GEOMETRY 9: Connectedness

Definition 9.1. Let M be a topological space. A closed and open at the same time subset $W \subset M$ is called **clopen**. M without proper clopen subsets is called **connected**. A subset $Z \subset M$ is called **connected** if it is connected in the induced topology.

Exercise 9.1. Is $\mathbb{R}$ connected?

Exercise 9.2 (!). Let X, Y be connected. Show that $X \times Y$ is connected.

Hint. Let $U \subseteq X \times Y$ be clopen. Consider $U \cap X \times \{y\}$. Show that $X \times \{y\}$ (in induced topology) is homeomorphic to X, and $U \cap X \times \{y\}$ is clopen there.

Exercise 9.3. Is $\mathbb{R}^n$ connected (in its natural topology)?

Exercise 9.4. Assume that it is possible to connect any two points x, y in M by a path, that is, to find a continuous mapping $[0,1] \xrightarrow{\varphi} M$ satisfying $\varphi(0) = x$, $\varphi(1) = y$. Show that M is connected.

Remark. Such an M is called **path-connected**.

Exercise 9.5. Remove a point from a circle or the plane. Show that the result is connected.

Exercise 9.6 (!).

a. Remove a finite number of points from $\mathbb{R}^2$. Show that the result is connected.

b. Remove a point from an interval. Show that the result is not connected.

Exercise 9.7 (!). Show that the following spaces are not homeomorphic to each other: $\mathbb{R}$, $\mathbb{R}^2$, the circle.

Exercise 9.8 (!). Show that the following spaces are not homeomorphic to each other: closed interval, half-open interval, open interval.

Exercise 9.9. Let $f : X \longrightarrow Y$ be continuous and X be connected. Show that $f(X)$ is connected.

Exercise 9.10 (!). Let $U \subseteq [0,1]$ be connected. Show that U is either a closed interval, or a half-open interval, or an open interval.

Exercise 9.11. Let $f : X \longrightarrow \mathbb{R}$ be continuous and X be connected. Assume that f takes positive as well as negative values. Show that $f(x) = 0$ for some $x \in X$.

Exercise 9.12 (*). Let M be a connected metrizable countable topological space. Show that M consists of one point.

Exercise 9.13. Show that the union of two connected subsets of a topological space M is connected, provided that their intersection is nonempty.

Exercise 9.14 (!). Let $x \in M$ and W be the union of all the connected subsets of M containing x. Show that W is connected.

Definition 9.2. In such a situation W is called **the connected component** of x (or just **a connected component**).

Exercise 9.15. Show that $W \subset M$ is a connected component if and only if any connected subset containing W coincides with W.

Exercise 9.16. Show that M is the disjoint union of its connected components.

Exercise 9.17. Show that each connected component of M is closed.

Totally disconnected spaces

Definition 9.3. A topological space M is called **totally disconnected** if each connected component of M consists of one point.

Exercise 9.18. Show that $\mathbb{Q}$, the space of rational numbers, in the topology induced by $\mathbb{R}$, is totally disconnected, but not discrete.

Exercise 9.19 (*). Show that $\mathbb{Q}_p$, the space of p-adic numbers, is totally disconnected.

Exercise 9.20 (*). Show that the product of totally disconnected spaces is totally disconnected.

Exercise 9.21. Let S be a subbase in a Hausdorff topological space M, and all the elements of S clopen. Show that M is totally disconnected.

Exercise 9.22 (!). Consider the set $\{0,1\}$ equipped with the discrete topology. Let $\{0,1\}^I$ be the product of I copies of $\{0,1\}$ with Tychonoff topology, with I being an arbitrary index set. Show that $\{0,1\}^I$ is totally disconnected.

Hint. Use the preceding problem.

Exercise 9.23 (*). Let M be Hausdorff topological space, M_1 be the sets of connected components of M, and $M \xrightarrow{\pi} M_1$ the natural projection (each point is mapped to its connected component). On M_1 introduce the following topology: $U \subset M_1$ open if $\pi^{-1}(U) \subset M$ is open. Show that M_1 is totally disconnected. Show that any continuous mapping $M \xrightarrow{\pi_2} M_2$ from M to a totally disconnected space M_2 can be written as a composition of continuous mappings $M \xrightarrow{\pi} M_1 \longrightarrow M_2$.

Hint. If $S \subset M_1$ is connected then the preimage $\pi^{-1}(S)$ is connected, too. Indeed, if $W \subset \pi^{-1}(S)$ is clopen then $W = \pi^{-1}(W_1)$ (if W intersects a connected component of M, then W contains it). Thus W_1 is clopen.

Exercise 9.24. Let U be an open subset of a compact Hausdorff space and a collection of closed subsets $\{K_i\}$, so that their intersection is conatined in U. Show that $\{K_i\}$ contains a finite subcollection so that their intersection is contained in U.

Exercise 9.25 (*). Let M be a totally disconnected compact Hausdorff space. Show that, for each point $x \in M$, the intersection of all the clopen subsets of M containing x is $\{x\}$.

Hint. Let P be the intersection of the clopen subsets containing x. Obviously P is closed. Show that P is either $\{x\}$ or disconnected. In the latter case P is the disjoint union of two nonempty closed subsets P_1, P_2. As T4 holds in M (show this), find for P_1, P_2 nonintersecting open neighbourhoods U_1, U_2. Derive from the preceding problem that $U_1 \cup U_2$ contains a clopen subset $W \subset M$ containing x. Show that $W \cap U_i$ are clopen, and derive from this that $P = \{x\}$.

Exercise 9.26 (*). Let M be a totally disconnected compact Hausdorff space. Show that the clopen subsets form a base of the topology of M.

Hint. Let $U \subset M$ be open and $x \in U$. For each point in $M \backslash U$ pick a clopen neighbourhood that does not contain x (show that this is always possible). This is a cover $\{U_\alpha\}$ of $M \backslash U$. As $M \backslash U$ is compact, $\{U_\alpha\}$ contains a finite subcover $U_1, ...U_n$. Show that the complement to $\cup U_i$ is clopen, contains x, and is contained in U.

Exercise 9.27 (*). Let M be a totally disconnected compact Hausdorff space. Let $x, y \in M$ be two distinct points. Show that M admits a continuous mapping to $\{0, 1\}$ (with discrete topology) such that x goes to 0 and y goes to 1.

Exercise 9.28 (*). Let M be a totally disconnected compact Hausdorff space. Let I be the set of all continuous mappings from M to $\{0, 1\}$. Define a natural mapping $M \longrightarrow \{0, 1\}^I$. Show that it is a continuous embedding, and that the image of M is closed.

Exercise 9.29 (*). Let M be a compact Hausdorff space. Show that the following statements are equivalent.

(i) M is totally disconnected.

(ii) M can be embedded into $\{0, 1\}^I$ for some set I of indices.

Remark. Recall that if a compact M admits a continuous injective mapping $f : M \to X$ into a Hausdorff space X then f is a homeomorphism between M and $f(M) \subset X$ with induced topology.