GEOMETRY 8: Pointwise and uniform convergence

During the work on this sheet, it is allowed to use Tychonoff's Theorem in the following form.

Theorem. Let X be a compact topological space, I an arbitrary set, X^I the topological space (in the pointwise convergence topology), of the mappings $I \to X$. Then X^I is compact.

Exercise 8.1. Consider the space of of functions from an interval to an interval. Show that the limit of a sequence of continuous functions need not be continuous.

Definition 8.1. Let X, Y be metric spaces, $\{f_\alpha\}$ a set of continuous functions $X \to Y$. Then $\{f_\alpha\}$ is called *uniformly continuous* if for any ε there exists δ such that the image of any δ-ball under any f_α is contained in an ε-ball B_α. (Note that B_α can depend upon α.)

Exercise 8.2. Let $f: X \longrightarrow Y$ be a mapping of metric spaces that maps each Cauchy sequence to a Cauchy sequence. Show that f is continuous as a mapping of topological spaces. Is it true that any continuous mapping maps each Cauchy sequence to a Cauchy sequence?

Exercise 8.3 (!). Let X, Y be metric spaces, $\{f_i\}$ a uniformly continuous sequence of continuous functions $X \to Y$. Suppose that $\{f_i\}$ converges to f in the pointwise convergence topology. Show that f is continuous.

Hint. Show that f is uniformly continuous, with the same ε, δ as $\{f_i\}$, and then use the preceding problem.

Pick compact metric spaces X, Y, and let $\operatorname{Map}(X, Y)$ be the set of continuous mappings $X \to Y$.

Exercise 8.4. For any $f, g \in \operatorname{Map}(X, Y)$ define

$$d_{\sup}(f, g) := \sup_{x \in X} d(f(x), g(x)).$$

Show the correctness of the definition of $d_{\sup}(f, g)$, and that it defines a metric on $\operatorname{Map}(X, Y)$.

Definition 8.2. The latter metric is called sup-metric on $\operatorname{Map}(X, Y)$.

Exercise 8.5 (!). Let a uniformly continuous sequence of mappings $\{f_i\} \subset \operatorname{Map}(X, Y)$ pointwise converge to f. Show that it converges to f in the topology induced by the sup-metric, too.

Hint. Let $\sup_{x \in X} d(f(x), f_i(x)) > C$ for any i. Find a converging subsequence of $\{x_i\}$'s satisfying $d(f(x_i), f_i(x_i)) > C$. Let $x = \lim_{i \to \infty} x_i$. Due to uniform convergence, $d(f_i(x_i), f_i(x)) \to 0$. Derive a contradiction from the triangle inequality

$$d(f_i(x), f(x)) + d(f_i(x_i), f_i(x)) \geqslant d(f(x), f_i(x_i)).$$

Exercise 8.6 (!). (Arzelà-Ascoli Theorem) Let $\Psi \subset \operatorname{Map}(X, Y)$ be closed (w.r.t. the sup-metric) and uniformly continuous. Show that Ψ is compact.

Hint. Use Tychonoff's Theorem and the preceding problem. As we have already said, we assume here that X and Y are compact!

Exercise 8.7 (**). Find an independent of Tychonoff's Theorem (and thus, of the axiom of choice) proof of Arzelà-Ascoli Theorem.

Exercise 8.8 (*). Let $K \subset X$ be compact and $V \subset Y$ open. Denote by $U(K,V) \subset \mathrm{Map}(X,Y)$ the set of all mappings sending K in V. Consider the topology on $\mathrm{Map}(X,Y)$, defined by the subbase of all $U(K,V)$. Show that it coincides with the topology induced by the sup-metric.

Definition 8.3. The latter topology on $\mathrm{Map}(X,Y)$ is called **topology of uniform convergence**.

Exercise 8.9. Show that the pointwise convergence topology is weaker than the uniform convergence topology; in other words that the identity map from $\mathrm{Map}(X,Y)$ endowed with the latter topology onto $\mathrm{Map}(X,Y)$ endowed with the former topology is continuous.

Definition 8.4. Let M be a metric space and $Z \subseteq M$. **Diameter** of Z is the number $\mathsf{diam}(Z) := \sup_{x,y\in Z} d(x,y)$.

