Non-algebraic Zariski geometries

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Abstract

The thesis deals with definability of certain Zariski geometries introduced by Zilber [37, 40] in the theory of algebraically closed fields. We axiomatise a class of structures, abstract linear spaces, that are a common reduct of these Zariski geometries, and describe what an interpretation of an abstract linear space in an algebraically closed field looks like. We give a new proof that the structure introduced in [40] is not interpretable in an algebraically closed field. We prove that a similar structure from an unpublished note [34] is not definable in an algebraically closed field and explain the non-definability of both structures in terms of geometric interpretation of the group law on a Galois cohomology group $H^1(k(x), \mu_n)$. We further consider quantum Zariski geometries introduced in Zilber [37] and give necessary and sufficient conditions that a quantum Zariski geometry is definable in an algebraically closed field.

Finally, we attempt to extend the results described above to a complex-analytic setting. We define what it means for quantum Zariski geometry to have a complex analytic model, an give a necessary and sufficient conditions for a smooth quantum Zariski geometry to have one. We then prove a theorem giving a partial description of an interpretation of an abstract linear space in the structure of compact complex spaces $\mathcal{A}$ (as defined e.g. in [24]) and discuss the difficulties that present themselves when one tries to understand interpretations of abstract linear spaces and quantum Zariski geometries in the structure $\mathcal{A}$.

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Chapter 1

Introduction

1.1 Overview

The study of strongly minimal sets is one of the fundamental topics of the geometric model theory. A strongly minimal definable set is a set of Morley rank and degree one. Understanding such sets is crucial for understanding an ω-stable theory: for example, in a uncountably categorical theory T the properties of any model M of the theory are tightly controlled by the properties of a strongly minimal set $\varphi(M)$ over which M is prime and minimal. This fact has strong model-theoretic implications, such as Morley's theorem and Baldwin-Lachlan theorem.

Therefore to understand uncountably categorical structures one has to understand strongly minimal sets in these structures. The notion of a *pregeometry* serves as an important tool for the study of such sets. Let φ be a formula. Then the model-theoretic algebraic closure, acl, is an an operator on subsets of $\varphi(M)$ such that:

- for any $A \subset \varphi(M)$, $A \subset \mathrm{acl}(A)$;

- for any $A, B \subset \varphi(M)$, $A \subset B$ implies $\mathrm{cl}(A) \subset \mathrm{cl}(B)$;

- for any $A \subset \varphi(M)$, $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$;

- if $A \subset X$ and $a \in \mathrm{cl}(A)$ then there is a finite $A_0 \subset A$ such that $a \in \mathrm{cl}(A_0)$.

If φ is strongly minimal, then additionally the so-called *exchange property* holds. Alltogether, these properties make up the axioms of a pregeometry.

The axioms of a pregeometry are satisfied by the field-theoretic algebraic closure and the span of a vector space over a division ring. A pregeometry can be seen as an abstract axiomatisation of a notion of dependence (such as algebraic or linear dependence). In particular, one can talk about a notions of a basis of a set in a pregeometry, about independence and dimension: a set A is called independent if $a \notin \mathrm{cl}(A \setminus \{a\})$ for all $a \in A$, a basis of a set is a maximal independent subset (using exchange property one can show that all such sets have the same cardinality), and the dimension of a set is the cardinality of its basis.

Another important example of a pregeometry is an affine space together with the affine span operation. Let A be an affine space over a vector space V. Pick a point $a \in A$ and define a closure operation $cl(X) = \mathrm{span}(X \cup \{a\})$. This new operation also satisfies the axioms of a pregeometry and is the pregeometry of a vector space corresponding to A, with the origin placed at a. The new pregeometry is called a *localisation* of (A, span) at a. More generally, a localisation of any pregeometry (M, cl) with respect to a subset $D \subset M$ is the pregeometry (M, cl_D) where $\mathrm{cl}_D(X) = \mathrm{cl}(X \cup D)$.

A pregeometry is called a *geometry* if $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(\{a\}) = \{a\}$. There is a canonical way to associate a geometry to any pregeometry (M, cl). Put $N = (M \setminus \mathrm{cl}(\emptyset))/\sim$ where the equivalence relation is defined as follows: $a \sim b$ if and only if $\mathrm{cl}(a) = \mathrm{cl}(b)$. Then $\mathrm{cl}'(A/\sim) = \{b/\sim \mid b \in cl(A)\}$ defines a closure operation on N and (N, cl) is a geometry. For example, taking (M, cl) to be a vector space one gets the geometry of the associated projective space.

A strongly minimal set $\varphi(M)$ of a model M can itself be regarded as a structure: take $\varphi(M)$ as the universe and declare traces of definable sets on $\varphi(M)$ definable. The structure thus obtained is strongly minimal and uncountably categorical. It has been a suggestion of Zilber that with the help of pregeometries it should be possible to classify strongly minimal structures up to bi-interpretability.

This classification is defined in terms of properties of pregeometries.

Let (M, cl) be a pregeometry. It is called *trivial* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$. It is called *modular* if for any finite dimensional closed $A, B \subset X$

$$\dim(A \cup B) = \dim A + \dim B - \dim(A \cap B)$$

It is called *locally modular* if a localisation at some $D \subset M$ is modular.

A *pseudo-plane* is a structure which consists of two sorts, P and L ("points" and "lines") and an incidence relation $I \subseteq P \times L$, subject to some geometric axioms: every point is incident to infinitely many lines, two lines have only finitely many points that are incident to both of them plus the dual axioms where lines and points exchange roles.

Uncountably categorical sets can be roughly classified according to the following theorem.

Theorem (Weak Trichotomy, Zil'ber [41], Theorem 3.1)**.** *Let M be an uncountably categorical structure. Then exactly one of the following holds:*

- *the pregeometry of every strongly minimal set in M is trivial;*
- *the pregeometry of every strongly minimal set in M is locally projective;*
- *a pseudo-plane is definable in every strongly minmal set in M.*

This trichotomy has structural consequences for a strongly minimal structure:

Theorem (Zil'ber [41], Theorem 6.1)**.** *If M is an uncountably categorical structure that has a strongly minimal set with a trivial pregeometry then no infinite group is definable in M.*

Theorem (Pillay [22], Chapter 5, Theorem 1.1)**.** *If M is an uncountably categorical structure that has a strongly minimal set with a locally modular non-trivial pregeometry then an infinite group is definable in M.*

Theorem (Zil'ber [41], Theorem 5.1)**.** *Any group definable in an uncountably categorical structure that has a locally modular strongly minimal set is abelian-by-finite.*

The prototypical examples of a trivial, locally modular and non-locally modular strongly minimal structures are respectively the trivial structure, a vector space, and an algebraically closed field. The Trichotomy conjecture stated in [41] suggested that a strongly minimal set that has a non-locally modular pregeometry has an algebraically closed field definable in it. Later Hrushovski [15] disproved this conjecture by constructing a strongly minimal structure with a non-locally modular pregeometry that does not interpret even a group.

In a consequent paper [16], Hrushovski and Zilber prove that the Trichotomy conjecture is true for a class of strongly minimal structures that they call Zariski geometries. A Zariski geometry on a set M is defined by giving a family of Noetherian topologies on Cartesian powers of M. The language contains a predicate for each closed subset. The axioms of a Zariski geometry include some conditions on how the dimension of definable sets should behave.

The main conclusion of the paper is that an algebraically closed field F is interpretable in every non-locally modular Zariski geometry M. Moreover, there is a curve C defined in F and a definable continuous map $f : M \to C$ such that the fibres of f are finite. Under an additional assumption (M is "very ample") the map f is a bijection, so this gives a characterisation of those Zariski geometries that are obtained by considering Zariski topologies on Cartesian powers of algebraic varieties.

Interestingly, not all Zariski geometries are of this form, or are reducts of structures of this form. An example exhibited already in [16] shows that there exist Zariski geometries that are not interpretable in an algebraically closed field. In the series of papers [37, 40, 38] such structures have been investigated.

The axiomatics of Zariski geometries presented in [16] is tailored towards strongly minimal, i.e. dimension 1, case. A more general axiomatics of which this one is a particular case has been circulating in Zilber's webprint and finally appeared in the book [39]. Zilber's approach is that a Zariski geometry is an axiomatisation where it is possible to do geometry. In the new axiom system the dimension is not necessarily the Krull dimension as in the paper [16], but a function defined on constructible sets. The essence of the axiomatics is the properties of this dimension function. Many geometric objects are Zariski geometries: algebraic varieties over an algebraically closed field, compact complex spaces and rigid analytic spaces.

Zilber [37] suggested that there is a connection between the field of non-commutative geometry and certain classes of non-algebraic Zariski geometry. Methodologically, in algebraic and differential geometry one associates to a space a commutative object — the ring of functions or the structure sheaf — and this object encapsulates all the relevant information about the geometry. In non-commutative geometry (as developed e.g. in the works of Connes) one considers a non-commutative object, for example, a non-commutative algebra, and treats it as a ring of functions on a non-existent "non-commutative space".

In [37] Zilber considers a class of associative algebras with large centres ("quantum algebras at roots of unity") and to each such algebra A associates a Zariski geometry: the structure is essentially a collection of A-modules fibred over a variety, the spectrum of the centre of A, where A is the input algebra. He gives examples of some quantum algebras and associated Zariski geometries, and shows that one of them (associated to the "quantum torus" algebra) is not definable in an algebraically closed field. This fact somehow supports the view that this Zariski geometry is not an object of "classical" algebraic geometry.

In [38] and [40] the program is pursued: the structures presented there carry representations of non-commutative algebras in some form, and non-definability proofs are given. The approaches used in non-definability proofs in the three mentioned papers ([37, 38, 40]) are completely different. In the paper [37] the proof uses a computation in the field of Puiseux series, in [38] the proof is based on the classification of algebraic curves and their groups of automorphisms, in [40] the proof uses some Kummer theory (but in fact contains a gap).

The motivation for the work presented in this thesis stems from the desire to develop a unified approach to (non-)definability proofs. The thesis deals with definability of some of the Zariski geometries discussed above in algebraically closed fields, and makes some steps towards understanding definability in the structure $\mathcal{A}$ of compact complex spaces (as defined in Pillay [24], for example). I treat the structures presented in the two papers that have been mentioned — the quantum Zariski geometries of [37] and the structure Q_n of [40], the "quantum harmonic oscillator" structure. For the latter I give a new proof of non-definability in an algebraically closed field, and for quantum Zariski geometries I develop a general definability criterion which subsumes in particular the proof of non-definability of "quantum torus" structure.

These structures have one thing in common — they consist of a sort with a projection to a variety defined in a separate field sort. The main sort is a disjoint union of vector spaces, such that each vector space is a fibre of the projection. More concretely

- let X be the main sort;
- let S be a variety over an algebraically closed field k
- let $p : X \to k$ be a map with the image S;
- let $a : X \times X \to X$ which is defined only for x, y such that $p(x) = p(y)$ and $p(a(x, y)) = p(x) = p(y)$;
- let $m : S \times X \to X$ be a map such that $p(m(a, x)) = p(x)$
- for any $s \in S$ the fibre $X_s = p^{-1}(s)$ with operations a, m restricted to $X_s \times X_s$ and $k \times X_s$ respectively is a k-vector space where a is addition and m is multiplication by a scalar.

I call these structures *abstract linear spaces.* In Chapter 3 I prove that an interpretation of an abstract linear space in an algebraically closed field is a trivial vector bundle if one restricts to some constructible dense subset of the base variety. Having established that one has a better idea of how a piece of extra structure on top of an abstract linear space (e.g. a map $X \to X$ preserving fibres and acting linearly on them) can be interpreted.

The structure Q_n is an abstract linear space with fibres of dimension 1 over the affine line, with the following extra structure:

- a predicate B of "base vectors" such that in each fibre the elements of B are conjugated by multiplication by n-th roots of unity, n even;
- two maps $\mathbf{a}, \mathbf{a}^\dagger$ that linearly map a fibre over s to a fibre over $s + 1$ ($s - 1$ respectively) for any $s \in S$.

The structure is defined in such a way that $\mathbf{a}$ and $\mathbf{a}^\dagger$ considered as linear operators on the direct sum of the fibres of the abstract linear space define a representation of the Weyl algebra, the algebra on two generators $\mathbf{a}, \mathbf{a}^\dagger$ with the relation $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. The predicate B over the generic point of S is a principal homogeneous space for the group μ_n of n-th roots of unity. The proof of non-definability of Q_n in ACF_0 is mainly based on figuring out how the predicate B can be interpreted: all the possible ways correspond to cohomology classes of the Galois cohomology group $H^1(k(x), \mu_n)$ which by Kummer theory is just $k(x)^\times/(k(x)^\times)^n$. From the way the map $\mathbf{a}$ is defined a contradiction is derived. This contradiction can be interpreted in terms of a geometric construction with principal homogeneous spaces that corresponds to the group operation on $H^1(k(x), \mu_n)$. The same proof strategy works for a structure defined by Vinesh Solanki in an unpublished note [34], a structure that carries a representation of the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$.

A quantum Zariski geometry is a structure that is associated to some input data, which most importantly includes an associative algebra A that is finite over its centre. The structure is a linear space over a variety X (the coordinate ring of X is the centre of A) such that each fibre is an A-module.

I associate to each such structure a certain algebra, an *Azumaya algebra*. I then prove that the definability of the structure in an algebraically closed field depends on a certain property of this algebra (being split). In fact, one can associate to an Azumaya algebra a quantum Zariski geometry, but the correspondence is not bijective.

Azumaya algebras are a generalisation of central simple algebras. An algebra A over a field k is called central simple if for some field extension k'/k, $A \otimes k' \cong M_n(k')$, where $M_n(k')$ is the algebra of matrices over k'. Azumaya algebra is a relative analogue of this notion for algebras over a commutative ring (and more generally for sheaves of algebras over a scheme, complex analytic space or just a topological space) with tensor product replaced by base change. Just as central simple algebras are described by cohomology classes in $H^1(k, \mathrm{PGL}_n)$, Azumaya algebras over a variety are described by cohomology classes of Čech cohomology $\check{H}^1(X_{\text{ét}}, \mathrm{PGL}_n)$ in étale topology. This correspondence is established using Grothendieck's theory of descent.

