Meausre theory 1: Volumes of polytopes

1.1 Rings of subsets and finitely additive measures

Definition 1.1. Let S be a set. The set of all its subsets is denoted by 2^S. Let $\mathfrak{U} \subset 2^S$. $\mathfrak{U}$ is called *ring* if $A \cup B$, $A \cap B$, and $A \backslash B$ belong to $\mathfrak{U}$ for any $A, B \in \mathfrak{U}$. In this case $\mathfrak{U}$ is a called a *subring* in 2^S.

Problem 1.1. Let S be finite. Describe all the subrings in 2^S and count them when $|S| = 5$.

Definition 1.2. The *characteristic function* of $U \subset S$ is a function

$$\chi_U : S \longrightarrow \{0,1\} \mid \chi_U(x) = 1, \text{ if } x \in U \qquad\qquad \chi_U(x) = 0, \text{ if } x \notin U$$

Problem 1.2. Let $\mathfrak{U} \subset 2^S$ and $R_{\mathfrak{U}} = \{\chi_U\}$ be the set of characteristic functions for all $U \in \mathfrak{U}$. Consider $\{0,1\}$ as the 2-element field $\mathbb{Z}/2\mathbb{Z}$. This specifies a natural additive and multiplicative structure on the set of maps from S to $\{0,1\}$ (coodinate-wise addition and multiplication). Shown that $R_{\mathfrak{U}}$ forms a ring (maybe, without unity) iff $\mathfrak{U}$ is a ring.[1]

Definition 1.3. Let $\mathfrak{V} \subset 2^S$. The minimal subring in 2^S, containing $\mathfrak{V}$, is called *subring, generated by* $\mathfrak{V}$.

Problem 1.3 (*)**.** Let $\mathfrak{V} \subset 2^S$ have N elements. What is the maximal carinality of the subring generated by $\mathfrak{V}$?

Problem 1.4 (*)**.** Let $\mathfrak{U}_1 \subset 2^{S_1}$, $\mathfrak{U}_2 \subset 2^{S_2}$ be rings of subsets. Consider the ring $\mathfrak{U} \subset 2^{S_1 \times S_2}$ generated by all subsets of the form $U_1 \times U_2$, $U_1 \in \mathfrak{U}_1, U_2 \in \mathfrak{U}_2$,. Show that the corresponding rings $R_{\mathfrak{U}_1}$, $R_{\mathfrak{U}_2}$, $R_{\mathfrak{U}}$ relate to each other as follows.

$$R_{\mathfrak{U}_1} \otimes_{\mathbb{Z}/2\mathbb{Z}} R_{\mathfrak{U}_2} \cong R_{\mathfrak{U}}.$$

Definition 1.4. Let $S \subset \mathbb{R}^n$. The *convex closure* of S is the smallest convex set[2] containing S.

Problem 1.5 (!)**.** Show that the convex closure of S is the set of vectors of the form $\sum \alpha_i s_i$, where $\{s_i\}$ is a finite set of points from S, and α_i real numbers satisfying $0 \leqslant \alpha_i \leqslant 1$, $\sum \alpha_i = 1$.

Definition 1.5. *Closed simplex* in $\mathbb{R}^n$ is the convex closure of $\{x_0, \ldots, x_n\} \subset \mathbb{R}^n$. Such a simplex is called *spanned by* $x_0, \ldots, x_n$. *Simplex* is a convex set, whose closure is a closed simplex.

Problem 1.6. Describe all the simplices in $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$.

Definition 1.6. Let $\Delta(x_0, \ldots, x_n)$ be a simplex spanned by $\{x_0, \ldots, x_n\}$. The convex closure of $k+1$ points from $\{x_0, \ldots, x_n\}$ is called a *face* of Δ of dimension k.

Problem 1.7. The edges (i.e. 1-dimensional faces) of the n-dimensional simplex $\Delta(x_0, \ldots, x_n)$ form a graph. Suppose that the x_i's are pairwise different. How many edges does this graph have? Draw it. How many k-dimensional faces does $\Delta(x_0, \ldots, x_n)$ have?

Definition 1.7. Ring of polyhedra is the ring of subsets in $\mathbb{R}^n$ generated by closed simplices. An element of this ring is called polyhedron.

Problem 1.8. Show that each polyhedron can be represented as a disjoint union of finitely many simplices.