Exercise 8.10. Let $f \in \mathrm{Map}(X,Y)$ be an arbitrary mapping, ε be a real number, and $\delta(f,\varepsilon)$ be the supremum of $\mathsf{diam}(f(B))$ over all the ε-balls B in X. Show that $\lim\limits_{\varepsilon \longrightarrow 0} \delta(f,\varepsilon) = 0$.

Hint. Assume that for a convergent to 0 sequence ε_i, a collection of points $x_i \in X$ and a positive constant C one has $\mathsf{diam} f(B_{\varepsilon_i}(x_i))) > C$. Consider a limit point x of $\{x_i\}$. Then each ε-ball around x contains $B_{\varepsilon_i}(x_i)$ (for sufficiently large i), implying that the image of this ε-ball has diameter greater than C. Thus, f is not continuous.

Exercise 8.11 (!). Let $f \in \mathrm{Map}(X,Y)$ be continuous. Show that f is uniformly continuous.

Hint. The claim is tautologically equivalent to $\lim\limits_{\varepsilon \longrightarrow 0} \delta(f,\varepsilon) = 0$.

Exercise 8.12. Let $\Psi \subset \mathrm{Map}(X,Y)$. Show that Ψ is uniformly continuous if and only if

$$\lim_{\varepsilon \longrightarrow 0} \sup_{f\in\Psi} \delta(f,\varepsilon) = 0.$$

Exercise 8.13 (*). Let $d_{\sup}(f,g) < \gamma$. Show that $\delta(f,\varepsilon) < \delta(g,\varepsilon) + \gamma$.

Exercise 8.14 (*). Let $\{f_i\}$ be a Cauchy sequence in $(\mathrm{Map}(X,Y), d_{\sup})$. Show that it is uniformly continuous.

Hint. We shall show that

$$\lim_{\varepsilon \longrightarrow 0} \sup_i \delta(f_i,\varepsilon) = 0.$$

Using the preceding problem, check that for all f_i in an γ-ball in $(\mathrm{Map}(X,Y), d_{\sup})$ the numbers $\delta(f_i,\varepsilon)$ differ by no more than γ. Derive from this that $\sup_i \delta(f_i,\varepsilon) < \delta(f_N,\varepsilon) + \gamma$ for a fixed N, and thus

$$\sup_i \delta(f_i,\varepsilon) < \gamma + \max_{i\leqslant N} \delta(f_i,\varepsilon)$$

The limit of the latter, as $\varepsilon \longrightarrow 0$, cannot be greater than γ, as all f_i are uniformly continuous.

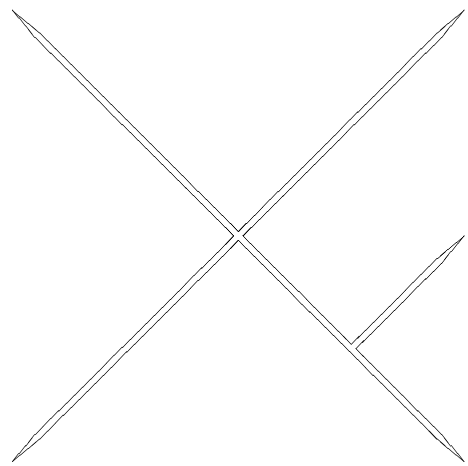
Exercise 8.15 (*). Show the completness of the metric space $(\mathrm{Map}(X,Y), d_{\sup})$.

Exercise 8.16 (*). Is the space $(\mathrm{Map}(X,Y), d_{\sup})$ locally compact?

