Lebesgue measure

2.1 Boolean algebras

Definition 2.1. **Lattice** is a set L endowed with algebraic binary operations $\wedge$ and $\vee : L \times L \longrightarrow L$, called respectively **meet** and **join**, satisfying the following:

a. idempotent laws: $a \wedge a = a \vee a = a$.

b. commutativity: $a \wedge b = b \wedge a, a \vee b = b \vee a$.

c. associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$,

d. absorbtion laws: $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$.

Problem 2.1. Let $(S, \preceq)$ be a poset such that for any $x, y \in S$ there exists **supremum** $t \in S$, denoted $t := x \vee y$ (i.e. such an element $t \succeq x, y$ that any z satisfying $z \succeq x, y$ satisfies $z \succeq t$), and **infimum** $u \in S$, denoted $u := x \wedge y$, (i.e. such $t \preceq x, y$ that any z satisfying $z \preceq x, y$ satisfies $z \preceq u$). Show that it is a lattice.

Problem 2.2 (!)**.** Let L be a lattice. Introduce on L a binary relation $x \preceq y$, that holds true whenever $x \wedge y = x$.

a. Show that $x \preceq y$ iff $x \vee y = y$.

b. Show that $x \preceq y$ is a partial order relation.

c. Consider $(L, \preceq)$ as a poset. Show that it has supremum and infimum. Show that they can be expressed as $(x \vee y)$, resp. $(x \wedge y)$.

d. Show that any lattice can be obtained from a poset using the construction described in Problem 2.1.

Problem 2.3. Let R be a factorial ring. Construct a lattice from it, using operations of taking the least common multiple and of taking the greatest common divisor.

Problem 2.4. Consider the following partial order relation on 2^S: $x \preceq y$, if $x \subset y$. Show that $(2^S, \preceq)$ have supremum and infimum. Show that they correspond to intersections and unions of sets.

Definition 2.2. Boolean algebra is a construction to axiomatise the operations of intersection and union in the algebra of subsets. Boolean algebras are named after an English mathematician George Bool, 1815-1864.

Boolean algebra $(A, \vee, \wedge)$ is a lattice satisfying the following conditions:

a. Boundedness from below: A contains an element 0 such that $x \wedge 0 = 0$.

b. Boundedness from above: A contains an element 1 such that $x \vee 1 = 1$.

c. Distributivity: $(a \vee b) \wedge c = (a \wedge c) \vee (a \wedge c)$

d. Existence of complements: for any $x \in A$ there exists $\neg x$ such that $x \wedge \neg x = 0$, $x \vee \neg x = 1$.

Problem 2.5. Show that 0, 1, $\neg x$ are uniquely determined by the lattice structure on A.

Problem 2.6. Show that $\neg 0 = 1$, $\neg 1 = 0$.

Problem 2.7. Prove de Morgan laws: $\neg(a \vee b) = (\neg a) \wedge (\neg b)$, $\neg(a \wedge b) = (\neg a) \vee (\neg b)$.

Problem 2.8. (Boolean algebra duality) Let $(A, \vee, \wedge)$ be a Boolean algebra. Consider operations $\vee_1 := \wedge$, $\wedge_1 := \vee$. Show that $(A, \wedge_1, \vee_1)$ is a Boolean algebra, too.

Problem 2.9. Construct a two-element Boolean algebra.

Problem 2.10. Let R be a commutative ring, and V its set of idempotents (i.e. $a \in R$ satisfying $a^2 = a$). Consider operations $e \vee f = e + f - ef$, $e \wedge f = ef$. Show that they give a Boolean algebra structure on V.

Definition 2.3. Symmetric difference in a Boolean algebra is defined by setting $a \triangle b := (a \vee b) \wedge \neg(\wedge b)$

Problem 2.11 (!)**.** a. Show that the symmetric difference is associative.

b. Show that $\wedge$ is distributive w.r.t. $\triangle$.

c. SHow that $(A, \wedge, \triangle)$ is a ring, where addition is given by $\triangle$ and multiplication by $\wedge$.

d. Show that all the elements of this ring are idempotent.