The correspondence between quantum Zariski geometries and Azumaya algebras is based, roughly, on the observation that the data describing a quantum Zariski geometry can be used for descent which in turn yields an Azumaya algebra.

An Azumaya algebra over a ring R is called split if it is isomorphic to the algebra of endomorphisms of a locally free module over R. The Azumaya algebras A, B over a ring R are called equivalent if there exist locally free modules M, N such that $A \otimes \mathrm{End}(M) \cong B \otimes \mathrm{End}(N)$. The equivalence classes of Azumaya algebras form a group under the tensor product, called the *Brauer group* $\mathrm{Br}(R)$ of the ring. An Azumaya algebra is split if and only if its class in the Brauer group is trivial. The study of Brauer groups of rings using cohomolgical methods has been initiated by Grothendieck [11]. The Brauer group of the ring naturally injects into the torsion part of the étale cohomology group $H^2((\mathrm{Spec}\, R)_{\text{ét}}, \mathbb{G}_m)$. The Brauer group of a field can be computed using Galois cohomolgy: it is isomorphic to the torsion part of $H^2(k, \mathbb{G}_m)$.

I prove that a necessary and sufficient condition for a quantum Zariski geometry over a smooth irreducible variety to be definable in an algebraically closed field is that the image of the associated Azumaya algebra under the restriction map $\mathrm{Br}(R) \to Br(Q(R))$ (where $Q(R)$ is the the field of fractions of R) is trivial.

A Zariski geometry is a geometric object similar both to algebraic varieties and compact complex spaces. It seems natural to try to generalise the results about definability in algebraically closed fields to results about definability in the structure $\mathcal{A}$ of compact complex spaces. I have obtained some partial results in this direction.

The structure $\mathcal{A}$ is a multi-sorted structure that consists of all compact complex spaces, a complex space per sort, with the language including all analytic subsets of all products of sorts. I prove that interpretations of abstract linear spaces in $\mathcal{A}$, contrary to the situation with interpretations in algebraically closed fields, are only locally (in complex topology) piecewise isomorphic to vector bundles. Since it is unclear what an interpretation in $\mathcal{A}$ might look like globally, I have introduced a notion weaker than "being definable in $\mathcal{A}$", tailored towards the class of quantum Zariski geometries.

Recall that a quantum Zariski geometry V over an algebraic variety X is an abstract linear space with a linear action of an algebra A on its fibres. Say that V has a *complex analytic model* if there is a complex space $\overline{V}$ fibred over X with holomorphic maps corresponding to all abstract linear space operations (addition and multiplication by scalar on fibres), and a holomorphic endomorphism of $\overline{V}$ for each element of A, linear on fibres, so that the modules on the fibres of $\overline{V}$ thus defined are isomorphic to the modules in the structure V.

For this weaker notion it is possible to prove a "definability criterion", i.e. give necessary and sufficient condition for a quantum Zariski geometry to have a complex analytic model. It is similar to the algebraic case, but is formulated in terms of the *analytic Brauer group*. Surprisingly, the "quantum torus" structure which is not definable in an algebraically closed field *does* have a complex analytic model (although it does not necessarily imply that it is interpretable in $\mathcal{A}$).

1.2 Outline of the thesis

Chapter 2 covers background material on Zariski geometries, the theory of algebraically closed fields, Galois cohomology and complex geometry. It contains no new material and serves mainly to state the results required later in the convenient form, give references and set up the notation. Chapters 3-4 contain the main results. In Chapter 3 I prove that the structure introduced in Zilber and Solanki [40] is not interpretable in an algebraically closed field (the original proof contained a gap). I also prove that a similar structure from an unpublished note [34] is not definable in an algebraically closed field and explain the non-definability of both structures in terms of the group law on a Galois cohomology group. Then I consider quantum Zariski geometries introduced in Zilber [37] and give necessary and sufficient conditions that a quantum Zariski geometry is definable in an algebraically closed field. I apply this definability criterion to the "quantum torus" and "quantum plane" structures described in [37].

Chapter 4 is an attempt to extend the non-interpretability results of Chapter 3 to a complex-analytic setting. I define what it means for quantum Zariski geometry to have a complex analytic model, and give necessary and sufficient conditions for a smooth quantum Zariski geometry to have one. I then prove a theorem that gives a partial description of an interpretation of an abstract linear space in the structure $\mathcal{A}$ and discuss the difficulties that present themselves when one tries to understand interpretations of abstract linear spaces and quantum Zariski geometries in the structure $\mathcal{A}$.

Chapter 2

Preliminaries

2.1 Zariski geometries

The data that defines a Zariski geometry is a set M together with a collection $\{\tau_n\}$ of topologies on Cartesian powers M^n of M, subject to a list of axioms. This can be turned into a first-order structure in the language that has a predicate for every closed set on every Cartesian power of M.

Definition 2.1.1. A *locally closed* set in a topological space is a set of the form $Z \setminus X$ where Z, X are closed. A *constructible* set is a finite union of locally closed sets.

Definition 2.1.2 (Noethereian Zariski geometry, Zilber [39]). A *Noetherian Zariski geometry* is defined by the data $(M, \{\tau_n\}, \dim)$ where τ_n is a topology on M^n and dim is a function that puts into correspondence to every definable set a non-negative number if the following axioms hold:

Topological axioms

- the topologies τ_n are Noetherian, i.e. any descending chain $X_0 \supset X_1 \supset \ldots X_n \supset \ldots$ of closed sets stabilizes for some k;

- singletons of all sorts M^n are closed, products of closed sets are closed, diagonals $x_i = x_j$ are closed;

- all permutations of coordinates are homeomorphisms;

- for a tuple $\bar{a} \in M^k$ and a closed set $Z \subset M^{n+k}$ the set

$$Z(a, y) = \{y \in M^n \mid (a, y) \in Z\} \subset M^n$$

is closed;

Dimension axioms

- singletons are zero-dimensional;

- $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$;

- a closed irreducible subset of a locally closed irreducible set has a strictly smaller dimension;

- let X be an irreducible locally closed subset of M^{n+k} and let $\pi : M^{n+k} \to M^n$ be a projection onto first n coordinates. Then

$$\dim X = \dim \pi(X) + \min_{a \in \pi(X)} \dim(\pi^{-1}(a) \cap X)$$

(this property is called *Addition formula*) and there is a relatively open $O \subset \pi(X)$ such that

$$\dim(\pi^{-1}(a) \cap X) = \min_{a \in \pi(X)} \dim(\pi^{-1}(a) \cap X)$$

(this property is called *Fibre condition*). In particular, the dimension function is upper semi-continuous.

Remark 2.1.3. Zilber [39] have also introduced a notion of Analytic Zariski geometry. In this thesis I only deal with Noetherian Zariski geometries, and I will further use the term "Zariski geometry" to refer to them since no ambiguity can occur.

Definition 2.1.4. A Zariski geometry M is called *pre-smooth* if for any constructible $X, Y \subset M^n$ and for any irreducible component Z of the intersection $X \cap Y$, $\dim Z \geq \dim X + \dim Y - n$.

Definition 2.1.5. A Zariski geometry M is called *complete* if for any n, k, any projection map $\pi : M^{n+k} \to M^n$ and any closed set $X \subset M^{n+k}$ the projection $\pi(X)$ is closed.

Definition 2.1.6 (Semi-properness)**.** A Zariski geometry is *semi-proper* if for any projection map $\pi : M^{n+k} \to M^n$ and any closed irreducible set $X \subset M^{n+k}$ there exists a closed set $Z \subset M^n$ such that $\pi(X) \supset \overline{\pi(X)} \setminus Z$ (i.e. $\pi(X)$ contains a set relatively open in $\overline{\pi(X)}$).

Lemma 2.1.7. *Semi-properness is equivalent to quantifier elimination.*

Proof. The left to right implication is Theorem 3.2.1 in Zilber [39].

Suppose that M eliminates quantifiers. Then $\pi(X) = \cup Y_i \setminus Y_i'$ where Y_i, Y_i' are irreducible closed and the union is finite. Therefore $\overline{\pi(X)}$ is $\cup Y_i$ and $\pi(X) \supset \overline{\pi(X)} \setminus \cup Y_i'$. □

Remark 2.1.8. The definition of a Zariski geometry that appears in Zilber [39] (Sections 2.1 and 3.1) only stipulates that a Zariski geometry has the property of semi-properness but does not require quantifier elimination. As we have just seen, Definition 2.1.2 is equivalent to one in [39].

Remark 2.1.9. The definition of a Zariski geometry that has appeared in the paper [16], is different from Definition 2.1.2. It presupposes that the dimension is the "Krull dimension", i.e. $\dim(X)$ for an irreducible X is the length of a maximal chain of proper inclusions $X \supset X_1 \supset \ldots X_n$ of irreducibles. It also presupposes that the dimension of the whole structure is one and that it is irreducible (i.e. the structure is strongly minimal); the definition also stipulates that the structure be pre-smooth. A pre-smooth irreducible one-dimensional Zariski geometry in the sense of Definition 2.1.2 is a Zariski geometry in the sense of [16] (see [39], Section 3.3), .

Definition 2.1.10 (Ample and very ample)**.** Let M be a one-dimensional Zariski geometry. Let O be an open subset of M^2 and let C be a relatively closed subset of $O \times T$. Then C is called a *family of plane curves* if for any $t \in T$, C_t is one-dimensional subset of O and for some open $U \subset T$ the fibre C_t is irreducible.

A Zariski geometry M is called *ample* if there is a family of plane curves $C \subset O \times T$ such that for any $a, b \in O$ there exists $t \in T$ such that $a, b \in C_t$.

If there is a two-dimensional family C of plane curves in M and additionally for any $a, b \in M^2$ there is a curve C_t that passes through a but does not pass through b then such Zariski geometry is called *very ample.*

Theorem 2.1.11 (Theorem A, Hrushovski and Zilber [16]). *Let M be an irreducible pre-smooth ample Zariski geometry. Then M interprets an algebraically closed field k. There is a closed continuous surjective map from $M \setminus A$ onto $\mathbb{P}^1(k) \setminus B$, where A, B are finite, and the map has finite fibres.*

Theorem 2.1.12 (Theorem B, Hrushovski and Zilber [16]). *Let M be an irreducible pre-smooth very ample Zariski geometry. Then M is isomorphic to the Zariski geometry of a quasi-projective algebraic curve over a field k interpretable in M.*

First example of a non-algebraic Zariski geometry

Let M be an irreducible one-dimensional Zariski geometry, and $Aut(M)$ be the group of bijective maps whose graphs are closed irreducible. Call an action $\rho : G \to Aut(M)$ *semi-free* if it is free on $M \setminus M_0$ where M_0 is the set of the fixed points p such that the stabilizers of p are the entire group G.

Consider a short exact sequence of groups

$$1 \to H \to G^* \xrightarrow{\pi} G \to 1$$

with H finite. Suppose that G acts on M non-trivially, then M_0 is a proper closed subset of M. Then M is a disjoint union

$$\bigsqcup_{\alpha} x_\alpha G \sqcup M_0$$

where x_α are representatives of the orbits of G.

Consider the set $M^* = \bigsqcup_\alpha x_\alpha G^* \sqcup M_0$ on which G^* acts in the natural way. The homomorphism $p : G^* \to G$ of groups induces a natural map $p : M^* \to M$

$$p(x_\alpha \cdot g) = x_\alpha \cdot \pi(g),\ p(x) = x \text{ for } x \in M_0$$

The language of the structure M^* consists of graphs of elements of G^* acting on M^* and pullbacks by p of closed sets of M.

Call the sets defined by positive Boolean combinations of the above predicates closed. This defines a Noetherian topology on every Cartesian power of M^*.

Proposition 2.1.13 (Proposition 10.1, [16])**.** *The structure described above is a one-dimensional Zariski geometry. If M is pre-smooth then M^* is pre-smooth too, If M is complete then M^* is complete.*

Let M be an elliptic curve with transcendental j-invariant. Consider two translations t_a, t_b that generate a group isomorphic to $\mathbb{Z}^2$ in $Aut(M)$. Consider an exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to G^* \xrightarrow{\pi} \mathbb{Z}^2 \to 1 \qquad (*)$$

where G^* is generated by elements T_a, T_b with relations $[T_a, T_b]^2 = [T_a, [T_a, T_b]] = [T_b, [T_a, T_b]] = 1$ (square brackets denote the commutator of two elements). The homomorphism $\pi : G^* \to G$ sends T_a to t_a and T_b to t_b.

Theorem 2.1.14 (Theorem C, Hrushovski and Zilber [16])**.** *The Zariski geometry that can be constructed by Proposition 2.1.13 using the exact sequence* $(*)$ *is not definable in an algebraically closed field.*

Elementary extensions of Zariski geometries

Let M' be a saturated elementary extension of a Zariski geometry M.

Definition 2.1.15 (Generic points)**.** Let X be an irreducible Zariski closed set in a Zariski geometry M. The *generic type* is defined as:

$$p_X(x) = x \in X \cup \{x \notin Z \mid Z \text{ a proper closed subset of} X\}$$

A realisation of the generic type p_X in M' is called a *generic point of* X. Every tuple in M' is a generic point of a closed irreducible set which we will call its *locus*.

A set X definable in M' over a tuple of parameters $\bar{s} \in M'$ has the form

$$X(M') = \{y \in Y \mid (\bar{s}, y) \in Z(M')\}$$

where Y and Z are definable subsets of some Cartesian power of M, $Z \subset X \times Y$.

Thus any definable set X in M' can be presented in the form $Y(M', \bar{a})$ where Y is a set definable in M and $\bar{a}$ a is the generic point of some closed irreducible

$Z \subset M^n$ and $Y \subset T \times Z$ for some T definable in M. Call a set of the form $Y(M', \bar{a})$ closed if X is closed in M. Closed sets in M' define a Noetherian topology, and with a suitably defined dimension function, a Zariski geometry (Zilber [39, Section 3.7]).

2.2 Algebraically closed fields

In this section K will be assumed to be an algebraically closed field and k will assumed to be a not necessarily algebraically closed subfield of K.