Problem 1.9 (*)**.** Show that each convex polyhedron can be represented as the intersection of finitely many simplices.

[1]You might also like to think how to describe the unity in $\mathfrak{U}$, assuming it exists. (DP)

[2]Recall that S is convex when $\{\alpha x + (1-\alpha)y \mid 0 \leqslant \alpha \leqslant 1\} \subseteq S$ for any $x, y \in S$. (DP)

Definition 1.8. Two polyhedra A, B are called *equi-decomposable* if they can be cut into simplices $A_1, \dots, A_k$, $B_1, \dots, B_k$ such that the closure of each A_i is congruent to the closure of B_i.

Problem 1.10. Show that to be equi-decomposable is an equivalence relation.

Problem 1.11. Show that any triangle A is equi-decomposable to a parallelogram of the same base and of height equal to the half of the height of A.

Problem 1.12. Show that each parallelogram is equi-decomposable to a rectangle with the same base and height.

Problem 1.13. (*) Show that $a \times b$- and $c \times d$-rectangles are equi-decomposable when $ab = cd$.

Definition 1.9. Let $\mathfrak{U} \subset 2^S$ be a ring of subsets. Map $\mu: \mathfrak{U} \longrightarrow \mathbb{R}$ is called *finitely additive measure*, or *additive function of a set*, or *valuation*, if for any $A, B \in \mathfrak{U}$,

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

A valuation is called *nonnegative* if takes only nonnegative values. Obviously valuations form a linear space over $\mathbb{R}$.

Problem 1.14. Let $S = [0, 1]$, and $\mathfrak{U}$ be the set of finite unions of segments and intervals. Show that $\mathfrak{U}$ is a ring. Show that the map $\coprod_i A_i \longrightarrow \sum |A_i|$ (disjoint union of segments mapped to the sum of its lengths) is a nonnegative valuation.

Problem 1.15 (!)**.** Let $\mathfrak{U} = 2^S$, where S is a finite set. Denote by L the linear space of all valuations on $\mathfrak{U}$. Find the dimension of L over $\mathbb{R}$.

Problem 1.16. Let $\mathbb{R} \xrightarrow{\xi} \mathbb{R}$ be a $\mathbb{Q}$-linear homomorphism, [3] S be a set and $\mathfrak{U} \subset 2^S$ be a ring of subsets. Show that for any valuation $\mu: \mathfrak{U} \longrightarrow \mathbb{R}$ the composition $\mu \circ \xi$ is a valuation.

Problem 1.17. Let a point $x \in S$, a ring of subsets $\mathfrak{U} \subset 2^S$ and a function $\mu: \mathfrak{U} \longrightarrow \mathbb{R}$, taking values $\mu(U) = 1$ for $x \in U$ and $\mu(U) = 0$ for $x \notin U$ be given. Show that μ is a valuation.

Remark. Recall that an *isometry* in $\mathbb{R}^n$ (or any other metric space) is any isometric bijection. Two sets are *congruent* if one is the image of the other under an isometry.

Definition 1.10. Let $\mathfrak{U} \subset 2^{\mathbb{R}^n}$ be a ring of sets. A valuation $\mu: \mathfrak{U} \longrightarrow \mathbb{R}$ is called *invariant* if $\mu(A) = \mu(B)$ for conguent $A, B \subset \mathbb{R}^n$.

Problem 1.18. Let $\mathfrak{U} \subset 2^{\mathbb{R}^n}$ be a ring of polyhedra, and let $\mu: \mathfrak{U} \longrightarrow \mathbb{R}$ be an invariant valuation.

a. *Degenerate simplex* is a simplex lying entirely within a hyperplane. Show that a simplex $\Delta(x_0, x_1, \dots, x_n)$ is degenerate iff the vectors $x_1 - x_0$, $x_2 - x_0$, $\dots$, $x_n - x_0$ are linearly dependent.

b. Show that $\mu(I) = 0$ for any degenerate simplex I.

c. Show that $\mu(A) = \mu(\overline{A})$ for any polyhedron A (here $\overline{A}$ denotes its closure).

d. Show that $\mu(A) = \mu(B)$ for equi-decomposed A and B.

Problem 1.19 (*)**.** Do the statements of the previous problem remain valid without the nonnegativity assumption on μ?