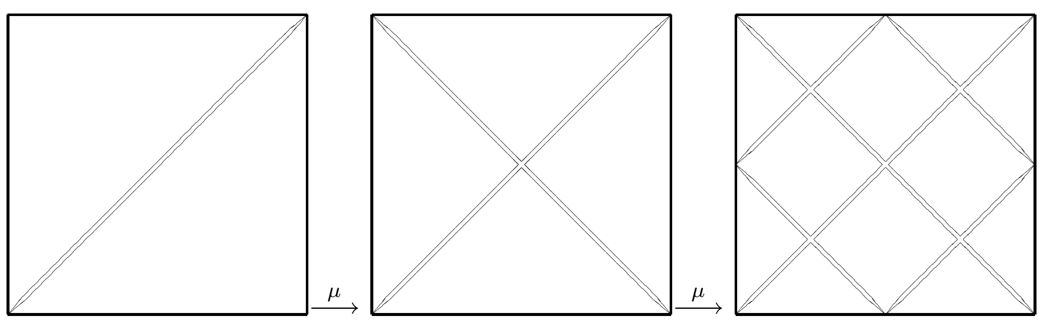
Peano curve

Let $[a,b] \subset \mathbb{R}$. The mapping $[a,b] \xrightarrow{f} \mathbb{R}^n$ is called **linear**, if $f(\lambda a + (1-\lambda)b) = \lambda f(a) + (1-\lambda)f(b)$, for any $0 < \lambda < 1$. It is called **piecewise linear** if $[a,b]$ is partitioned into subsegments $[a,a_1], [a_1,a_2], [a_2,a_3], \ldots$, and f is linear on each of $[a_\ell, a_{\ell+1}]$. The image of $[a,b]$ under a piecewise linear map is, certainly, a polygonal curve.

Let f be a piecewise linear map $f : [0,1] \to [0,1] \times [0,1]$ satisfying the following property; all the segments of $f([0,1])$ are parallel either to the line $x = y$ or to the line $x = -y$.



In other words, for any subsegment $[a,a_1]$, on which f linear, f maps $[a,a_1]$ onto a diagonal of a square Q, with the sides parallel to the coordinate axes. Let $\mathcal{P}l$ be the space of such piecewise linear mappings. Let us define an operation μ that produces from an $f \in \mathcal{P}l$ with k linear segments a piecewise linear map with $4k$ linear segments.



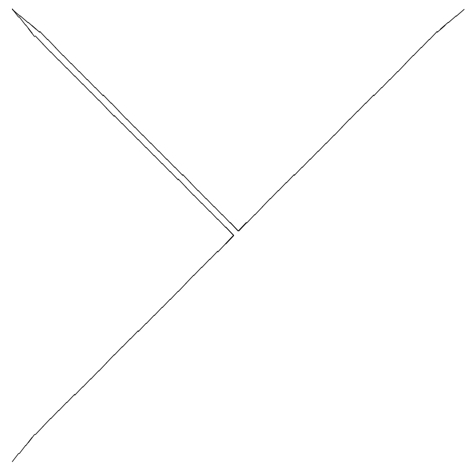
Namely $\mu(f)$ is defined as follows.

1. Denote by $a_0, a_1, \ldots, a_k$ the ends of the segments where f was linear. Then $\mu(f)$ maps a_i to $f(a_i)$.

2. Partition each segment $[a_i, a_{i+1}]$ into 4 equal parts:
$$[b_{4i}, b_{4i+1}], [b_{4i+1}, b_{4i+2}], [b_{4i+2}, b_{4i+3}], [b_{4i+3}, b_{4i+4}].$$
$\mu(f)$ maps $[b_{4i}, b_{4i+1}]$ linearly to $[f(a_i), f\left(\frac{a_i+a_{i+1}}{2}\right)]$, and $[b_{4i+3}, b_{4i+4}]$ to $[f\left(\frac{a_i+a_{i+1}}{2}\right), f(a_{i+1})]$.

3. Consider the square with a diagonal $[f(a_i), f(a_{i+1})]$, and number its vertices clockwise: $f(a_i)$, A, $f(a_{i+1})$, B. Then $\mu(f)$ maps $[b_{4i+1}, b_{4i+2}]$ linearly to $[f\left(\frac{a_i+a_{i+1}}{2}\right), B]$, and $[b_{4i+2}, b_{4i+3}]$ to $[B, f\left(\frac{a_i+a_{i+1}}{2}\right)]$.

We obtain the following polygonal curve:



Exercise 8.17. Consider the segment and the square as metric spaces endowed with the standard metric. Let $f \in \mathcal{P}l$, and the biggest straight segment $[f(a_i), f(a_{i+1})]$ of the corresponding polygonal curve is of length k. Then $d_{\sup}(f, \mu(f)) \leqslant \frac{k}{\sqrt{2}}$.