Abstract algebraic varieties

Definition 2.2.1. Let $I \subset K[x_1, \dots, x_n]$ be an ideal in the ring of polynomials over a field K. The set $X = V(I) = \{(x_1, \dots, x_n) \mid \forall f \in I\ f(x_1, \dots, x_n) = 0\}$ of zeroes of polynomials that belong to the ideal I is called an *affine algebraic variety.* Subsets of the form $V(I)$ are called *Zariski closed* subsets of the affine n-space $\mathbb{A}^n$, which consists of n-tuples of elements of the field K. The subspace topology induced on an affine algebraic variety X is called a *Zariski topology* on X. An affine variety is *defined over* k if the ideal I is generated by polynomials in $k[x_1, \dots, x_n]$.

Open subsets of affine algebraic varieties are called *quasi-affine varieties.* The subspace topology on a quasi-affine variety is also called a Zariski topology.

Definition 2.2.2. Let X be a quasi-affine variety. A function $f : X \to k$ is called *regular at* $x_0 \in X$ if there is a Zariski open subset $U \subset X$ that contains x_0, and polynomials $p(\bar{x}), q(\bar{x})$ such that $f(\bar{x}) = p(\bar{x})/q(\bar{x})$ and $q(\bar{x}) \neq 0$ for all $\bar{x} \in U$. Let X, Y be quasi-affine algebraic varieties. A map $f : X \to Y$ is called a *morphism* if there exist functions $f_i(\bar{x})$ regular on all X such that $f(\bar{x}) = (f_1(\bar{x}), \dots, f_n(\bar{x}))$ for all $\bar{x} \in X$. A morphism f is *defined over* k if all f_i belong to $k[x_1, \dots, x_n]$. If $\operatorname{char} K = p > 0$ then a composition of a morphism with a power of an inverse of the Frobenius morphism is called a *p-morphism.*

Definition 2.2.3. An *abstract algebraic variety* over a field k is a set X together with a cover $\cup U_i = X$ and bijective maps $f_i : U_i \to V_i$ (V_i-s are called *charts* and the data (U_i, f_i) is called an *atlas*) where V_i are affine algebraic varieties and $f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ are isomorphisms of affine varieties. The notion of Zariski topology is extended to abstract algebraic varieties: sets Z such that $f_i(Z \cup U_i)$ are closed in all V_i are declared closed in this topology. An abstract algebraic variety is *defined over* k if all its charts and transition maps are defined over k.

Let X and Y be abstract algebraic varieties with atlases (U_i, g_i), (O_j, h_j). A continuous map $f : X \to Y$ is called a *morphism* if the following additional data is specified. Let $W_{ij} = f^{-1}(O_j) \cap U_i$, then a collection of morphisms of quasi-affine varieties $f_{ij} : g_i(W_{ij}) \to h_j(O_j)$ specifies the morphism f if $h_j \circ f \circ g_i^{-1} = f_{ij}$ on W_{ij}. The notion of a *p-morphism* of abstract algebraic varieties is defined similarly to the quasi-affine case.

Remark 2.2.4. One gets an equivalent definition if one allows a quasi-affine atlas. First notice that principal quasi-affine varieties, i.e. those that are defined by a single inequality $f(x) \neq 0$ are isomorphic to affine varieties defined by $f(x)y = 1$. Then observe that any quasi-affine variety is a union of finitely many principal quasi-affine varieties.

Theorem 2.2.5 (Zilber [39], Section 3.4.1)**.** *An abstract algebraic variety with its Zariski topology and Krull dimension is a Zariski geometry.*

This definition in the spirit of Weil allows to adopt a point of view that (sets of points of) varieties over algebraically closed fields can be seen as definable sets. Given an abstract algebraic variety, one can consider the disjoint union of V_i-s factored by the equivalence relation that identifies $f_i(U_i \cap U_j)$ with $f_j(U_i \cap U_j)$ using isomorphisms $f_j \circ f_i^{-1}$. Due to elimination of imaginaries in algebraically closed fields this is a definable set. We are going to tacitly assume that we work with this definable set if we speak about a variety as a model-theoretic object. Then one can naturally associate a definable map with any morphism between abstract varieties. Reciprocally, by quantifier elimination, any definable map $f : X \to Y$ is piecewise morphism or p-morphism, i.e. there is a cover $X = \cup X_i$ by quasi-affine sets such that f agrees with (p-)morphisms $f_i : X_i \to Y$ on every X_i.

By virtue of quantifier elimination an arbitrary definable set in an algebraically closed field is a union of quasi-affine varieties. There is a subtlety in treating definable sets as objects of algebraic geometry that consists in that there is no canonical abstract algebraic variety structure on a definable set.

For example, $\mathbb{A}^2_k \setminus \{0\} = k^2 \setminus \{0\}$ is a definable set and a quasi-affine variety. Let L be a line passing through 0. Then $\mathbb{A}^2_k \setminus L$ and $L \setminus \{0\} \cong \mathbb{A}^1_k \setminus \{0\}$ constitute an atlas (both charts are quasi-affine, but isomorphic to affine varieties), one gets different atlases for each different L.

Forms and descent

Definition 2.2.6 (Galois action on varieties and morphisms)**.** Let X be a variety with an atlas (U_i, f_i) and let K be an extension of k. The group of field automorphisms $\mathrm{Aut}(K/k)$ acts on the atlas of X: for each $\sigma \in \mathrm{Aut}(K/k)$ denote by ${}^\sigma X$ the variety with the atlas $(U_i, \sigma^{-1} \circ f_i)$.

Note that if $h : X \to Y$ is a morphism of algebraic varieties defined by a collection of morphisms h_{ij} (see the Definition 2.2.3) then for the morphism ${}^\sigma h = \sigma^{-1} \circ h \circ \sigma$: ${}^\sigma X \to {}^\sigma Y$, the collection of affine morphisms that defines it is $\sigma^{-1} \circ h_{ij} \circ \sigma$.

Definition 2.2.7 (Form)**.** Let X be a variety defined over a field k. We say that a variety Y over k is a *form* of X, if there is a finite extension of fields K/k and an isomorphism $f : Y \xrightarrow{\sim} X$ defined over K.

Remark 2.2.8. The notion of a form can be extended to many other objects, such as algebraic groups, principal homogeneous spaces, algebras etc.

Definition 2.2.9 (Non-commutative group cohomology, finite group case)**.** Let G be a group. Consider a group A, not necessarily Abelian, endowed with a (left) action of G, $G \to \mathrm{Aut}(A)$ (such groups are also called G-groups). Let us denote the group action as ${}^\sigma a$ for $\sigma \in G, a \in A$. The first cohomology set $H^1(G, A)$ is defined to be the set of *cocycles*, collections of elements $\{h_\sigma\}_{\sigma \in G} \in A$ that satisfy

$$h_{\sigma\tau} = h_\sigma {}^\sigma(h_\tau)$$

modulo the equivalence relation defined as follows. Two cocycles g, h are equivalent (they are then called *cohomologous*) if there is an element $a \in A$ such that

$$h_\sigma = a^{-1} g_\sigma {}^\sigma a$$

The 0-th cohomology group is defined to be $H^0(G, A) = A^G = \{a \in A \mid \forall g \in G \; ga = a\}$.

From now till the end of this section K will always be a finite Galois extension of k.

Let Y be a K-form of a k-variety X and $f : Y \to X$ be an isomorphism defined over K. Consider the action of the Galois group $G = \mathrm{Gal}(K/k)$ on the group of automorphisms of X defined over K, $\mathrm{Aut}_K(X)$ given in the Definition 2.2.6, and define the cocycle

$$h_\sigma = {}^\sigma f \circ f^{-1}$$

(this makes sense as X is defined over k and so is invariant under $\mathrm{Gal}(K/k)$). If we pick a different morphism $\tilde{f} : X \to Y$ and let $s = \tilde{f} \circ f^{-1}$, then we get a cohomologous cocycle

$$\begin{aligned} {}^\sigma s \circ h_\sigma \circ s^{-1} &= ({}^\sigma \tilde{f} \circ {}^\sigma f^{-1}) \circ ({}^\sigma f \circ f^{-1}) \circ (f \circ \tilde{f}^{-1}) \\ &= {}^\sigma \tilde{f} \circ \tilde{f}^{-1} \\ &= \tilde{h_\sigma} \end{aligned}$$

We have defined an injective map from the set of k-isomorphism classes of K-forms of X to $H^1(G, \mathrm{Aut}_K(X))$. Now suppose a cocycle is given and let us construct the corresponding form.

The necessary construction is given by the descent theory.

Definition 2.2.10 (Descent datum)**.** Let Y be a variety defined over K. A collection of morphisms $h_\sigma : Y \to {}^\sigma Y, \sigma \in \mathrm{Gal}(K/k)$ satisfying

$$h_{\sigma\tau} = {}^\sigma h_\tau \circ h_\sigma$$

is called a *descent datum*.

Since the extension K/k is finite and separable there exists a primitive element α, $K = k(\alpha)$. The stabiliser subgroup of α in $\mathrm{Gal}(K/k)$ is trivial. If Y is an affine variety then given a descent datum one can define a natural action of $\mathrm{Gal}(K/k)$ on $\sqcup {}^\sigma Y = \bigsqcup_{\sigma \in \mathrm{Gal}(K/k)} {}^\sigma Y \times \{\sigma(\alpha)\}$:

$$\sigma \cdot : {}^\alpha Y \times \mathrm{Gal}(K/k)\alpha \to Y^{\sigma(\alpha)} \times \mathrm{Gal}(K/k)\alpha \quad (y, \tau(\alpha)) \cdot \sigma = ({}^\sigma h_\tau(y), \sigma(\alpha))$$

The condition in the definition of descent datum ensures that this is indeed a group action. For $y \in Y^\alpha$:

$$\begin{aligned} \sigma \cdot (\tau \cdot (y, \alpha)) &= (\ {}^\sigma h_\tau(h_\tau(y)),\ \sigma\tau(\alpha)\ ) \\ &= (\ h_{\sigma\tau}(y),\ \sigma\tau(\alpha)\ ) = (\sigma\tau) \cdot (y, \alpha) \end{aligned}$$

The quotient $(\sqcup^\sigma Y)/\mathrm{Gal}(K/k)$ is a quotient of an affine variety by an action of a finite group, and hence is an affine variety X (by, for example, [33], III.§12, Proposition 18). Moreover, by construction, X is $\mathrm{Gal}(K/k)$-invariant and hence defined over k.

We have thus obtained an isomorphism $f : Y \to X$ defined over K. Construction of this morphism is an example of *descent* (or more precisely, Galois descent). Descent theory has been developed by Grothendieck in a much more general setting (see, for example, Bosch et al. [1], Chapter 6) and can be applied to algebraic groups, algebras, vector spaces with a distinguished tensor, sheaves of modules, sheaves of algebras etc.

Theorem 2.2.11 (Serre [33], V.§4)**.** *Let Y be a quasi-projective variety over K and let (h_σ) be a descent datum. Then there exist a quasi-projective variety X over k and a K-isomorphism $f : Y \to X$ such that*

$$h_\sigma = f^\sigma \circ f^{-1}$$

Theorem 2.2.12 (Serre [32], III.§5, Proposition 5)**.** *Let X be a quasi-projective variety defined over k. Then the set of k-isomorphism classes of K-forms of X is in bijective correspondence with cohomology classes in $H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(X))$.*

Proof. Construction of a cocycle from a K-form was described earlier.

A cocycle that represents a class from $H^1(G, \mathrm{Aut}_K(X))$, $G = \mathrm{Gal}(K/k)$, constitutes a descent datum on X (since ${}^\sigma X = X$). Then one can apply Theorem 2.2.11 to get a k-variety X' which is a K-form of X.

□

Theorem 2.2.13. *Let $(G, \cdot)$ be an algebraic group defined over k and let K be a finite extension of K. Then the set of k-isomorphism classes of K-forms of G is in* *bijective correspondence with cohomology classes in* $H^1(\mathrm{Gal}(K/k), \mathrm{Aut}_K(G))$ *where* $\mathrm{Aut}_K(G)$ *is the group of regular automorphisms of the variety* G *that are defined over* K *and also preserve its group structure.*

Proof. Descend the multiplication map together with the variety. □

Proposition 2.2.14 (Serre [32], III.§1, Proposition 3)**.** *Let* K *be a Galois extension of* k *and* $G = \mathrm{Gal}(K/k)$*. Then*

$$H^1(G, \mathrm{GL}(n, K)) = \{1\}$$

Definition 2.2.15 (Principal homogeneous space)**.** A variety X is called a *principal homogeneous space* for an algebraic group G if G acts freely and transitively on X. A morphism of principal homogeneous spaces under the same group G is a map that preserves the action of G.

Theorem 2.2.16. *Let* G *be an algebraic group defined over* k*. Then the set of* k*-isomorphism classes of principal homogeneous spaces for* G *that have a* K*-point is in bijective correspondence with cohomology classes in* $H^1(\mathrm{Gal}(K/k), G(K))$ *where* $G(K)$ *is the group of* K*-points of* G*.*

Proof. The set of k-isomorphism classes of principal homogeneous spaces for G that have a K-point is the same as the set of K-forms of a variety that has $\mathrm{Aut}_K(X) = G(K)$. □

Definition 2.2.17. A topological group G is called *profinite* if it is an inverse limit of an inverse system of finite groups seen as discrete groups. The topology of a profinite group has a base that consists of all subgroups of finite index.

Let G be a profinite group. A G-group A is called a *discrete G-group* if the action of G is continuous (A is considered as a space with discrete topology). In equivalent terms, the stabiliser of any element g is a subgroup of G of finite index.

Definition 2.2.18 (Non-commutative group cohomology, pro-finite group case)**.** Let G be a profinite group. Consider a discrete G-group A, not necessarily Abelian. The cohomology sets $H^q(G, A), q = 0, 1$ are defined similarly to the finite group case, but the cocycles in H^1 are required to be continuous.

Remark 2.2.19. Let A be a G-group and let U be a subgroup of G. Then there exists a natural inclusion $H^1(G/U, A^U) \hookrightarrow H^1(G, A)$. If G is a profinite group then a cocycle in $H^1(G, A)$ is a map that factors through a finite quotient of G.