[3] $\mathbb{R}$ is considered here as a vector space over $\mathbb{Q}$.

1.2 Volume

Definition 1.11. Let $V = \mathbb{R}^n$. Consider the 1-dimensional vectorspace $\Lambda^n(V)$ of the antisymmetric forms of highest degree. This space is also called *the space of volume forms.*

Fix a nonzero vector $\nu \in \Lambda^n(V)$.

For a simplex $\Delta = \Delta(x_0, ...x_n) \subset V$ its *volume* Δ is the following nonnegative number:

$$\int_\Delta \nu := |\nu(x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0)|.$$

Problem 1.20. (!) Under these assumptions show that

a. $\int_\Delta \nu > 0$ iff Δ is nondegenerate.

b.
$$\int_{\Delta(x_0,....x_n)} \nu = \int_{\Delta(x_{\sigma 0},....x_{\sigma n})} \nu,$$
where $(x_0, ...x_n) \longrightarrow (x_{\sigma_0}, ...x_{\sigma_n})$ an arbitrary permutation.

c. If the simplices Δ and Δ' are congruent then $\int_\Delta \nu = \int_{\Delta'} \nu$

d. If a simplex Δ is a disjoint union of simplices: $\Delta = \Delta_1 \coprod \Delta_2$ then
$$\int_\Delta \nu = \int_{\Delta_1} \nu + \int_{\Delta_2} \nu.$$

Hint. All these properties follow from the well-known properties of the determinant.

Problem 1.21. Let a simplex Δ be a disjoint union of simplices: $\Delta = \coprod_i \Delta_i$. Show that

$$\int_\Delta \nu = \sum \int_{\Delta_i} \nu.$$

Hint. This follows from the properties of the volume listed in the Problem 1.20.

Problem 1.22 (*)**.** Show that the properties of the volume listed in the Problem 1.20. 1.20 uniquely, up to a constant positive multiplier, determine the mapping $\Delta \longrightarrow \int_\Delta \nu$.

Problem 1.23 (!)**.** Let a polyhedron $C \subset V$ be given together with its partition $C = \coprod A_i = \coprod B_i$ into disjoint simplices. Show that

$$\sum \int_{A_i} \nu = \sum \int_{B_i} \nu.$$

This number is called *the volume of the polyhedron* C and is denoted by $\int_C \nu$. Show that this function defines a nonnegative invariant valuation on the ring of polyhedra.

Problem 1.24 (!)**.** Let V be a Euclidean vectro space and C a unit cube (cube with edgelength 1) there.[4] Show that there exists unique invariant nonnegative valuation μ on the ring of polyherda satisfying $\mu(C) = 1$. Give an explicit formula for it.

Definition 1.12. This valuation is called *Euclidean volume of the polyhedron.*

[4]A cube can be defined, for instance, as follows. Fix an orthonormal basis $\{\xi_1, \ldots, \xi_n\}$ in V. Consider the set of linear combinations of the form $\sum \alpha_i \xi_i$, where $0 \leqslant \alpha_i \leqslant 1$. This set is called *unit cube* in V.

1.3 Hilbert's 3rd Problem

Hilbert's 3rd Problem is formulated as follows.

> In two letters to Gerling, Gauss expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, i. e., in modern phraseology, upon the axiom of continuity (or upon the axiom of Archimedes).
>
> Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now the analogous problem in the plane has been solved. Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts.
>
> Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility.
>
> This would be obtained, as soon as we succeeded in specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra.
>
> (quoted from V. G. Boltianskii. Hilbert's Third Problem, Winston, Halsted Press, Washington, New York, 1978.)

Simplifying a bit, this boils down to

Hilbert's 3rd Problem Construct two polyhedra of the same volume that are not equi-decomposable.

Problem 1.25 (*)**.** Let A be B planar polyhedra of the same volume (planar polyhedra are usually called *polygons*). Show that they are equi-decomposable.

Remark. This is known as *Bolyai-Gerwien Theorem.*

Remark. Suppose that there exists a valuation $\mu : \mathfrak{U} \longrightarrow \mathbb{R}$ on the ring of polyhedra satisfying $\mu(A) \neq \mu(B)$, whereas A and B have the same volume. Then A and B are not equi-decomposable.