Exercise 8.18. Let $f \in \mathcal{P}l$, and the biggest straight segment $[f(a_i), f(a_{i+1})]$ of the corresponding polygonal curve is of length k. Then the biggest straight segment in $\mu(f)$ is of length $k/2$.

Exercise 8.19. Let $f_0 \in \mathcal{P}l$, $f_1 = \mu(f_0), \ldots, f_n = \mu(f_{n-1})$, and the biggest straight segment of the polygonal curve $f([0,1])$ has length k. Show that

$$d_{\sup}(f_n, f_{n+1}) < \frac{k}{2^n\sqrt{2}}$$

Exercise 8.20 (!). Show that $\{f_i\}$ is a Cauchy sequence in the metric $d_{\sup}$.

Exercise 8.21. Let $f \in \mathcal{P}l$, and for all straight segments $[a_i, a_{i+1}]$ f the length of $[f(a_i), f(a_{i+1})]$ is at most

$$\rho(a_{i+1} - a_i),$$

where $\rho > 0$ is a real. Show that $\delta(f, \varepsilon) \leqslant \rho\varepsilon$, where $\delta(f, \varepsilon)$ is the function defined above.

Exercise 8.22. Let $f_0 \in \mathcal{P}l$, $f_1 = \mu(f_0), \ldots, f_n = \mu(f_{n-1})$, and for all straight segments $[a_i, a_{i+1}]$ f_0 the length of $[f(a_i), f(a_{i+1})]$ is at most $\rho(a_{i+1} - a_i)$. Show that $\delta(f_n, \varepsilon) \leqslant \rho 2^n \varepsilon$.

Exercise 8.23. Let $f \in \mathcal{P}l$, and the longest straight segment $[f(a_i), f(a_{i+1})]$ of $f([0,1])$ has length k. Show that $\delta(\mu(f), \varepsilon) \leqslant 2\frac{k}{\sqrt{2}} + \delta(f, \varepsilon)$.

Exercise 8.24. Let $f_0 \in \mathcal{P}l$, $f_1 = \mu(f_0), \ldots, f_n = \mu(f_{n-1})$, and the longest straight segment of $f_0([0,1])$ has length k. Show that

$$\delta(f_n, \varepsilon) \leqslant 4\frac{k}{2^{n-m}\sqrt{2}} + \rho 2^m \varepsilon \tag{8.1}$$

for any n, m $(n > m)$

Exercise 8.25. In the previous problem take $\varepsilon < 2^{-2m}$, $n > 2m$. Derive from (8.1) that

$$\delta(f_n, \varepsilon) \leqslant \frac{4k\sqrt{2} + \rho}{2^{-m}}.$$

Show that for any i the following holds.

$$\delta(f_i,\varepsilon) \leqslant \max\left(\frac{4k\sqrt{2}+\rho}{2^{-m}}, \rho 2^{2m}\varepsilon\right).$$

Exercise 8.26 (!). Let f_0 linearly map $[0,1/2]$ to the segment $[(0,0),(1,1)]$, and $[1/2,1]$ – to $[(1,1),(0,0)]$. Show that $\{f_i\}$ is uniformly continuous.

Hint. Derive from the preceding problem that $\lim\limits_{\varepsilon\longrightarrow 0} \sup_i(\delta(f_i,\varepsilon)) = 0$.

Exercise 8.27. Derive from Arzelà-Ascoli Theorem the existence of $\lim f_i$ (in sup-metric) and continuity of it as a function $\mathcal{P}:\ [0,1] \longrightarrow [0,1]\times[0,1]$.

Definition 8.5. The function $\mathcal{P}$ defined above is called a **Peano curve**.

Exercise 8.28. Find $\mathcal{P}(q)$, for $q = \frac{a}{2^n}$ $(a \in \mathbb{Z})$. (Such numbers are called binary-rational.)

Exercise 8.29. Let Q_2 be the set of binary-rational numbers. Show that $\mathcal{P}(Q_2)$ is dense on the unit square.

Exercise 8.30 (!). Show that $\mathcal{P}([0,1])$ is the whole unit square.

Hint. Use the fact that the image of a compact is compact.

Exercise 8.31 (!). Is it possible to map, surjectively and continuously, $[0,1]$ onto a cube ? Onto a cube with one point removed?