Problem 2.12 (!)**.** Let R be a commutative ring over $\mathbb{Z}/2\mathbb{Z}$, with all its elements idempotent. Consider the structure of Boolean algebra on R, defined in Problem 2.10. Show that R can be obtained from this Boolean algebra by the construction described above.

Definition 2.4. Ideal in a Boolean algebra is a subset $I \subset A$, closed w.r.t. $\vee$, that satisfies $a \wedge i \in I$ for any $a \in A, i \in I$.

Problem 2.13. Show that a Boolean algebra with more than 2 elements contains a nontrivial ideal.

Problem 2.14 (!)**.** Let $(A, \wedge, \vee)$ be Boolean algebra with an ideal $I \subset A$. On A, define the relation $a \sim_I b := a \triangle b \in I$. Show that $\sim_I$ is an equivalence relation. Show that $\wedge$ and $\vee$ preserve the corresponding equivalence classes, and induce on the set A' of the classes the structure of a Boolean algebra.

Definition 2.5. Under these operations, A' is called a **quotient algebra (modulo I)**, and is denoted by A/I. The ideal I is called **maximal** when A/I consists of 2 elements.

Problem 2.15 (*)**.** Show that any nontrivial ideal of a Boolean algebra is contained in a maximal ideal.

Definition 2.6. Representation, or **injective representation** of a Boolean algebra A is an injective homomorphism $A \longrightarrow 2^S$, defined for a set S. In orther words, a representation of A is its realisation as a (sub)algebra of sets.

Problem 2.16 (*)**.** a. Show that each Boolean algebra admits an injective representation.

b. Let A be a finite Boolean algebra. Show that A cosists of 2^n elements. Show that A is isomorphic to the algebra of all subsets of $S = \{1, \dots, n\}$.

2.2 External measure

From now on S will denote a set, and $\mathfrak{U} \subset 2^S$ will denote a ring of subsets that contains S. (Such a ring is called **algebra of subsets**, or **subalgebra of subsets in** 2^S). Consider 2^S as a Boolean algebra, with operations $\vee = \cup$ and $\wedge = \cap$. Obviously $\mathfrak{U}$ is a Boolean subalgebra of 2^S.

Consider a function $\mu: \mathfrak{U} \longrightarrow \mathbb{R} \cup \{\infty\}$. On the set $\mathbb{R} \cup \{\infty\}$ an addition operation is defined, so that $x + \infty = \infty$ and $\infty + \infty = \infty$.

Definition 2.7. Function $\mu: \mathfrak{U} \longrightarrow \mathbb{R} \cup \{\infty\}$ is called a **finitely additive measure**, if for any nonintersecting $A, B \in \mathfrak{U}$ one has $\mu(A \coprod B) = \mu(A) + \mu(B)$. A measure is called **nonnegative**, if $\mu(A) \geqslant 0$ holds for any A.

Definition 2.8. Under these assumptions, let $X \subset S$. Define **external measure** $\mu^*(X)$ as

$$\mu^*(X) := \inf_{\{A_i\}} \sum \mu(A_i),$$

where inf is taken over all countable collections $\{A_i\} \subset \mathfrak{U}$ covering X. We say that X is of **measure** 0 if $\mu^*(X) = 0$. Wer say that μ is σ-additive if $\mu^*(A) = \mu(A)$ for any $A \in \mathfrak{U}$.

Problem 2.17. Show that $\mu^*(A \cup B) \leqslant \mu^*(A) + \mu^*(B)$.

Problem 2.18 (*)**.** Give an example of a nonadditive external measure (i.e. of a measure μ for which $\mu^*(A \coprod B) = \mu^*(A) + \mu^*(B)$ fails).

Problem 2.19 (!)**.** Let A have measure 0. Show that $\mu^*(A \cup B) = \mu^*(B \backslash A) = \mu^*(B)$.

Problem 2.20 (!)**.** Show that a countable union of measure 0 sets has measure 0.

Problem 2.21. Show that the measure 0 sets make up a a Boolean ideal in the Boolean algebra 2^S.

Problem 2.22 (*)**.** Let $\mathfrak{U} \subset 2^S$ be the algebra generated by intervals on a closed segment or the line, with the usual measure. Give an example of uncountable measure 0 subset.

Problem 2.23. Consider a smooth diffeomorphism φ of closed segments (smooth on the endpoints, too). Show that φ maps sets of measure 0 to sets of measure 0.