Problem 1.26. (!) Derive the following from Bolyai-Gerwien Theorem. Let $\mu : \mathfrak{U} \longrightarrow \mathbb{R}$ be a valuation, with $\mathfrak{U}$ the ring of polygons (polyhedra in $\mathbb{R}^2$). Show that $\mu = \operatorname{Vol} \circ \xi$, where $\operatorname{Vol} : \mathfrak{U} \longrightarrow \mathbb{R}$ is a valuation given by the volume, and $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ - $\mathbb{Q}$-linear homomorphism of abelian groups.

Problem 1.27 (*)**.** Construct a nontrivial (not $\mathbb{R}$-linear) $\mathbb{Q}$-linear homomorphism $\xi : \mathbb{R} \longrightarrow \mathbb{R}$. Use the Axiom of Choice.

Problem 1.28. Show that such a ξ always maps some positive numbers to negative.

Definition 1.13. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\mathbb{Q}$-linear homomorphism mapping π to 0, and C - polyhedron $\mathbb{R}^3$, with edges of lengths $d_1, \dots, d_n$ and the angles $\alpha_1, \dots, \alpha_n$ between the corresponding facets (given in radians). Dehn invariant $D_\varphi(C)$ is given by

$$D_\varphi(C) := \sum_{i=1}^{n} d_i \varphi(\alpha_i).$$

Problem 1.29. (!) Show that the space of $\mathbb{Q}$-linear homomorphisms mapping π to 0 can be identified with the tensor product $(\mathbb{R}/\mathbb{Q}\pi)^* \otimes_{\mathbb{Q}} \mathbb{R}$.

Definition 1.14. This set affords a vector space structure over $\mathbb{R}$:

$$\lambda(\varphi)(c) = \lambda\varphi(c).$$

It is called the *space of Dehn invariants.*

Problem 1.30. (*) Show that space of Dehn invariants is ininite-dimensional over $\mathbb{R}$. Show that for any $\lambda \in R$ there exists a homomorphism $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\varphi(\lambda) \neq 0$ whenever λ/π is irrational. Use the Axiom of Choice.

Problem 1.31. (!) Let a simplex Δ be represented as a disjoint union of simplices $\Delta = \coprod_i \Delta_i$. Show that

$$D_\varphi(\Delta) = \sum_i D_\varphi(\Delta_i)$$

Problem 1.32 (*). Derive from this that Dehn invariant D_φ is a valuation on the space of polyhedra in $\mathbb{R}^3$.

Problem 1.33. Consider a regular tetrahedron. Show that its dihedral angles equal $\arccos(1/3)$.

Problem 1.34. Let $\cos(\pi\alpha) = 1/n$ and $\alpha \in \mathbb{Q}$. Derive that

$$e^{\sqrt{-1}\,\pi k\alpha} = \left(\frac{1}{n} + \sqrt{-1}\,\frac{\sqrt{n^2-1}}{n}\right)^k = 1$$

for an integer $k > 0$.

Problem 1.35 (*). Let $n = 3$, and $\left(\frac{1}{n} + \sqrt{-1}\,\frac{\sqrt{n^2-1}}{n}\right)^k = 1$. Show that $k = 0$.

Hint. Show that the ring $\mathbb{Z}[\sqrt{-2}]$ is UFD (i.e. factorisation there is unique) and utilise this.

Problem 1.36 (*). Denote by α the dihedral angle of a regular tetrahedron. Show that $\frac{\alpha}{\pi} \notin \mathbb{Q}$.

Problem 1.37 (*). Find an element φ of the space of Dehn invariants $(\mathbb{R}/\mathbb{Q}\pi)^* \otimes_{\mathbb{Q}} \mathbb{R}$ such that $\varphi(\alpha) \neq 0$, where α is the dihedral angle of a regular tetrahedron.

Problem 1.38 (*). In conditions of the previous problem, show that $D_\varphi(\Delta) \neq 0$, where Δ a regular tetrahedron.

Problem 1.39. Show that $D_\varphi(C) = 0$ for any parallelepiped.

Problem 1.40 (*). Show that a regular tetrahedron and a regular cube with the same volume are not equi-decomposable.

Problem 1.41 (**). (Dehn-Sydler Theorem) Let polyhedra A and B in $\mathbb{R}^3$ have the same volume, and $D_\varphi(A) = D_\varphi(B)$ for any Dehn invariant φ. Show that A and B are equi-decomposable.