## **GEOMETRY 8:** Pointwise and uniform convergence

During the work on this sheet, it is allowed to use Tychonoff's Theorem in the following form.

**Theorem.** Let X be a compact topological space, I an arbitrary set,  $X^I$  the topological space (in the pointwise convergence topology), of the mappings  $I \to X$ . Then  $X^I$  is compact.

**Exercise 8.1.** Consider the space of of functions from an interval to an interval. Show that the limit of a sequence of continuous functions need not be continuous.

**Definition 8.1.** Let X, Y be metric spaces,  $\{f_{\alpha}\}$  a set of continuous functions  $X \to Y$ . Then  $\{f_{\alpha}\}$  is called *uniformly continuous* if for any  $\varepsilon$  there exists  $\delta$  such that the image of any  $\delta$ -ball under any  $f_{\alpha}$  is contained in an  $\varepsilon$ -ball  $B_{\alpha}$ . (Note that  $B_{\alpha}$  can depend upon  $\alpha$ .)

**Exercise 8.2.** Let  $f: X \longrightarrow Y$  be a mapping of metric spaces that maps each Cauchy sequence to a Cauchy sequence. Show that f is continuous as a mapping of topological spaces. Is it true that any continuous mapping maps each Cauchy sequence to a Cauchy sequence?

**Exercise 8.3 (!).** Let X, Y be metric spaces,  $\{f_i\}$  a uniformly continuous sequence of continuous functions  $X \to Y$ . Suppose that  $\{f_i\}$  converges to f in the pointwise convergence topology. Show that f is continuous.

**Hint.** Show that f is uniformly continuous, with the same  $\varepsilon$ ,  $\delta$  as  $\{f_i\}$ , and then use the preceding problem.

Pick compact metric spaces X, Y, and let Map(X, Y) be the set of continuous mappings  $X \to Y$ .

**Exercise 8.4.** For any  $f, g \in Map(X, Y)$  define

$$d_{\sup}(f,g) := \sup_{x \in X} d(f(x),g(x)).$$

Show the correctness of the definition of  $d_{sup}(f,g)$ , and that it defines a metric on Map(X,Y).

**Definition 8.2.** The latter metric is called sup-metric on Map(X, Y).

**Exercise 8.5 (!).** Let a uniformly continuous sequence of mappings  $\{f_i\} \subset Map(X, Y)$  pointwise converge to f. Show that it converges to f in the topology induced by the sup-metric, too.

**Hint.** Let  $\sup_{x \in X} d(f(x), f_i(x)) > C$  for any *i*. Find a converging subsequence of  $\{x_i\}$ 's satisfying  $d(f(x_i), f_i(x_i)) > C$ . Let  $x = \lim_{i \to \infty} x_i$ . Due to uniform convergence,  $d(f_i(x_i), f_i(x)) \to 0$ . Derive a contradiction from the triangle inequality

$$d(f_i(x), f(x)) + d(f_i(x_i), f_i(x)) \ge d(f(x), f_i(x_i)).$$

**Exercise 8.6 (!).** (Arzelà-Ascoli Theorem) Let  $\Psi \subset \operatorname{Map}(X, Y)$  be closed (w.r.t. the sup-metric) and uniformly continuous. Show that  $\Psi$  is compact.

**Hint.** Use Tychonoff's Theorem and the preceding problem. As we have already said, we assume here that X and Y are compact!

**Exercise 8.7** (\*\*). Find an independent of Tychonoff's Theorem (and thus, of the axiom of choice) proof of Arzelà-Ascoli Theorem.

**Exercise 8.8 (\*).** Let  $K \subset X$  be compact and  $V \subset Y$  open. Denote by  $U(K,V) \subset Map(X,Y)$  the set of all mappings sending K in V. Consider the topology on Map(X,Y), defined by the subbase of all U(K,V). Show that it coincides with the topology induced by the sup-metric.

**Definition 8.3.** The latter topology on Map(X, Y) is called **topology of uniform convergence**.

**Exercise 8.9.** Show that the pointwise convergence topology is weaker than the uniform convergence topology; in other words that the identity map from Map(X, Y) endowed with the latter topology onto Map(X, Y) endowed with the former topology is continuous.

**Definition 8.4.** Let *M* be a metric space and  $Z \subseteq M$ . **Diameter** of *Z* is the number diam $(Z) := \sup_{x,y \in Z} d(x,y)$ .

**Exercise 8.10.** Let  $f \in Map(X, Y)$  be an arbitrary mapping,  $\varepsilon$  be a real number, and  $\delta(f, \varepsilon)$  be the supremum of diam(f(B)) over all the  $\varepsilon$ -balls B in X. Show that  $\lim_{\varepsilon \to 0} \delta(f, \varepsilon) = 0$ .