Problem 2.24. Let φ be a diffeomorphism from an interval to the line. Show that φ maps sets of measure 0 to sets of measure 0.

2.3 Measurable sets

Definition 2.9. Consider the measure 0 subsets as a Boolean ideal in the Boolean algebra 2^S. If $\mu^*(A \triangle B) = 0$ for $A, B \subset S$ then we say that A **and** B **coincide almost everywhere**.

The quotient algebra modulo the ideal of measure 0 subsets is called **the algebra of subsets of** S **modulo the measure 0 subsets**. In the remainder of this set of problems we denote this algebra by $2^S/\sim$.

Problem 2.25. Fix $x \in S$. Assume that $\{x\} \in \mathfrak{U}$. Let the measure of a subset $X \subset S$ is given by the following rule: $\mu(X) = 1$ if $x \in X$, and $\mu(X) = 0$ otherwise. Find $\mathfrak{U}/\sim$.

Problem 2.26 (!)**.** Define a function $d: \ 2^S \times 2^S \longrightarrow \mathbb{R}$ by $d(A, B) := \mu^*(A \triangle B)$. Show that d satisfies the triangle inequality: $d(A, B) \leqslant d(A, C) + d(B, C)$.

Problem 2.27 (!)**.** Let $\mu^*(A_1 \triangle A_2) = 0$. Show that $\mu^*(A_1 \triangle B) = \mu^*(A_2 \triangle B)$, for any $B \in 2^S$.

Remark. The latter problem implies that $d(A, B) = \mu^*(A \triangle B)$ is well-defined on $2^S/\sim$.

Problem 2.28. Show that the function $d(A, B) = \mu^*(A \triangle B)$ defines a metric on $2^S/\sim$.

Problem 2.29 (!)**.** Consider the completion of $2^S/\sim$ with respect to this metric. Show that it is a Boolean algebra.

Definition 2.10. Let $\{X_i\}$ be a sequence of subsets of S. **The inverse limit** of $\{X_i\}$ is the set

$$\lim_{\leftarrow} \{X_i\} := \bigcup_i \left(\bigcap_{j>i} X_j \right).$$

Problem 2.30. Show that a reordering of $\{X_i\}$ does not change the inverse limit.

Problem 2.31. Let $A \in 2^S$ and $\{X_i\} \subset 2^S$, whereas $d(A, X_i) = \lambda_i$. Show that

$$d(A, \underleftarrow{\lim}\{X_i\}) \leqslant \sum \lambda_i.$$

Problem 2.32 (!)**.** Let $\{X_i\}$ be a Cauchy sequence in $2^S/\sim$. Show that it converges to $\underleftarrow{\lim}\{X_i\}$.

Hint. Replacing $\{X_i\}$ by a subsequence, ensure that

$$d(X_i, X_j) < 2^{-\min(i,j)}.$$

Using the previous problem, check that

$$d(X_i, \underleftarrow{\lim}\{X_i\}) \leqslant \frac{1}{2^{i-1}}.$$

Definition 2.11. The set $X \subset S$ is called **measurable** if it lies in the completion of $\mathfrak{U}/\sim$ with respect to the metric defined above.

Problem 2.33 (!)**.** Show that the measureable sets for a subalgebra in 2^S.

Problem 2.34 (**)**.** Using the Axiom of Choice, give an example of unmeasurable subset of $[0, 1]$ (with the standard measure).

Problem 2.35 (!)**.** (Lebesgue Theorem) Show the finite additivity of the function μ^* on the measurable sets (i.e. $\mu^*(A \coprod B) = \mu^*(A) + \mu^*(B)$ holds).

Hint. Use the fact that the aglebra of measurable sets is the completion of $\mathfrak{U}/(\sim \cap \mathfrak{U})$, and there μ is additive.

Definition 2.12. Let μ be σ-additive. In this case the function μ^* on the algebra of measurable sets is called **continuation** of μ. We denote it by μ, as well.

Problem 2.36 (!)**.** Let $\{A_i\} \subset \mathfrak{U}$ be a countable sequence of nonintersecting sets, such that the series $\sum \mu(A_i)$ converges. Show that $\bigcup A_i$ is measurable.