Proposition 2.2.20. *Let G be profinite and A be a G-group. Then*

$$H^1(G, A) = \varinjlim H^1(G/U, A^U)$$

where U runs through all normal subgroups of finite index and A^U is the subgroup of A fixed by the action of U.

Remark 2.2.21. As a corollary, Theorems 2.2.12, 2.2.13 and 2.2.16 all hold for k perfect and $K = k^s$, the separable closure of k (in which case the Galois group is profinite). The group cohomology sets for G-groups where $G = \mathrm{Gal}(k^s/k)$ are denoted $H^q(k, A)$.

Galois descent doesn't use much of the algebraic structure of the field, it only relies on the existence of finite quotients. Indeed, descent can be carried out in a purely model-theoretic setting as shown in Pillay [23]. In particular, Theorem 2.2.12 has an analogue.

Let X be a set defined in some algebraically closed field K over a set of parameters A, $K = \mathrm{acl}(A)$. Let $\mathrm{Aut}(K/A)$ be the profinte group of automorphisms. If $k = dcl(A)$, then $\mathrm{Aut}(K/A) = \mathrm{Gal}(k^{sep}/k))$. Pillay [23] defines a *definable cocycle* which in this case (K is the algebraic closure of A) coincides with the notion of continuous cocycle.

Theorem 2.2.22 (Pillay [23], Proposition 4.1)**.** $H^1(\mathrm{Aut}(M/A), \mathrm{Aut}_{\mathrm{def}}(X))$ *is in bijective correspondence with the set of A-forms of X up to A-definable bijection.*

Corollary 2.2.23. *If X is a definable group and* $\mathrm{Aut}_{\mathrm{def}}(X)$ *is the group of definable automorphisms of X that preserve group structure then* $H^1(\mathrm{Aut}(M/A), \mathrm{Aut}_{\mathrm{def}}(X))$ *is in bijective correspondence with the set of A-forms of X as a group.*

Principal homogeneous spaces for cyclic groups

In case A is Abelian, the cohomology sets that have been defined in an *ad hoc* manner in the previous section carry a natural structure of groups, and the correspondence between G-groups (for finite and profinite G) and cohomology groups is in fact functorial in A. It can be shown (see Serre [32], I.§2) that it is a derived functor of the functor $A^G = \{a \in A \mid ga = a\}$ on the category of (discrete) Abelian G-groups. In particular there is a long exact sequence of cohomology groups associated to every short exact sequence of G-groups.

In case A is not Abelian, G-groups do not form an Abelian category, H^0 is still a group and although H^1 does not have a group structure, it is still a pointed set with the distinguished class of cocycles cohomologous to the identity cocycle. There is a long exact sequence of pointed sets up to H^1 for any A.

Let k be a perfect field such that $\mu_n \subset k^\times$ for some $n \geq 1$, char $k \nmid n$, and let K be the separable closure of k with absolute Galois group G. Then the following is a short exact sequence:

$$1 \to \mu_n \to K^\times \xrightarrow{x \mapsto x^n} K^\times \to 1$$

Form the long exact cohomology sequence

$$1 \to \mu_n \to k^\times \xrightarrow{x \mapsto x^n} k^\times \to H^1(k, \mu_n) \to H^1(k, \mathbb{G}_m) \xrightarrow{n^*} H^1(k, \mathbb{G}_m)$$

where n^* is the morphism induced on the cohomology by the morphism $x \mapsto x^n$. But since by Proposition 2.2.14 $H^1(k, \mathbb{G}_m) = \{1\}$, we have

$$1 \to \mu_n \to k^\times \xrightarrow{x \mapsto x^n} k^\times \xrightarrow{\delta} H^1(k, \mu_n) \to 1$$

It follows that $H^1(k, \mu_n) \cong k^\times/(k^\times)^n$.

Proposition 2.2.24. *Let K be an algebraically closed field and let $X \subset K^m$ be a principal homogeneous space for the algebraic group μ_n such that* char $K \nmid n$*, and let X be defined over a subfield $k \subset K$. Then X is definably isomorphic over k to* *the principal homogeneous space* $Y \subset K$ *defined by the equation* $y^n = a$ *where* $a \in k$ *if* $\operatorname{char} k = 0$ *or* $a \in k^{p^{-\infty}}$ *where* $k^{p^{-\infty}}$ *is the perfect closure of* k*.*

Proof. This roughly follows the proof of the main statement of Kummer theory.

Passing to the perfect closure we can suppose k perfect. By Theorem 2.2.16 and Remark 2.2.21, X is determined by a cohomology class from $H^1(k, \mu_n)$. Since the action of the absolute Galois group of k on μ_n is trivial, $H^1(k, \mu_n) = \operatorname{Hom}(\operatorname{Gal}(k^s/k), \mu_n)$, the set of continuous homomorphisms from $\operatorname{Gal}(k^s/k)$ to μ_n. A cocycle f representing a class in $H^1(k, \mu_n)$ factors through some finite quotient G of $\operatorname{Gal}(k^s/k)$. Moreover it defines an inclusion $H = G/\operatorname{Ker} \sigma \hookrightarrow \mu_n$ where H is the Galois group of some finite extension $k' = (k^s)^H$ of k.

Let σ be a generator of $\operatorname{Gal}(k'/k) \cong \mu_l, l|n$. Since σ is a linear operator from $\operatorname{End}_k(k')$ with characteristic polynomial $\sigma^l - 1$, it has an eigenvector b, $\sigma b = \zeta b$, $\zeta^l = 1$. The product $\prod_{\sigma \in \operatorname{Gal}(k'/k)} \sigma(b) = \zeta^N b^l$ is invariant under $\operatorname{Gal}(k'/k)$, and hence in k. So, $k' = k(b), b^n = a \in k$.

The k'-form that corresponds to the cocycle f is the quotient of $\sqcup \mu_n^\sigma = \mu_n \times \operatorname{Gal}(k'/k) \cdot b$ by the following action of $\operatorname{Gal}(k'/k)$:

$$\sigma \cdot (\zeta^m, \tau \cdot b) = (f_\sigma \cdot \zeta^m, \sigma\tau \cdot b)$$

The quotient of $\sqcup \mu_n^\sigma$ by the action of $\operatorname{Gal}(k'/k)$ is k-isomorphic to $\operatorname{Gal}(k'/k)\, b \subset K$ via the map $(x, y) \mapsto y/x$. But $\operatorname{Gal}(k'/k)\, b$ is the set defined by the equation $y^n = a$.

□

2.3 Compact complex spaces

It has been observed by Zilber [36] that a compact complex manifold as a structure in the language with predicate for every analytic subset has elimination of quantifiers and has a finite Morley rank. Recall that an analytic set is defined to be a set which locally is the vanishing set of finitely many holomorphic functions.

Following Pillay [24] we consider all complex manifolds as one multi-sorted structure. It is also convenient to pass to the more general notion of a complex space.

Model-theoretically, compact complex spaces enjoy the same nice properties as manifolds: there is elimination of quantifiers and a compact complex space is of finite Morley rank.

We will now recall basic definitions and facts from the theory of compact complex spaces. We refer to Fischer [5], Grauert et al. [9] and Chapters I, II and VII of Grauert et al. [10] for more detailed relevant background material.

Definition 2.3.1 (Ringed spaces)**.** A *ringed space* is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ called the *structure sheaf.* A *locally ringed space* is a ringed space such that each stalk $\mathcal{O}_{X,x} = \varinjlim_{U \subset_{op} X} \mathcal{O}_X(U)$ is a local ring. It is called a *locally ringed space of* $\mathbb{C}$*-algebras* if additionally $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras.

Definition 2.3.2 (Subspace of a ringed space)**.** Let X be a locally ringed space and let f be a section of the structure sheaf $\mathcal{O}_X$ over a set U. One says that f *vanishes* at $x \in U$ if the image of f in the stalk $\mathcal{O}_{X,x}$ belongs to the maximal ideal $\mathfrak{m}_x \in \mathcal{O}_{X,x}$.

Let $(X, \mathcal{O}_X)$ be a ringed space and let $\mathcal{I}$ be a sheaf of ideals, i.e. for every open U, $I(U)$ is an ideal of $\mathcal{O}_X(U)$. A set of points where all functions from I vanish is called the *support* of $\mathcal{I}$. A set Z such that for any point $z \in Z$ there is a neighbourhood $U \ni z$ such that all functions $f \in \mathcal{I}(U)$ vanish exactly at the points of $Z \cap U$ is called *analytic.* A restriction of the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ to Z endows it with a locally ringed space structure. One says that $(Z, \mathcal{O}_X/\mathcal{I})$ is a *closed subspace* of X corresponding to the ideal sheaf $\mathcal{I}$.

Definition 2.3.3 (Morphisms of locally ringed spaces)**.** Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. Let $f : X \to Y$ be a continuous map. Then the sheaf $f_*\mathcal{O}_X$ on Y is defined as the sheafification of the presheaf

$$U \mapsto \mathcal{O}_X(f^{-1}(U))$$

A *morphism of ringed spaces* is a pair (f, f') where f is a continuous map $f : X \to Y$ and $f' : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings. A *morphism of locally ringed space* is additionally *local*, i.e. on stalks it maps maximal ideals to maximal ideals.

Definition 2.3.4 (Germs of sets)**.** Let X be a locally $\mathbb{C}$-ringed space and let $x \in X$. A germ of a space at x is an equivalence class of sets under the equivalence relations that identifies two sets that coincide on an open neighbourhood of x.

Definition 2.3.5 (Complex space)**.** A *domain* is an open connected subset of $\mathbb{C}^n$ for some n. A *model space* is a closed subspace of a domain in $\mathbb{C}^n$.

A *complex space* is a locally $\mathbb{C}$-ringed space such that every point has a neighbourhood isomorphic (as a local ringed space) to a model space. A complex space is called *reduced* if all stalks of the structure sheaf are reduced rings.

If a point $x \in X$ has a neighbourhood isomorphic to an open set $W \subset \mathbb{C}^n$ then we say that X is *non-singular* an x. If a complex space is non-singular at every point it is called a *complex manifold.*

Theorem 2.3.6 (Nullstellensatz for germs of analytic sets)**.** *For an ideal $J \subset \mathcal{O}_{\mathbb{C}^n,0}$ denote $V(J)$ the germ associated with an analytic set defined by J. For a germ of an analytic set Z at x denote $I(X)$ the set of functions $f \in \mathcal{O}_{X,x}$ such that $X \subset V(f)$.*

Let $J \subset \mathcal{O}_{\mathbb{C}^n,0}$ be an ideal, then $I(V(J)) = \sqrt{J}$.

Definition 2.3.7 (Germs of spaces)**.** Let $x \in X$ be a point of a complex spaces. A germ of a complex subspace at x is an equivalence class of complex subspaces of X passing through X under the following equivalence relation: two spaces Z, Z' are equivalent if there exists an open neighbourhood and an isomorphism of complex spaces $Z|_U \cong Z|'_U$.

Morphisms of germs are just morphisms of spaces that are representatives of germs and which preserve the point x.

Definition 2.3.8 (Analytic local algebra)**.** A quotient of $\mathcal{O}_{\mathbb{C}^n,0}$ by an ideal is called an *analytic local ($\mathbb{C}$-)algebra.*

Note that $\mathcal{O}_{\mathbb{C}^n,0}$ is just the ring of converging power series in n variables (otherwise denoted $\mathbb{C}\{z_1, \dots, z_n\}$).

Theorem 2.3.9 (Fischer [5], Proposition 0.21)**.** *Let X be a complex space and x a point on Z. Correspondence $Z \mapsto \mathcal{O}_{X,x}/I(Z)$ is an anti-equivalence between the category of germs of complex spaces at x and the category of analytic local algebras.*

Analytic local algebras are in some ways similar to polynomial rings. Here are some basic facts about them from Grauert et al. [9], Chapter 2, §0.

Theorem 2.3.10. *Any analytic local algebra is local, Noetherian and has a residue field* $\mathbb{C}$.

Proposition 2.3.11. *Let* R *be an analytic local algebra with a maximal ideal* $\mathfrak{m}$. *For any set of elements* $f_1, \ldots, f_n \in \mathfrak{m}$ *there exists a unique homomorphism from* $\mathbb{C}\{z_1, \ldots, z_n\}$ *to* R *such that* $z_i \mapsto f_i$ *for all* i.

Corollary 2.3.12. *A morphism of analytic local algebras* $R \to S$ *is completely determined by its values on the generators on the maximal ideal of* R.

Proposition 2.3.13. *Let* $R = \mathcal{O}_n/I$, $S = \mathcal{O}_m/J$ *and let* $\varphi : R \to S$. *Then there exists a lifting* $\bar{\varphi} : \mathcal{O}_n \to \mathcal{O}_m$ *such that* $\pi_S \circ \bar{\varphi} = \varphi \circ \pi_R$ *where* π_R, π_S *are respective quotient projections.*

In view of Nullstellensatz and Noetherianity of $\mathcal{O}_{\mathbb{C}^n,0}$, one can give the following definition of a reduced space.

Definition 2.3.14 (Reduced complex space)**.** A *reduced complex space* is a topological space X together with a cover $\cup U_i = X$ and homeomorphisms $f_i : U_i \to V_i$ where V_i are model spaces and $f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ are biholomorphic maps. The sets U_i are called *charts* and the data (U_i, f_i) is called an *atlas*.

A subset $Z \subset X$ is called *closed* or *analytic* if it is locally biholomorphically equivalent to a model space. These closed sets are closed sets of a topology which we will call *Zariski topology* on the complex space X. The original topology of X will be called *fine* or *complex topology*.

Let X and Y be complex spaces with atlases (U_i, g_i), (V_j, h_j). A continuous (in the complex topology) map $f : X \to Y$ is called a *morphism* or a *holomorphic map* if the following additional data is specified. Let $W_{ij} = f^{-1}(V_j) \cap U_i$, then a collection of holomorphic maps of analytic sets $f_{ij} : W_{ij} \to V_j$ specifies the morphism f if $h_j \circ f \circ g_i^{-1} = f_{ij}$ on W_{ij}.

