite subcover. Alexander's theorem uses the Axiom of Choice and is equivalent to it (that was shown by Cayley).

Exercise 7.40 (*). Deduce that the Tychonoff cube $[0,1]^I$ is compact for any index set I.

Hint. Consider a prebase of the topology on the Tychonoff cube that consists of subsets of the form $[0,1] \times [0,1] \times \cdots \times]a,b[\times [0,1] \times \dots$ (an open interval occurs once). Use Alexander's theorem.

Remark. Compactness of the Tychonoff cube is equivalent to the following statement. Consider a space $\mathrm{Map}(I,[0,1])$ of mappings from a set I to the interval $[0,1]$, endowed with the topology of the pointwise convergence. Then $\mathrm{Map}(I,[0,1])$ is compact. In particular, every sequence $\{a_i(x)\}$ of mappings has a subsequence $\{a_{i_k}(x)\}$ such that $\{a_{i_k}(x)\}$ converges for all $x \in I$.

Definition 7.7. Consider a topological space M, a set I and M^I, the set of all mappings from I to M, that is, the product of I copies of M. For an arbitrary $x \in I$ and an open set $U \subset M$ consider a subset $U(x) \subset M^I$ which consists of all mappings that map x to an element of U. Define a topology on M^I using the prebase that consists of all $U(x)$. This topology is called **Tychonoff topology** (or **weak topology** or **topology of pointwise convergence**).

Exercise 7.41 (*). Consider a compact space M. Deduce from Alexander's theorem that M^I endowed with Tychonoff topology is compact.

Fundamental theorem of algebra

Consider a polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of a positive degree with complex coefficients. We look at P as a function from $\mathbb{C}$ to $\mathbb{C}$. $\mathbb{C}$ is identified with $\mathbb{R}^2$ as a topological space.

Exercise 7.42. Prove that P is continuous.

Exercise 7.43 (!). Prove that if $|x| > 2\max(1, \sum |a_i|)$, then $\frac{|P(x)-x^n|}{|x^n|} < 1/2$.

Exercise 7.44 (!). Prove that if $|x| > 2R\max(1, \sum |a_i|)$, then $|P(x)| > R^n$.

Exercise 7.45 (!). Deduce that $|P|$ reaches its local minimum at a point $a \in \mathbb{C}$.

Hint. We approximated the polynomial $|P|$ with the polynomial x^n, for which we know how fast it grows. We deduce that $|P(x)| > R^n$, when $|x|$ is big enough. That's why the minimum of $|P|$ on the disc $|x| \leqslant R$ is reached inside the disc and not on its boundary.

In order to simplify the notation we will suppose that $|P|$ reaches its minimum at zero. We want to prove that the minimum of $|P|$ is zero. Suppose it is not true. Then let k be the smallest number among $1, 2, 3, \dots, n$, such that $a_k \neq 0$. Multiply P by a_0^{-1} and perform the substitution $x = z\sqrt[k]{a_k^{-1}}$, so we get a polynomial of the form

$$Q(z) = 1 + z^k + b_{k+1}z^{k+1} + b_{k+2}z^{k+2} + \dots$$

Exercise 7.46. Prove that for any complex z, such that $|z| < 1$, the following holds:

$$|Q(z) - 1 - z^k| < |z^{k+1}|(\sum |b_i|).$$

Exercise 7.47 (!). Prove that for any complex number z, such that $|z| < \frac{1}{2} \max\left(1, \sum |b_i|\right)^{-1}$, the following holds:

$$\frac{|Q(z) - 1 - z^k|}{|z^k|} < \frac{1}{2}.$$

Exercise 7.48 (!). Deduce that for any positive real $\varepsilon < \frac{1}{2} \max\left(1, \sum |b_i|\right)^{-1}$ and any complex z, such that $z^k = -\varepsilon$, the following holds:

$$|Q(z) - 1 + \varepsilon| < \varepsilon/2.$$

Remark. We approximated Q with the polynomial $1 - z^k$ in a neighbourhood of zero. We can use this approximation to deduce that $|Q(\sqrt[k]{-\varepsilon})| < |Q(0)|(1 - \frac{1}{2}\varepsilon)$ for ε that is small enough. It follows that the local minimum of the polynomial is 0.

Exercise 7.49 (!). Prove the Fundamental Theorem of Algebra: every polynomial P of positive degree has a root in $\mathbb{C}$.