**Hint.** Assume that for a convergent to 0 sequence  $\varepsilon_i$ , a collection of points  $x_i \in X$  and a positive constant C one has  $\operatorname{diam} f(B_{\varepsilon_i}(x_i)) > C$ . Consider a limit point x of  $\{x_i\}$ . Then each  $\varepsilon$ -ball around x contains  $B_{\varepsilon_i}(x_i)$  (for sufficiently large i), implying that the image of this  $\varepsilon$ -ball has diameter greater than C. Thus, f is not continuous.

**Exercise 8.11 (!).** Let  $f \in Map(X, Y)$  be continuous. Show that f is uniformly continuous.

**Hint.** The claim is tautologically equivalent to  $\lim_{\varepsilon \to 0} \delta(f, \varepsilon) = 0$ .

**Exercise 8.12.** Let  $\Psi \subset \operatorname{Map}(X, Y)$ . Show that  $\Psi$  is uniformly continuous if and only if

$$\lim_{\varepsilon \longrightarrow 0} \sup_{f \in \Psi} \delta(f, \varepsilon) = 0.$$

**Exercise 8.13 (\*).** Let  $d_{\sup}(f,g) < \gamma$ . Show that  $\delta(f,\varepsilon) < \delta(g,\varepsilon) + \gamma$ .

**Exercise 8.14 (\*).** Let  $\{f_i\}$  be a Cauchy sequence in  $(Map(X, Y), d_{sup})$ . Show that it is uniformly continuous.

Hint. We shall show that

$$\lim_{\varepsilon \to 0} \sup_{i} \delta(f_i, \varepsilon) = 0.$$

Using the preceding problem, check that for all  $f_i$  in an  $\gamma$ -ball in  $(\operatorname{Map}(X, Y), d_{\sup})$  the numbers  $\delta(f_i, \varepsilon)$  differ by no more than  $\gamma$ . Derive from this that  $\sup_i \delta(f_i, \varepsilon) < \delta(f_N, \varepsilon) + \gamma$  for a fixed N, and thus

$$\sup_{i} \delta(f_i, \varepsilon) < \gamma + \max_{i \leq N} \delta(f_i, \varepsilon)$$

The limit of the latter, as  $\varepsilon \longrightarrow 0$ , cannot be greater than  $\gamma$ , as all  $f_i$  are uniformly continuous.

**Exercise 8.15** (\*). Show the completness of the metric space  $(Map(X, Y), d_{sup})$ .

**Exercise 8.16 (\*).** Is the space  $(Map(X, Y), d_{sup})$  locally compact?

## Peano curve

Let  $[a,b] \subset \mathbb{R}$ . The mapping  $[a,b] \xrightarrow{f} \mathbb{R}^n$  is called **linear**, if  $f(\lambda a + (1-\lambda)b) = \lambda f(a) + (1-\lambda)f(b)$ , for any  $0 < \lambda < 1$ . It is called **piecewise linear** if [a,b] is partitioned into subsegments  $[a,a_1], [a_1,a_2], [a_2,a_3], \ldots$ , and f is linear on each of  $[a_\ell, a_{\ell+1}]$ . The image of [a,b] under a piecewise linear map is, certainly, a polygonal curve.

Let f be a piecewise linear map  $f: [0,1] \to [0,1] \times [0,1]$  satisfying the following property; all the segments of f([0,1]) are parallel either to the line x = y or to the line x = -y.



In other words, for any subsegment  $[a, a_1]$ , on which f linear, f maps  $[a, a_1]$  onto a diagonal of a square Q, with the sides parallel to the coordinate axes. Let  $\mathcal{P}l$  be the space of such piecewise linear mappings. Let us define an operation  $\mu$  that produces from an  $f \in \mathcal{P}l$  with k linear segments a piecewise linear map with 4k linear segments.



Namely  $\mu(f)$  is defined as follows.

- 1. Denote by  $a_0, a_1, \ldots, a_k$  the ends of the segments where f was linear. Then  $\mu(f)$  maps  $a_i$  to  $f(a_i)$ .
- 2. Partition each segment  $[a_i, a_{i+1}]$  into 4 equal parts:

 $[b_{4i}, b_{4i+1}], [b_{4i+1}, b_{4i+2}], [b_{4i+2}, b_{4i+3}], [b_{4i+3}, b_{4i+4}].$ 

 $\mu(f)$  maps  $[b_{4i}, b_{4i+1}]$  linearly to  $[f(a_i), f\left(\frac{a_i+a_{i+1}}{2}\right)]$ , and  $[b_{4i+3}, b_{4i+4}]$  to  $[f\left(\frac{a_i+a_{i+1}}{2}\right), f(a_{i+1})]$ .

3. Consider the square with a diagonal  $[f(a_i), f(a_{i+1})]$ , and number its vertices clockwise:  $f(a_i)$ ,  $A, f(a_{i+1}), B$ . Then  $\mu(f)$  maps  $[b_{4i+1}, b_{4i+2}]$  linearly to  $[f\left(\frac{a_i+a_{i+1}}{2}\right), B]$ , and  $[b_{4i+2}, b_{4i+3}]$  to  $[B, f\left(\frac{a_i+a_{i+1}}{2}\right)]$ .