If a complex space X has an atlas (U_i, f_i) such that U_i are homeomorphic to the unit ball of $\mathbb{C}^{n_i}$ then X is called a *complex manifold*.

Definition 2.3.15 (Dimension)**.** The *(algebraic) dimension of a complex space X at a point x*, denoted $\dim_x X$, is the Krull dimension of the local ring $\mathcal{O}_{X,x}$. The dimension of a constructible set X is defined to be $\max_{x \in X} \dim_x X$.

Theorem 2.3.16 (Zilber [39], Section 3.4.2)**.** *A reduced complex space with its Zariski topology and the dimension as defined above is a Zariski geometry.*

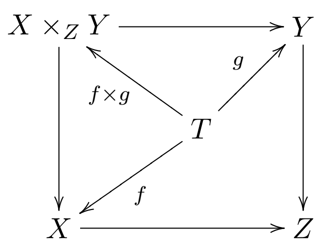
Remark 2.3.17. Let X be a constructible set of a compact complex space. The Krull dimension of X in the sense of the Zariski topology is dominated by the complex dimension.

An analytic subset of a complex space has a natural reduced complex space structure.

An arbitrary compact complex space is not necessarily an analytic subspace of a complex manifold, but the following characterisation holds in the compact case.

Theorem 2.3.18 (Hironaka; Theorem 7.13 in Peternell [21])**.** *Every compact complex space is a holomorphic image of a compact complex manifold.*

Definition 2.3.19 (Fibre product)**.** Let $f : X \to Z, g : Y \to Z$ be two morphisms of complex spaces. The fibre product is defined as the product in the category of complex spaces, i.e. it is a space such that for any morphism $T \to Z$ and any pair of morphisms $f : T \to X, g : T \to Y$ commuting with it there is a unique morphism $f \times g : X \times Y \to T$ that makes the following diagram commute:



The operation on the algebraic level dual to the fibre product is the analytic tensor product.

Definition 2.3.20 (Analytic tensor product)**.** Let R, S be analytic local algebras over an analytic local algebra A. The analytic tensor product $R \hat{\otimes}_A S$ is defined as the coproduct in the category of local algebras and local morphisms.

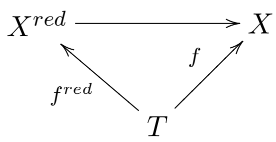
By Theorem 2.3.9, the local ring of a point $x \times y$ in the fibre product $X \times_S Y$ (x and y map to the same point s) is the tensor product $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{Y,y}$.

A fibre product of reduced spaces is not necessarily reduced.

Example 2.3.21. Let X be the cuspidal curve defined by the equation $y^2 = x^3$ and let $f : \mathbb{A}^1 \to X$ be the map defined by $x \mapsto (x^2, x^3)$. Consider the algebras of germs of functions at 0. Algebraically it corresponds to an inclusion $R = \mathbb{C}\{x, y\}/(y^2 - x^3) \hookrightarrow \mathbb{C}\{t\}$ such that x maps to t^2 and y maps to t^3 (by Proposition 2.3.11 such an map exists).

Using Propositions 2.3.11 and 2.3.13 and the universal property of analytic tensor product one can check that $R \otimes_{\mathbb{C}\{t\}} R \cong \mathbb{C}u, v/(u^2 - v^2, u^3 - v^3)$, which is not reduced: $(u - v)^3 = (u^2 - v^2)(3u + 3v) - 2(u^3 - v^3) = 0$, but $(u - v)$ is non-zero.

One can turn a non-reduced space into a reduced space by quotienting out the sheaf of nilpotent elements from the structure sheaf. Moreover, a morphism between not necessary reduced spaces naturally lifts to a morphism between their reductions, i.e. reduction is a functor from the category of complex space to the category of reduced spaces. The following property can be used as a definition of reduction: for any reduced space T and the map $T \to X$ factors through X^{red}:



The many-sorted structure of complex spaces

Definition 2.3.22 (The structure $\mathcal{A}$)**.** The sorts of the many-sorted structure $\mathcal{A}$ are irreducible compact complex spaces. Predicates are closed subsets of products of sorts.

Definition 2.3.23 (Meromorphic maps)**.** A holomorphic map $f : X \to Y$ is a *modification* if there exist proper Zariski open subsets $O \subset X$ and $U \subset Y$ such that f restricts to biholomorphic map between O and U.

By a *meromorphic map* $f : X \to Y$ we mean an irreducible Zariski closed set $\Gamma(f) \subset X \times Y$ such that the projection $\Gamma(f) \to X$ is a modification (so on some Zariski open set $O \subset X$ f is a well-defined map).

If the projection on Y is also a modification, then f is called *bimeromorphic.* That means that if two varieties are bimeromorphic, then some dense opens of these varieties are biholomorphic.

It follows immediately from the definition that meromorphic maps are definable. By quantifier elimination the maps definable in $\mathcal{A}$ are “piecewise meromorphic”, i.e. for a definable $f : X \to Y$ there exists a cover $\cup U_i = X$ where U_i-s are Zariski open subsets of compact spaces X_i, and meromorphic maps $f_i : X_i \to Y$ such that their restrictions on U_i-s coincide with f.

In addition to the structure $\mathcal{A}$ itself one might consider its saturated elementary extensions. Let $\mathcal{A}'$ be a κ-saturated model, for a very large cardinal κ (a monster model). Since $\mathcal{A}$ has a Zariski structure on every sort, one can introduce the natural Zariski structure on $\mathcal{A}'$, as described in Section 2.1 (with obvious modifications pertaining to a multi-sorted structure).

Submersions and sections

Let $f : X \to Y$ be a morphism of complex spaces. In Chapter 4 we will need to take sections of f locally in a neighbourhood of a given point $y \in Y$ passing through a given point $x \in X$, i.e. to find a morphism $g : U \to X$ such that $f \circ g$ is identity on $U \ni y$ and $g(y) = x$. Submersions are a class of morphisms that have this property.

Definition 2.3.24 (Mersions and submersions)**.** Let $f : X \to Y$ be a morphism of complex spaces. It is called a *k-mersion* at $x \in X$ if there exist open neighbourhoods U, V of x and $f(x)$ and there exists a complex space X' and an open $O \subset \mathbb{C}^k$ such that U and $X' \times O$ are biholomorphic and such that the following diagram commutes

$$\begin{array}{ccccc} X & \hookleftarrow & U & \xrightarrow{\sim} & X' \times O \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi_1} \\ Y & \hookleftarrow & V & \hookleftarrow & X' \end{array}$$

If the inclusion $X' \hookrightarrow V$ is biholomorphic the k-mersion is called a *submersion.*

If a map is a submersion at x then clearly it has a section in a neighbourhood of $f(x)$. Note that a map that is both a 0-mersion and a submersion at x is a biholomorphic in a neighbourhood of x.

The notion of flatness is crucial for the characterisation of submersions.

Definition 2.3.25 (Flat module)**.** An R-module M is flat if for every exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

the tensored sequence

$$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

is exact.

Definition 2.3.26 (Fibre of a sheaf)**.** Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. A *fibre* of $\mathcal{F}$ at a point x is the module $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$ (where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x). It is naturally a $\mathbb{C}$-vector space (since $\mathcal{O}_{X,x}/\mathfrak{m}_x \cong \mathbb{C}$). The dimension of this space is called the dimension of a fibre of $\mathcal{F}$ at x, $\dim_x \mathcal{F}$.

Definition 2.3.27 (Flat morphism)**.** Let $f : X \to Y$ be a morphism of complex spaces. A sheaf $\mathcal{F}$ on X called *flat* at x if for any $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module. The morphism f is called flat at x if $\mathcal{O}_X$ is flat at x.

Theorem 2.3.28 (Fischer [5], Theorem 3.21)**.** *Let $f : X \to Y$ be a morphism of complex spaces. Then f is a submersion at x if and only if the fibre $f^{-1}(f(x))$ is non-singular and f is flat at x.*

Here is a useful criterion for flatness for morphisms between non-singular complex spaces.

Theorem 2.3.29 (Fischer [5], Corollary 3.20)**.** *Suppose that $f : X \to Y$ be a holomorphic map between connected complex manifolds, and let every fibre of f is of pure dimension* $\dim X - \dim Y$*. Then f is flat.*

Now that we have found out in which circumstances we can take local sections of a morphism, the following theorems tell us that we can guarantee these circumstances after throwing out some analytic sets.

Theorem 2.3.30 (Fischer [5], Theorem 3.18)**.** *Let $f : X \to Y$ be a morphism and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then the non-flatness locus*

$$\{x \in X \mid \mathcal{F}_x \text{ is not flat over } \mathcal{O}_{f(x)}\}$$

is a proper analytic subset of X.

To find out if a space is (non-)singular one has to look at its *tangent space.*

Let X be a complex space. A *tangent vector at x* is a derivation $\nu : \mathcal{O}_{X,x} \to \mathbb{C}$, i.e. a $\mathbb{C}$-linear map that satisfies the Leibniz rule, $\nu(fg) = \nu(f)g + f\nu(g)$. The set of all tangent vectors at x is a $\mathbb{C}$-vector space, called the *tangent space at x* and denoted T_xX. There exists a natural isomorphism $T_xX \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

Let Z be a model space defined by the holomorphic functions $f_1, \dots, f_m$ on $U \subset \mathbb{C}^n$. Then $TZ = \sqcup_{z \in Z} T_zZ$ can be given the structure of a complex space. Namely, it is a subset of the trivial bundle $U \times \mathbb{C}^n$ defined by the equations in the coordinates $(z_1, \dots, z_n, x_1, \dots, x_n)$

$$f_i(z) = 0 \qquad \sum_{j=1}^{n} \frac{\partial f_i}{\partial z_j}(z) \cdot x_j = 0, \qquad 1 \le i \le m$$

A point $(z, x) \in TZ$ corresponds to the derivation $\nu_{(z,x)} \in T_zZ$

$$\nu_{(z,x)} : \mathcal{O}_{Z,z} \to \mathbb{C} \qquad h \mapsto \sum_j \frac{\partial h}{\partial z_j}(z)x_j$$

Each fibre of the natural projection $TZ \subset U \times \mathbb{C}^n \to Z \subset U$ is a linear subspace of $\mathbb{C}^n$.

A gluing construction is used to define the tangent space TX for a complex space X so that restrictions to subsets U biholomorphically equivalent to model spaces, are isomorphic to TU_α, but we will gloss over the details (which can be consulted in the section 2.5 of Chapter II, Fischer [5]).

Given a morphism of complex spaces $h : X \to Y$ there is an associated morphism of tangent spaces, the *differential.* It sends a derivation $\nu : \mathcal{O}_{X,x} \to \mathbb{C}$ to the derivation $Dh(\nu) : \mathcal{O}_{Y,h(x)} \to \mathbb{C}$, $Dh(\nu) = \nu \circ h^*$ where $h^* : \mathcal{O}_{Y,h(x)} \to \mathcal{O}_{X,x}$ is the map of local rings associated to h.

The differential is an actual holomorphic map of tangent spaces, which in local coordinates is defined as follows. Let $Z \subset X$ be biholomorphically equivalent to the model space which is the zero set of analytic functions $f_1, \dots, f_m$ on a domain in $\mathbb{C}^n$ and $V \subset Y$ be biholomorphically equivalent to the model space which is the zero set of analytic functions $g_1, \dots, g_k$ on a domain in $\mathbb{C}^k$, such that $f(U) \subset V$. Then Dh is

$$Dh : TU \to TV \qquad (x,t) \mapsto (h(x), \left(\frac{\partial h_i}{\partial x_j}\right)_{ij} t)$$

This is well-defined: indeed, by applying the chain rule one sees that Dh maps TU to TV.

Definition 2.3.31 (Vector bundle)**.** Let Y be a complex space. We say that X a *vector bundle* of rank n over Y if

- every fibre $f^{-1}(y)$ is isomorphic to $\mathbb{C}^n$;
- there exists an open covering $\cup U_i = Y$ such that and for any i an isomorphism $\phi_i : f^{-1}(U_i) \xrightarrow{\sim} U_i \times R$;
- the maps $\varphi_j \circ \varphi_i^{-1}$ induce linear transformations on the fibres.

Proposition 2.3.32 (Fischer [5], Theorem 2.15)**.** *A point x of a complex space X is non-singular if and only if T_xX is a vector bundle in a neighbourhood of x.*

Theorem 2.3.33 (Fischer [5], Corollary 2.14)**.** *Let be a sheaf of $\mathcal{O}_X$-modules. Then the singularity locus*

$$\{p \in X \mid X \textit{ is not a vector bundle at } p\}$$

is an analytic subset of X.

Chapter 3

Non-algebraic Zariski geometries

3.1 Abstract linear spaces

Zariski geometries that will be discussed further all have as a reduct a structure of the kind that we will axiomatise below.

Definition 3.1.1 (Abstract linear space)**.** Let $M = (X, S, F, p)$ be a multisorted structure such that

- F is an algebraically closed field;

- S is an algebraic variety over F with the natural Zariski geometry structure;

- $p : X \to S$ is a surjective map;

- there are maps $a : X \times_S X \to X$ and $m : F \times X \to X$ such that the following diagrams commute

$$\begin{array}{ccc} X \times_S X & \xrightarrow{\quad a \quad} & X \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle p} & \\ & S & \end{array} \qquad \begin{array}{ccc} F \times X & \xrightarrow{\quad m \quad} & X \\ & {\scriptstyle p\circ\pi_2}\searrow \qquad \swarrow{\scriptstyle p} & \\ & S & \end{array}$$

  (where the fibre product is understood in the set-theoretic sense) and such that $(X_s = p^{-1}(s), a, m)$ is a vector space for any $s \in S$.

The structure M is called an *abstract linear space* over S.

For the ease of notation we will often use $+$ and $\cdot$ as synonyms of a and m.

Note that since the sorts S and F are inter-definable and one can as well consider a structure with just one of them without losing any information.

Definition 3.1.2 (Interpretation)**.** Let M be a structure in a language L and N be a structure in a language L'. An *interpretation* of M in N is a structure with the universe being a definable set (further denoted $M(N)$) in N^{eq} and such that the predicates of L are definable relations in N^{eq} (further denoted $P(N)$ where P is a predicate of L) such that $M(N)$ is isomorphic to M as an L-structure. The notation $M(N)$ will be used to denote both the definable set and the structure with such universe. If X is definable set in M^k then the image of it under the isomorphism is denoted $X(N)$.