Problem 2.37 (!)**.** Show that on the measurable sets the function μ^* is **countably additive**, i.e. satisfies $\mu(\coprod X_i) = \sum \mu(X_i)$

2.4 Lebesgue measure

Definition 2.13. Let $\mathfrak{W} \subset 2^S$ be an algebra of subsets. $\mathfrak{W}$ is called **σ-algebra** if it is closed w.r.t. taking countable unions: $\bigcup X_i$ belongs to $\mathfrak{W}$ for any countable collection of subsets $\{X_i\} \subset \mathfrak{W}$.

Problem 2.38 (!)**.** Let $\mathfrak{U} \subset 2^S$ be an algebra of subsets equipped with a countably additive and nonnegative measure $\mu : \mathfrak{U} \longrightarrow \mathbb{R} \cup \{\infty\}$. Assume that $\mu(S) < \infty$. Show that its algebra of measurable subsets is a σ-algebra.

Definition 2.14. **Measure** on a σ-algebra $\mathfrak{W} \subset S$ is a countably additive, nonnegative function $\mathfrak{W} \longrightarrow \mathbb{R} \cup \{\infty\}$.

Problem 2.39 (*)**.** Give an example of a finitely additive, but not countably additive, measure. Show that a measure is countably additive iff $\mu^*(A) = \mu(A)$ for any $A \in \mathfrak{W}$.[1]

[1] I.e. countable additivity and σ-additivity are in fact the same thing.

Problem 2.40 (!)**.** Let S_1, S_2 be sets equipped with algebras

$$\mathfrak{U}_i \subset 2^{S_i}, \quad i = 1, 2$$

and finitely additive nonnegative measures

$$\mu_i : \mathfrak{U}_i \longrightarrow \mathbb{R} \cup \{\infty\}.$$

Consider the subalgebra $\mathfrak{U}_1 \times \mathfrak{U}_2$ in $2^{S_1 \times S_2}$, generated by the subsets of the form $A_1 \times A_2$, where $A_i \in \mathfrak{U}_i$. On each such subset define

$$\mu(A_1 \times A_2) := \mu_1(A_1)\mu_2(A_2).$$

a. Show that μ can be extended to a finitely additive nonnegative measure on the ring $\mathfrak{U}_1 \times \mathfrak{U}_2$.

** Show that this extension is σ-addtitive whenever each μ_i is σ-additive.

Definition 2.15. Consider a subalgebra $\mathfrak{U} \subset 2^{\mathbb{R}^n}$ generated by open sets of the form $I_1 \times I_2 \times \ldots I_n$, where I_k - segments, intervals, or half-open intervals. We will call such an algebra of sets **an algebra generated by parallelepipeds**. Extend the function

$$\mu(I_1 \times I_2 \times ...I_n) \longrightarrow \prod |I_k|$$

to a finitely additive measure μ on $\mathfrak{U}$. Let $\mathfrak{M}$ denote the completion of $\mathfrak{U}$ w.r.t. $d(A, B) := \mu^*(A \triangle B)$, i.e. the set of the measurable subsets corresponding to $\mathfrak{U}$ and μ. The elements of $\mathfrak{M}$ are called **Lebesgue measurable** and the extension of μ^* to $\mathfrak{M}$ is called the **Lebesgue measure**.

Problem 2.41 (!)**.** a. Show that the Lebesgue measure is σ-additive on an algebra generated by parallelepipeds.

b. Show that each open subset of $\mathbb{R}^n$ is measurable.

Definition 2.16. Let M be a topological space. The elements of the σ-algebra generated by its open sets are called **Borel subsets** of M.

Problem 2.42 (!)**.** Show that the Borel subsets of $\mathbb{R}^n$ are Lebesgue measurable.

Problem 2.43 (!)**.** Show that for each measurable $A \subset \mathbb{R}^n$ there exists a Borel subset $B \subset \mathbb{R}^n$ satisfying $\mu(B \triangle A) = 0$.

Problem 2.44 (*)**.** Let $V \subset B \subset \mathbb{R}^n$ be a subset of an open ball B. Show that V is measurable iff $\mu^*(V) + \mu^*(B \backslash V) = \mu(B)$