We obtain the following polygonal curve:



**Exercise 8.17.** Consider the segment and the square as metric spaces endowed with the standard metric. Let  $f \in \mathcal{P}l$ , and the biggest straight segment  $[f(a_i), f(a_{i+1})]$  of the corresponding polygonal curve is of length k. Then  $d_{\sup}(f, \mu(f)) \leq \frac{k}{\sqrt{2}}$ .

**Exercise 8.18.** Let  $f \in \mathcal{P}l$ , and the biggest straight segment  $[f(a_i), f(a_{i+1})]$  of the corresponding polygonal curve is of length k. Then the biggest straight segment in  $\mu(f)$  is of length k/2.

**Exercise 8.19.** Let  $f_0 \in \mathcal{P}l$ ,  $f_1 = \mu(f_0), \ldots, f_n = \mu(f_{n-1})$ , and the biggest straight segment of the polygonal curve f([0, 1]) has length k. Show that

$$d_{\sup}(f_n, f_{n+1}) < \frac{k}{2^n \sqrt{2}}$$

**Exercise 8.20** (!). Show that  $\{f_i\}$  is a Cauchy sequence in the metric  $d_{sup}$ .

**Exercise 8.21.** Let  $f \in \mathcal{P}l$ , and for all straight segments  $[a_i, a_{i+1}]$  f the length of  $[f(a_i), f(a_{i+1})]$  is at most

$$\rho(a_{i+1} - a_i),$$

where  $\rho > 0$  is a real. Show that  $\delta(f, \varepsilon) \leq \rho \varepsilon$ , where  $\delta(f, \varepsilon)$  is the function defined above.

**Exercise 8.22.** Let  $f_0 \in \mathcal{P}l$ ,  $f_1 = \mu(f_0), \ldots, f_n = \mu(f_{n-1})$ , and for all straight segments  $[a_i, a_{i+1}]$  $f_0$  the length of  $[f(a_i), f(a_{i+1})]$  is at most  $\rho(a_{i+1} - a_i)$ . Show that  $\delta(f_n, \varepsilon) \leq \rho 2^n \varepsilon$ .

**Exercise 8.23.** Let  $f \in \mathcal{P}l$ , and the longest straight segment  $[f(a_i), f(a_{i+1})]$  of f([0, 1]) has length k. Show that  $\delta(\mu(f), \varepsilon) \leq 2\frac{k}{\sqrt{2}} + \delta(f, \varepsilon)$ .

**Exercise 8.24.** Let  $f_0 \in \mathcal{P}l$ ,  $f_1 = \mu(f_0), \ldots, f_n = \mu(f_{n-1})$ , and the longest straight segment of  $f_0([0, 1])$  has length k. Show that

$$\delta(f_n,\varepsilon) \leqslant 4 \frac{k}{2^{n-m}\sqrt{2}} + \rho 2^m \varepsilon$$
(8.1)

for any n, m (n > m)

**Exercise 8.25.** In the previous problem take  $\varepsilon < 2^{-2m}$ , n > 2m. Derive from (8.1) that

$$\delta(f_n,\varepsilon) \leqslant \frac{4k\sqrt{2+\rho}}{2^{-m}}.$$

Show that for any i the following holds.

$$\delta(f_i,\varepsilon) \leqslant \max\left(\frac{4k\sqrt{2}+\rho}{2^{-m}},\rho 2^{2m}\varepsilon\right).$$

**Exercise 8.26 (!).** Let  $f_0$  linearly map [0, 1/2] to the segment [(0, 0), (1, 1)], and [1/2, 1] – to [(1, 1), (0, 0)]. Show that  $\{f_i\}$  is uniformly continuous.

**Hint.** Derive from the preceding problem that  $\lim_{\varepsilon \to 0} \sup_i (\delta(f_i, \varepsilon)) = 0.$ 

**Exercise 8.27.** Derive from Arzelà-Ascoli Theorem the existence of  $\lim f_i$  (in sup-metric) and continuity of it as a function  $\mathcal{P}: [0,1] \longrightarrow [0,1] \times [0,1]$ .

**Definition 8.5.** The function  $\mathcal{P}$  defined above is called a **Peano curve**.

**Exercise 8.28.** Find  $\mathcal{P}(q)$ , for  $q = \frac{a}{2^n}$   $(a \in \mathbb{Z})$ . (Such numbers are called binary-rational.)

**Exercise 8.29.** Let  $Q_2$  be the set of binary-rational numbers. Show that  $\mathcal{P}(Q_2)$  is dense on the unit square.

**Exercise 8.30 (!).** Show that  $\mathcal{P}([0,1])$  is the whole unit square.

Hint. Use the fact that the image of a compact is compact.

**Exercise 8.31 (!).** Is it possible to map, surjectively and continuously, [0, 1] onto a cube ? Onto a cube with one point removed?