If moreover $M(N)$ is a definable set in N^k for some k, one says that M is *defined* in N.

Remark 3.1.3. We adopt a similar notation for realisations of definable sets. If X is a definable set and M is a model the $X(M)$ will mean the set of M-tuples that belong to the realisation of X in M.

Let $(F, +, \cdot, 0, 1)$ be an infinite field and let K be an algebraically closed field. Let F be interpreted (or rather defined, since ACF_p has elimination of imaginaries) in K. Then $F(K)$ in a definable set in the field K with definable field operations.

Theorem 3.1.4 (Pillay [25], Theorem 4.13; Poizat [26],Theorem 4.15)**.** *There is a bijection between $F(K)$ and K, definable in K, which is a field isomorphism.*

In this section we will prove some basic facts about interpretations of abstract linear spaces in algebraically closed fields. We will need several auxiliary statements about groups definable in algebraically closed fields to do that.

Lemma 3.1.5 (Pillay [25], Lemma 4.9)**.** *Let G, H be algebraic groups definable in an algebraically closed field K, and let $f : G \to H$ be a definable homomorphism of groups. If* $\mathrm{char}\, K = 0$ *then f is a morphism of algebraic varieties, if* $\mathrm{char}\, K = p > 0$ *then f is a p-morphism.*

Lemma 3.1.6. *Let m be a definable action of $\mathbb{G}_m(k) \times (\mathbb{G}_a)^n(k) \to (\mathbb{G}_a)^n(k)$ for some algebraically closed field k. Then for any $a \in \mathbb{G}_m(k)$, m_a is a linear map.*

Proof. Suppose $\operatorname{char} k = 0$, and consider some m_a. Then by Lemma 3.1.5 m_a is regular. Then $m_a(\alpha \cdot x) = (f_1(\alpha \cdot x), \ldots, f_n(\alpha \cdot x))$ and since m_a is bijective all polynomials f_i must be linear. Since m_a is addive, the polynomials must be homogeneous, hence $m_a(\alpha \cdot x) = \alpha \cdot m_a(x)$.

If $\operatorname{char} k = p > 0$ then each m_a is a p-morphism, i.e. $m_a(x) = (\mathrm{Fr}^{-k_1} \circ f_1(x), \ldots, \mathrm{Fr}^{-k_n} \circ f_n(x)))$ where Fr is the Frobenius morphism. Since m_a is bijective and additive, f_i are also bijective and additive and can only contain terms of the form $a \cdot x_i^{p^n}$. However, if $\mathrm{Fr}^{-k_i} f_i$ contains terms of degree not equal to 1, the iterates of m_a will have unbounded degree. Therefore $\mathrm{Fr}^{-k_1} \circ f_1(x)$ are homogeneous polynomials, and m_a are linear. □

Let $M = (X, S, F, p)$ be a linear space over S with fibres of constant dimension n, and suppose that S is irreducible. Suppose that M is interpreted over a set of parameters A in an algebraically closed field K. By Theorem 3.1.4 $F(K)$ is definably isomorphic (as a field) to K. Consequently, there exists a bijection between $S(F(K))$ and $S(K)$ definable over K, and we can identify $S(K)$ with a variety defined over K. Recall that under the adopted notation (Definition 3.1.2) $S(F(K))$ means the image under the isomorphism $F(K) \to K$ of the definable set S in some Cartesian power of K.

Denote X_s a generic fibre of X, that is $p^{-1}(s)$ where s is a point of S in some extension of K, generic over K. Suppose that s as a tuple of elements consists of $s_1, \ldots, s_m$ and denote the definable closure of K and s_i-s in this extensions as K'. We will further write $K' = \operatorname{dcl}(\bar{s}K)$ to mean $K' = \operatorname{dcl}(\cup\{s_i\} \cup K)$ for a tuple $\bar{s}$.

Proposition 3.1.7. *Let L be the algebraic closure of K'. Then L is an elementary extension of K. Consider $(X_s(L), L, +, \cdot)$ as a vector space structure (with two sorts: that of vector space and that of a field). Then $(X_s(L), L, +, \cdot)$ is definably isomorphic to $(L^n, L, +, \cdot)$ over K' as a vector space.*

Proof. Pick a basis $\{e_i\} \subset X_s(L)$ and construct a definable isomorphism of groups

$$\eta : (\mathbb{G}_a)^n \to (X_s, +) \qquad (a_1, \ldots, a_n) \mapsto a_1 \cdot e_1 + \ldots + a_n \cdot e_n$$

Since the coordinates of the basis vectors $e_1, \ldots, e_n$ belong to L, the isomorphism is defined over L. So X_s is an L-form of the group $(\mathbb{G}_a)^n(L)$, which by Corollary 2.2.23 corresponds to an element of $H^1(\mathrm{Gal}(L/K'), \mathrm{Aut}_{\mathrm{def}}(X_s))$. By Lemma 3.1.6 the group of definable automorphisms of $X_s(L)$ is $\mathrm{GL}_n(L)$, and by Proposition 2.2.14 $H^1(\mathrm{Gal}(L/K')), \mathrm{GL}_n(L))$ is trivial. Therefore, $(X_s(L), +)$ is isomorphic to $(\mathbb{G}_a)^n(L)$ over K' as a group via an isomorphism η'. It is left to show that the isomorphism respects multiplication by a scalar.

Multiplication by an element $a \in L$ is a definable automorphism of $X_s(L)$, and via the isomorphism η' it induces an automorphism m_a of $(\mathbb{G}_a)^n(L)$. By Lemma 3.1.6 all m_a-s are linear transformations of L^n.

The correspondence $a \mapsto m_a$ defines an inclusion of rings $L \overset{\iota}{\hookrightarrow} \mathrm{End}(L^n)$. Since m_a commutes with all elements of $\mathrm{End}(L^n)$, this correspondence defines an isomorphism of fields $L \to Z(\mathrm{End}(L^n)) = L$. Therefore m_a is the scalar matrix $\sigma(a)I$ where σ is a definable automorphism of L defined over $\mathrm{dcl}(\bar{s}K)$. If $\mathrm{char}\, L = 0$ this can only be identity, if $\mathrm{char}\, L = p > 0$ this can be a power of Frobenius automorphism.

Let $f_1, \ldots, f_n$ be the basis of $X_s(L)$ such that $\eta'(f_1), \ldots, \eta'(f_n)$ is the standard basis of L^n. Define a map $\theta : L^n \to X_s(L)$:

$$\theta(\alpha_1, \ldots, \alpha_n) = \eta'(\sigma^{-1}(\alpha_1), \ldots, \sigma^{-1}(\alpha_n))$$

It is clearly an isomorphism of vector spaces. □

Corollary 3.1.8. *There is an open subset $U \subseteq S(K)$ such that $X(K) \cap p^{-1}(U)$ is definably isomorphic to $U \times \mathbb{A}^n_K$, and the restrictions of a and m on $p^{-1}(u)$ for any $u \in U$ coincide with vector space addition and scalar multiplication on K^n.*

Proof. Let f be the definable isomorphism between the generic fibre $X_s(L)$ and L^n defined over K constructed in the previous theorem. By Lemma 3.1.5, f is a p-morphism and without lack of generality we can assume that it is just a morphism (we can compose it with a big enough power of the Frobenius).

Consider the map h from X_s to $S \times \mathbb{A}^n_K$ defined as $h(x) = (p(x), f(x))$, so its image is $\{s\} \times \mathbb{A}^n_K$. Consider the formula $\varphi(x, y, z)$ such that $\varphi(x, y, s)$ defines the graph of h; we will then denote $h_u : X_u \to S \times \mathbb{A}^n_K$ the map defined by $\varphi(x, y, u)$.

Since the property of being an isomorphism of groups can be expressed by a first-order formula and by the definition of the generic type of S, h_u is an isomorphism of groups (we look at $\mathbb{A}_K^n$ as the set of K-points of an additive group $\mathbb{G}^n$) for $u \in U$ where Morley rank of U is equal to that of s. By elimination of quantifiers, one can assume U to be open. We have obtained a definable map $p^{-1}(U) \to U \times \mathbb{A}_K^n$ such that restriction to every fibre X_u is a regular map. By applying elimination of quantifiers and further restricting U we get a morphism $p^{-1}(U) \to U \times \mathbb{A}_K^n$. □

3.2 Case study

The structure Q_n

Consider the following structure from Zilber and Solanki [40].

Let n be a fixed even number. Define the following theory T_n in a two-sorted language. There are sorts B and F. The sort F is an algebraically closed field of characteristic 0. The sort B consists of a disjoint union of n copies of F with the natural projection $p : B \to F$. It has the following structure:

- the action $\cdot : mu_n \times B \to B$ of the group $\mu_n \subset F$ of n-th roots of unity is free and transitive on fibres of B;
- predicates $\mathbf{A}$ and $\mathbf{A}^\dagger$ on $B \times B \times F$ are defined as follows: for any $x \in F$ there are unique $a \in p^{-1}(x)$ and $b \in p^{-1}(x+1)$ such that

  $$\mathbf{A}(a, b, \sqrt{x}) \text{ and } \mathbf{A}^\dagger(b, a, \sqrt{x})$$

  where $\sqrt{x}$ is a square root of x;
- Since μ_n acts freely and transitively on fibres of B, any point of the fibre $p^{-1}(x)$ is of the form $\zeta^k \cdot a$ and any point of the fibre $p^{-1}(x+1)$ is of the form $\zeta^l \cdot b$, where ζ is the primitive n-th root of 1; define for arbitrary points of the fibres

  $$\mathbf{A}(\zeta^k \cdot a, \zeta^l \cdot b, \zeta^{l-k}\sqrt{x}) \text{ and } \mathbf{A}^\dagger(\zeta^l \cdot b, \zeta^k \cdot a, \zeta^{k-l}\sqrt{x})$$

This theory is the same as in Definition 2.1 of Zilber and Solanki [40] up to notational changes, and with the difference that [40] consider a projection from B to the projective line, not to the affine line. The same argument as in Propsotion 2.1 of [40] establishes that this theory is complete and categorical.

Note that the field sort F is interpretable in the sort B (by quotienting out the action of μ_n), so this structure can be regraded as one-sorted, and as such — a finite cover of affine line, similar to the Zariski geometry from the Section 10 of Hrushovski and Zilber [16].

Structure 1. Fix some field K of characteristic 0 and consider the model of the theory T_n such that the field sort is K. We will refer to this model as QHO_n.

Structure 2 (Q_n)**.** Let n be a fixed even number. The structure Q_n is a linear space (V, X, F, p) where F is the same field K of characteristic 0 that is used to define Structure 1, X is the affine line $\mathbb{A}^1$ over F and all fibres of V are of dimension 1, with the following extra structure:

- there is a predicate $B \subset V$ and an action of the group of n-th roots of unity μ_n on B, such that for any $x \in \mathbb{A}^1_F$ the set $p^{-1}(x) \cap B = B_x$ is finite of size n and the action of μ_n restricted to B_x is free and transitive;
- there are maps $\mathbf{a}, \mathbf{a}^\dagger : V \to V$ that send, respectively, V_x to V_{x+1} and V_{x+1} to V_x and are linear;
- for any x there is $e_x \in B_x$ and $e_{x+1} \in B_{x+1}$ such that $\mathbf{a}e_x = \sqrt{x}e_{x+1}$ and $\mathbf{a}^\dagger e_{x+1} = \sqrt{x}e_x$, where $\sqrt{x}$ is a square root of x, fixed for every e_x and e_{x+1}.

Proposition 3.2.1. *Structure 2 is interpretable in Structure 1.*

Proof. Interpret V as a quotient of $B \times F$ by the following equivalence relation:

$$(a, x) \sim (g \cdot a, g^{-1} x) \text{ for some } \exists g \in \mu_n$$

Define $\mathbf{a}$ and $\mathbf{a}^\dagger$ as follows:

$$\begin{array}{ll} \mathbf{a}(a, x) = (b, xy)/\sim & \text{if there is } y \text{ such that } \mathbf{A}(a, b, y) \\ \mathbf{a}^\dagger(b, x) = (a, xy)/\sim & \text{if there is } y \text{ such that } \mathbf{A}^\dagger(b, a, y) \end{array}$$

□

Remark 3.2.2. The $\mathbb{C}$-algebra $A_1 = \langle \mathbf{a}, \mathbf{a}^\dagger \mid [\mathbf{a}, \mathbf{a}^\dagger] = 1 \rangle_{\mathbb{C}}$ is called the *Weyl algebra*. Equivalently, this is the ring of differential operators on the affine line, generated by x and $\partial/\partial x$. It is the algebra of observables of the simplest quantum system: a single one-dimensional particle oscillating in quadratic potential, hence the title of the article of Zilber and Solanki [40]. The sections of p form an infinite-dimensional $\mathbb{C}$-vector space, and $\mathbf{a}$ and $\mathbf{a}^\dagger$ define the action of respective generators of the Weyl algebra on this vector space. One can check that this defines an infinite-dimensional representation of the Weyl algebra.

Theorem 3.2.3 (Zilber and Solanki [40], Theorem 3.1)**.** *There is a Zariski geometry structure on QHO_n and Q_n.*

The proof in Zilber and Solanki [40] of non-definability of the structure Q_n in an algebraically closed field contains a gap: in the fourth paragraph of the proof of Proposition 2.2 the assumption that $b^k = a$ is unfounded (in fact, by Kummer tehory b^k could be equal to any element obtained as the result of applying a rational function to a). The proof we propose below fills this gap; it is based on general lemmas about linear spaces proved in Chapter 3.1.

Theorem 3.2.4. *The structure Q_n is not definable in an algebraically closed field.*

Proof. Suppose Q_n is interpretable in an algebraically closed field K. By Theorem 3.1.4, $F(K)$ is definably isomorphic to K, so we will further suppose that $F = K$. In particular, $X(F) = X(K) = \mathbb{A}^1_K$.

Let x be a point of $\mathbb{A}^1$ generic over K and consider $V_x = p^{-1}(x)$. From now on let us work in the model $L = acl(\bar{x}K)$.

By Proposition 3.1.7 V_x is isomorphic to $\{x\} \times L$. By Proposition 2.2.24 the points of $B(L) \cap V_x$ satisfy the equation $y^n = f(x)$ where f is a rational function defined over k. Then the points of $B(L) \cap V_{x+1}$ satisfy the equation $y^n = f(x+1)$.

Since the map $\mathbf{a} : V_x \to V_{x+1}$ is linear and is defined over x it is just multiplication $\{x\} \times L$ by some $g(x) \in K(x)$.

From the definition of the structure Q_n we have that for some $a \in B(L) \cap p^{-1}(x)$, some $b \in B(L) \cap p^{-1}(x+1)$ and some square root of x

$$ag(x) = b\sqrt{x}$$

By raising to the power $n = 2m$ we get

$$f(x)g^n(x) = f(x+1)x^m$$

I claim that for $f(x) \in K(x)$ there does not exist a rational function $g(x) \in K(x)$ that satisfies this identity.

First of all we can suppose that $f(x)$ is not only a rational function but a polynomial. Indeed, if for $f(x) = \dfrac{P(x)}{Q(x)}$ where $P, Q \in K[x]$ there is a $g(x)$ that satisfies the identity then $g(x)\dfrac{Q(x+1)}{Q(x)}$ satisfies it for $f(x) = P(x)Q^{n-1}(x)$.

We need to prove that $\dfrac{f(x+1)x^m}{f(x)}$ is not an n-th power. A reduced fraction is an n-th power if and only if both its numerator and denominator are n-th powers. Let the reduction of $\dfrac{f(x+1)x^m}{f(x)}$ be $\dfrac{A(x)}{B(x)}$. Then the degrees of $A(x)$ and $B(x)$ differ by m, so they cannot both be n-th powers.

□

The structure associated to $U_q(sl_2)$ for generic q

The following structure similar to the structure Q_n has been defined by Solanki [34].

Let n be a fixed even number. Define the following theory U_n in a two-sorted language. There is a sort V and a sort F, so that (V, X, F, p) is a linear space and F is an algebraically closed field of characteristic 0, $X = \mathbb{A}^1_F$ and all fibres are of dimension 1, with the following extra structure:

- there is a predicate $B \subset V$, such that for any $x \in \mathbb{A}^1_F$ the set $p^{-1}(x) \cap B = B_x$ is finite of size n and the action of μ_n, the group of n-th roots of unity, on B_x is free and transitive;

- let q be some non-torsion element of the multiplicative group $F^{\times}$. There are linear maps $E, F : V \to V$ where E, F map respectively V_x to V_{q^2x} and vice versa. For each $a \in B \cap V_x$ there is $b \in B \cap V_{q^2x}$ such that

$$Ea = \lambda(\sqrt{x})b \text{ and } Fb = -\lambda(q\sqrt{x})a$$

where the coefficient $\lambda(z)$ is defined to be $\dfrac{z^{-1}+z}{q-q^{-1}}$ and $\sqrt{x}$ is a square root of x;

- the map K maps a fibre $p^{-1}(x)$ to itself, it acts as a multiplication by x on this fibre.

Proposition 3.2.5. *The theory U_n is categorical in uncountable cardinals.*

Proof. Similarly to the proof of Proposition 2.1 in Zilber and Solanki [40], let us split $X = \mathbb{A}^1_K$ into orbits under the action of $\mathbb{Z}$ where the generator acts by multiplication by q^2.

Let M and N be two models of U_n of the same cardinality. By categoricity of the theory of algebraically closed fields we may assume that $F(M) = F(N)$.

Pick a representative x of an orbit O_x, and choose some isomorphism $\iota : V_x(M) \to V_x(N)$. Pick some vector $y \in V_x(M)$ and consider its image under the isomorphism, $y' \in V_x(N)$. Go over $\{q^{2k}x \mid k \in \mathbb{Z}\}$ and extend the isomorphism ι in such a way that $\iota(E^k y) = E^k(y')$. It is easy to check that this defines a partial isomorphism. By repeating the procedure for all orbits O_x, we get a total isomorphism. □

For the rest of the section fix an algebraically closed field K of characteristic 0.

Structure 3. A model of U_n where F is the field K is called the structure $U_q(sl_2)$.

Remark 3.2.6. The maps E, F, K generate a $\mathbb{C}$-algebra $U_q(sl_2)$, a deformation of the universal enveloping algebra of sl_2 (see Brown and Goodearl [3]). The term "generic" here refers to the fact that q is not a root of unity.

Theorem 3.2.7. *The Structure 3 is not definable in an algebraically closed field.*

Proof. The beginning of the proof is similar to that of Theorem 3.2.4.

Recall that if x is a point of $\mathbb{A}^1$ generic over K and $V_x = p^{-1}(x)$ is the fibre over x then the points of $B(L) \cap V_x$ satisfy the equation $y^n = f(x)$ and the points of $B(L) \cap V_{q^2x}$ satisfy the equation $y^n = f(q^2x)$.

Since the map $E : V_x(L) \to V_{q^2x}(L)$ is linear and is defined over L it is a multiplication by a factor $g(x)$ where $g(x) \in K(x)$.

For some $a \in B(L) \cap V_x$, some $b \in B(L) \cap V_{q^2x}$ and some square root of x

$$ag(x) = b\lambda(\sqrt{x})$$

By raising to the power $n = 2m$ we get

$$f(x)g^n(x) = f(q^2x)\frac{(x+1)^n}{x^m(q-q^{-1})^n}$$

I claim that for $f(x) \in K(x)$ there does not exist a rational function $g(x) \in K(x)$ that satisfies this identity.

Reasoning as in the previous section we suppose that $f(x)$ is a polynomial.

We need to prove that $\frac{f(q^2x)(x+1)^n}{f(x)x^m(q-q^{-1})^n}$ is not an n-th power. A reduced fraction is an n-th power if and only if both its numerator and denominator are n-th powers. Let the reduction of the fraction be $\frac{A(x)}{B(x)}$. Then the degrees of $A(x)$ and $B(x)$ differ by m, so they cannot both be n-th powers.

□

Group law on principal homogeneous spaces

The non-algebraicity of the structures discussed in two previous sections can be explained in terms of forms of principal homogeneous spaces and cohomology classes representing them.

Recall that $H^1(G, A)$ has a natural group structure in case of Abelian A. An explicit geometric construction that gives rise to the group operation was given in Weil [35].

Let A be a finite Abelian algebraic group. Let $g, h \in H^1(G, A)$ be cocycles corresponding to principal homogeneous spaces P_g, P_h. Define the action of A on $P_g \times P_h$ by

$$\gamma \cdot (a, b) = (\gamma \cdot p, \gamma^{-1} \cdot p')$$

We would like to take the quotient by this action in the following sense: find a variety $P_{g \cdot h}$ and a morphism $P_g \times P_h \to P_{g \cdot h}$ such that fibres of the morphism are precisely the orbits of the action.

Such quotient exists as a definable set by elimination of imaginaries. Existence of a variety structure on the quotient is not immediate, however, it is well-known that a quotient of a quasi-projective variety by an action of a finite group is a variety (see, e.g. Serre [33], III.12). One can define the following action of A on the quotient:

$$\gamma \cdot (a,b)/\sim = (\gamma \cdot a, b)/\sim$$

This is well-defined, as $(\gamma \cdot a, b) \sim (\gamma \cdot \zeta \cdot a, \zeta \cdot a)$ for any $\zeta \in A$ (that's where we use the commutativity of A).

Proposition 3.2.8. *This action makes the quotient into a principal homogeneous space $P_{g\cdot h}$ represented by the cocycle $g \cdot h$. The principal homogeneous space corresponding to the cocycle g^{-1} is the space P_g with the opposite action: an element $\gamma \in A$ acts as γ^{-1}.*

Proof. Recall that a cocycle corresponding to a principal homogeneous space under A is obtained like this. Fix an element $x \in P_g$, then

$$g_\sigma = a \text{ such that } ax = \sigma x$$

Let h be the cocycle obtained from an element $y \in P_h$. Consider the element of $P_{g\cdot h}$ which is the image of (x, y). Then $(g \cdot h)_\sigma = g_\sigma \cdot h_\sigma$ since $\sigma \cdot (x, y) = (g_\sigma x, h_\sigma y)$ and $(g_\sigma x, h_\sigma y) \sim (g_\sigma h_\sigma x, y)$.

Similarly, let considering an opposite action on a principal homogeneous space P_g yields the cocycle g^{-1}: $\sigma \cdot x = g_\sigma^{-1} x$.

□

A similar construction works for principal homogeneous spaces under Abelian varieties, see Propositions 4 and 5 in Weil [35]; this more general construction requires more work because one needs to construct a quotient under the action of a group which is in general not finite.

Let k be a perfect field and $X \subset (k^s)^m$ be a principal homogeneous space defined over k for an algebraic group A with the corresponding cocycle $h \in H^1(k, A)$. Let σ be an automorphism of k, then it lifts to an automorphism $\bar\sigma$ of k^s. Then the cocycle corresponding to the principal homogeneous space $\bar\sigma(X)$ is $\bar\sigma(h)$.

Consider the set $B_x = p^{-1}(x) \cap B$ in an elementary extension of the structure Q_n where x is a point of X generic over its field of definition (generic in the model-theoretic sense, which is well-defined since the structure Q_n is categorical and hence stable). Then $B_{x+1} = \bar{\sigma}(B_x)$ where σ is an automorphism of $F(x)$ that sends x to $x+1$. Consider the function $f : B_x \times B_{x+1} \to \mu_n\sqrt{x}$ that sends (e, e') to $z \in F$ such that $\mathbf{a}e = z \cdot e'$. By construction of Structure 2, $f(\zeta \cdot e, \zeta \cdot e') = f(e, e')$ where ζ is a primitive n-th root of unity. In other words f is invariant under the following action of μ_n: $\zeta(e, e') = (\zeta \cdot e, \zeta \cdot e')$.

Hence the image $\mu_n\sqrt{x}$ is the factor of $B_x \times B_{x+1}$ by this action and by Proposition 3.2.8 must be a principal homogeneous space corresponding to the cocycle $g \cdot (\bar{\sigma}(g))^{-1}$. The definition of the structure Q_n prescribes that this principal homogeneous space be $\mu_n\sqrt{x}$ to which corresponds a cocycle, say, $h \in H^1(F(x), \mu_n)$. Therefore if there is no cocycle $g \in H^1(F(x), \mu_n)$ such that $g \cdot (\bar{\sigma}(g))^{-1} = h$ we come to a contradiction and the structure Q_n is not definable in an algebraically closed field. This is actually the statement that concludes the proofs of Theorems 3.2.4 and 3.2.7.

Quantum torus

Zilber [37] has proposed a construction that produces a model-theoretic structure given a collection of modules over a non-commutative algebra and some other data. He proved that this structure is a Zariski geometry.

Zilber considers several examples of such structures associated to various collections of modules over non-commutative algebras.

Structure 4 (Quantum torus)**.** The Zariski structure M associated to the "quantum torus" algebra (Example 1 in Section 2.3 of 37, in the "weak language") is a linear space (V, T, F, p) over the algebraic torus $T = \mathbb{G}_m \times \mathbb{G}_m$ with maps $\mathbf{u}$ and $\mathbf{v}$ from V to itself which are fibrewise linear and are constructed as described below.

For every point $x = (a, b) \in T$ pick μ and ν, some n-th roots of a and b respectively, and choose some basis $e_0, \dots e_{n-1}$ in the fibre $V_x = p^{-1}(x)$. Let the action of $\mathbf{u}$ and $\mathbf{v}$ in this basis be defined by the following matrices

$$\mathbf{u} = \begin{pmatrix} \mu & 0 & \dots & 0 \\ 0 & q\mu & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & \dots & q^{n-1}\mu \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0 & 0 & \dots & 0 & \nu \\ \nu & 0 & \dots & 0 & 0 \\ 0 & \nu & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & & \dots & \nu & 0 \end{pmatrix}$$

Fibres of V are then modules over the algebra

$$\langle\, \mathbf{u}, \mathbf{v}, \mathbf{u}^{-1}, \mathbf{v}^{-1} \mid \mathbf{uv} = q\mathbf{vu}, \mathbf{uu}^{-1} = \mathbf{u}^{-1}\mathbf{u} = 1, \mathbf{vv}^{-1} = \mathbf{v}^{-1}\mathbf{v} = 1 \,\rangle_F$$

where $q^n = 1$ for some n.

Lemma 2.4 of [37] states that Structure 4 does not depend on the particular choice of μ and ν in each fibre, and Theorem 2.5 asserts that the structure M is not interpretable in an algebraically closed field. Here is a different proof of the latter.

Theorem 3.2.9. *Structure 4 is not definable in an algebraically closed field.*

Proof. Similarly to the previous proofs, suppose M is interpretable in an algebraically closed field K, and by Theorem 3.1.4, conclude that K is definably isomorphic to F, and $T(F) = T(K) = T$.

By Proposition 3.1.7, the fibre V_x over a generic point $\bar{x} = (x_1, x_2)$ of $T(K)$ is definably isomorphic over $dcl(\bar{x}K)$ to $\mathbb{A}^n_K$. We work further in the field $L = \mathrm{acl}(\bar{x}K)$.

If $\mathbf{u}$ and $\mathbf{v}$ are definable in K then their restrictions to V_x are definable over $K(x_1, x_2) = \mathrm{dcl}(\bar{x}K)$. Using $+$ and $\cdot$ operations of the linear space we can define any linear combination of $\mathbf{u}$ and $\mathbf{v}$ and their compositions with coefficients in $K(x)$. Denote this $K(x)$-algebra of definable endomorphisms E_x. Let us show that $E_x \cong M_n(K(x))$.

Since $\mathbf{uv} = q\mathbf{vu}$ any element of the algebra E_x is a linear combination with coefficients in $K(x_1, x_2)$ of monomials of the form $\mathbf{u}^i\mathbf{v}^j$, $0 \le i \le n-1, 0 \le j \le n-1$. We will prove that these monomials are linearly independent as operators from $\mathrm{End}_L(V_x)$. It will follow that the vector space of linear combinations of these monomials with coefficients in $K(x_1, x_2)$ is n^2-dimensional, and hence equal to $M_n(K(x_1, x_2))$.

The matrix of the monomial $\mathbf{u}^i\mathbf{v}^j$ in the distinguished basis has the following form (here μ and ν are n-th roots of x_1 and x_2):

$$\begin{pmatrix}
0 & \dots & 0 & q^{(n-j)i}\mu^i\nu^j & 0 & \dots & 0 \\
0 & \dots & 0 & 0 & q^{(n-j+1)i}\mu^i\nu^j & \dots & 0 \\
\vdots & & & & & & \vdots \\
0 & & & & & & q^{(n-1)i}\mu^i\nu^j \\
\mu^i\nu^j & 0 & & & & \dots & 0 \\
0 & q^i\mu^i\nu^j & 0 & & & \dots & 0 \\
\vdots & & & & & & \vdots \\
0 & \dots & q^{(n-j-1)i}\mu^i\nu^j & 0 & & \dots & 0
\end{pmatrix}$$

where the non-zero terms occupy the j-th diagonal. The linear spans of $\{\mathbf{u}^i\mathbf{v}^j \mid 0 \le i < n\}$ are clearly pairwise linearly disjoint for different values of j. Let us show that for a fixed j, $\{\mathbf{u}^i\mathbf{v}^j, 0 \le i < n\}$ are linearly independent.

Consider the determinant of the matrix where the i-th column is the j-th diagonal of $\mathbf{u}^i\mathbf{v}^j$. After some simplification using the basic properties of determinant we get

$$(\prod_i \mu^i)\nu^{nj} \begin{vmatrix}
1 & 1 & 1 & \dots & 1 \\
1 & q & q^2 & \dots & q^{n-1} \\
1 & q^2 & q^4 & \dots & q^{2(n-1)} \\
\vdots & & & \vdots & \\
1 & q^{n-1} & q^{2(n-1)} & \dots & q^{(n-1)^2}
\end{vmatrix}$$

this determinant is a Vandermonde determinant, and the whole expression equals $(\prod_i \mu^i)\nu^{nj}\prod_{0\le i<j\le n-1}(q^i - q^j)$, which is clearly non-zero.

We have therefore proved that $E_x \cong M_n(K(x_1, x_2))$.

But from the definition of the structure it is clear that E_x is not the matrix algebra $M_n(K(x_1, x_2))$: for example, there is no n-th root of x_1 in $M_n(K(x_1, x_2))$, but there is one in E_x. We have come to a contradiction. □

Definition 3.2.10 (Central simple algebra)**.** An algebra A over a field k is called *central simple* if $A \otimes k^s$ is isomorphic to the full matrix algebra $M_n(k^s)$ over k^s.

3.3 Twisted forms

Let X be a variety over a field k and let K be an extension of k. From the point of view of scheme theory to consider the variety X as a variety over the field K is to consider the fibre product $X \times_{\operatorname{Spec} k} \operatorname{Spec} K$, the base change via the morphism

$\operatorname{Spec} K \to \operatorname{Spec} k$. The correspondence between forms and group cohomology classes (cf. Section 2.2) can be re-expressed in this language. As a consequence a generalisation to the situation where the base is not a field is possible.

Let X be a variety. The category of *varieties over* X is the category of varieties with a fixed morphism to X (the structure morphism), where a morphism between $Y \xrightarrow{f} X$ and $Z \xrightarrow{g} X$ is a morphism $Y \xrightarrow{h} Z$ such that $f = g \circ h$. A twisted form of a variety Y over X is a variety Y' over X that becomes isomorphic to Y after a base change (of a particular form). Similarly to the situation over a field a twisted form defines an element in certain cohomology group, and every cocycle of this cohomology group defines a descent datum in a sense to be defined, that allows to obtain the corresponding twisted form. A notion of a twisted form makes sense for any object for which the base change makes sense, e.g. sheaves of modules, sheaves of algebras etc.

In order to formulate this notion precisely we need some background on Grothendieck topologies, Čech cohomology and flat descent.

Grothendieck topologies and Čech cohomology

In many contexts one encounters objects that are not Zariski locally trivial but become trivial after a base change. It is a natural to think about such objects as "locally trivial", but of course it does not immediately make sense in Zariski topology. Grothendieck topologies provide a framework that allows to express formally such notion of locality. Bellow we will recall the notions and results about Grothendieck topologies, sheaves and cohomology of sheaves on the étale site of variety that will be necessary later in the chapter. See Milne [20] for a more detailed exposition of the subject.

Let us first recall the definition of sheaf in the Zariski topology.

Definition 3.3.1 (Sheaf on a topological space)**.** Let X be a topological space. Its open sets form a category $Op(X)$ where morphisms are inclusions.

A *presheaf (of sets, groups, etc.)* over a scheme X (more generally, over any topological space) is a contravariant functor from $Op(X)$ to the category of sets, groups, etc.

For any open $V \subset X$ and any cover of V by open sets $\{U_i\}$ let U be the disjoint union $\sqcup U_i$ and consider the natural morphism $\sqcup U_i \to V$. A *sheaf* is a presheaf $\mathscr{F}$ such that for any V and U as above the equalizer diagram

$$\mathcal{F}(V) \xrightarrow{g} \mathcal{F}(U) \overset{f_1,f_2}{\rightrightarrows} \mathcal{F}(U \times_V U)$$

commutes, i.e. $f_1 \circ g = f_2 \circ g$, and g is injective. The maps f_1, f_2 here are images of natural inclusions $U \times_V U = \cup(U_i \cap U_j) \hookrightarrow U$ under $\mathcal{F}$, the map g takes a section s of $\mathcal{F}$ over V to the section of $\mathcal{F}$ over U that coincides with $res^V_{U_i}(s)$ over U_i, where $res^V_{U_i}$ is the restriction map of the presheaf.

Remark 3.3.2. The definition of a sheaf of Abelian groups can be written using the exact sequence

$$0 \to \mathcal{F}(V) \to \mathcal{F}(U) \xrightarrow{f} \mathcal{F}(U \times_V U)$$

where $f = f_1 - f_2$.

This definition is formulated in terms of morphisms and fibre products and is not tied to a particular structure of the category $Op(X)$. Grothendieck has taken some properties of this category that are necessary to produce a good cohomology theory as defining axioms of a *Grothendieck topology.*

Definition 3.3.3 (Grothendieck topology)**.** Fix a category $\mathcal{C}$ which has fibre products. A *Grothendieck topology* E on $\mathcal{C}$ is a collection of distinguished families of morphisms $\{U_i \to U\}$ for each object $U \in \mathcal{C}$, called *coverings*, such that:

- a family consisting of a single identity map is a covering;
- if $\{f_i : U_i \to U\}$ and $\{g_{ij} : V_{ij} \to U_i\}$ are coverings then $\{g_{ij} \circ f_i : V_{ij} \to U\}$ is a covering (coverings are closed under composition);
- if $f : U \to V$ is a morphism and $\{V_i \to V\}$ is a covering then $\{U \times_V V_i \to V\}$ is also a covering (coverings are stable under base change).

A category with a Grothendieck topology is called a *site.*

Definition 3.3.4 (Zariski site)**.** Fix a variety X. Consider the category of all varieties over X such that the structure morphism is an open immersion. Call a collection of jointly surjective inclusions of open sets $\{U_i \hookrightarrow X\}$ a covering. This defines a *Zariski site* on the category, denoted X_{Zar}.

Definition 3.3.5 (Sheaf on a site)**.** Let X be a site. A *presheaf of sets, groups, etc.* $\mathcal{F}$ on X is just a contravariant functor from X to the category of sets, groups, etc.

Consider for any cover $\{U_i \to U\}$ the equaliser sequence

$$\mathcal{F}(V) \xrightarrow{g} \mathcal{F}(U) \overset{f_1,f_2}{\rightrightarrows} \mathcal{F}(U \times_V U)$$

where $U = \sqcup U_i$. A presheaf $\mathcal{F}$ is *separated* if g is injective and it is a *sheaf* if the equaliser sequence is exact, that is, g is injective and $f_1 \circ g = f_2 \circ g$.

A *morphism of (pre)sheaves* is a natural transformation of functors.

Let $\mathscr{U} = \{U_i \to X\}$ be a covering on a site X_E and let $\mathcal{F}$ be a presheaf of Abelian groups on X_E. We will write the fibre product $U_{i_0} \times_X \ldots {}_X \times U_{i_k}$ as $U_{i_0 \ldots i_k}$. If $s_{i_0 \ldots i_k}$ is a section of $\mathcal{F}$ over $U_{i_1 \ldots i_k}$ then we will denote the image of it under the restriction morphism as $s_{i_0 \ldots \hat{i_j} \ldots i_k}$ where the hat means omission of the index.

The Čech complex is defined as the sequence of abelian groups

$$C^k(\mathscr{U}, \mathcal{F}) = \prod_{(i_0,\ldots,i_k)\in I^{k+1}} \mathcal{F}(U_{i_0 \ldots i_k})$$

with a differential $d^k : C^k(\mathscr{U}, \mathcal{F}) \to C^{k+1}(\mathscr{U}, \mathcal{F})$ defined as

$$(d^k s_{i_0 \ldots i_k}) = \sum_{j=0}^{k+1} (-1)^j s_{i_0 \ldots \hat{i_j} \ldots i_{k+1}}$$

One checks that $d^k d^{k+1} = 0$ for all k.

Definition 3.3.6 (Čech cohomology relative to a covering)**.** The *Čech cohomology groups* of the presheaf $\mathcal{F}$ for the covering $\mathscr{U}$ are defined as

$$\check{H}^k(\mathscr{U}, \mathcal{F}) = \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}$$

for $k > 0$ and $\check{H}^0(\mathscr{U}, \mathcal{F}) = \operatorname{Ker} d^0$.

A covering $\mathscr{V} = \{V_i \to X\}$ is called a refinement of $\mathscr{U}$ if the morphisms $V_i \to X$ factor through the morphisms that belong to $\mathscr{U}$. One can show that given a refinement $\mathscr{V}$ of $\mathscr{U}$ there is a canonical map $\check{H}^k(\mathscr{U}, \mathcal{F}) \to \check{H}^k(\mathscr{V}, \mathcal{F})$ which only depends on the coverings (Lemma III.2.1 in Milne [20]). Thus the k-th Čech cohomology group is a functor from the direct system of coverings, and it makes sense to define a direct limit of $\check{H}^p$ over all such coverings.

Definition 3.3.7 (Čech cohomology)**.** The *Čech cohomology groups* of a presheaf $\mathcal{F}$ on X are the direct limits $\check{H}^k(X_E, \mathcal{F}) = \varinjlim \check{H}^k(\mathscr{U}, \mathcal{F})$ with the limit taken over all coverings of X in the given topology E.

Let G be a presheaf of (not necessarily Abelian) groups and let $\mathscr{U} = \{U_i \to X\}$ be a covering. A 0-cocycle $g \in \check{H}^0(\mathscr{U}, G)$ is a collection of sections $g_i \in G(U_i)$ such that $g_i|_{U_{ij}} = g_j|_{U_{ij}}$.

The following criterion is just a restatement of the gluing condition of a sheaf in terms of $\check{H}^0$.

Proposition 3.3.8. *A presheaf of groups $\mathcal{F}$ on X_E is a sheaf if and only if for any object U of the site and for any covering $\mathscr{U} = (U_i \to U)$, $\mathcal{F}(U) \cong \check{H}^0(\mathscr{U}, \mathcal{F})$.*

Theorem 3.3.9 (Theorem II.2.11 in Milne [20])**.** *Let P be a presheaf on a site X_E. Then there is a sheaf P' on X_E such that the following universal property holds. There is a morphism $\varphi : P \to P'$ such that for any morphism $\varphi' : P \to F$ there exists a morphism $\psi : P' \to F$ such that*

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P' \\ {\scriptstyle \varphi'}\downarrow & \swarrow {\scriptstyle \psi} & \\ F & & \end{array}$$

commutes.

Let us write down explicitly what it means to be 1-cocycle. As in the Galois cohomology case we then can ease the assumption that the groups are Abelian, and still get a meaningful definition of $\check{H}^1$.

Let $\mathscr{U} = \{U_i \to U\}$ be a covering and G be a presheaf on a site X_E. Then a 1-cocycle g of $\check{H}^1(\mathscr{U}, G)$ is a collection of elements $g_{ij} \in G(U_{ij})$ such that

$$(g_{ij}|_{U_{ijk}}) \cdot (g_{jk}|_{U_{ijk}}) = g_{ik}|_{U_{ijk}}$$

Two cocycles g, g' are called cohomologous if there is $h \in \check{H}^0(\mathscr{U}, G)$ such that

$$g_{ij} = (h_i|_{U_{ij}}) \cdot g'_{ij} \cdot (h_j|_{U_{ij}})^{-1}$$

Étale morphisms

Étale topology is traditionally defined on the category of schemes, so we will use the language of the theory of schemes in our exposition. In this language abstract algebraic varieties in the sense of Weil correspond to reduced schemes of finite type over a field.

Definition 3.3.10 (Flat morphism)**.** A (commutative) algebra S over a ring R is called *flat* if it is flat as a module, i.e. tensoring with S is an exact functor (see Definition 2.3.25).

A morphism of varieties $f : X \to Y$ is called *flat* if for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$. A morphism is called *faithfully flat* if it is flat and surjective.

One can show that a morphism of affine varieties $f : Y = \operatorname{Spec} S \to X = \operatorname{Spec} R$ is flat iff S is flat as an R-algebra (Milne [20], Proposition I.2.2).

It follows easily from the definition that any field extension is flat. Informally speaking, a flat morphism is a morphism such that the fibres vary continuously. For example, a finite morphism is flat if and only if its fibres have the same number of points, counting multiplicities (Milne [20], Theorem I.2.9).

Another important example of a flat morphism is a Zariski cover. Let $X = \cup X_i$ is an open cover of X, then there is a natural morphism $p : \sqcup X_i \to X$. The morphism p is flat, since flatness is a property local on the source and open immersions are flat (Milne [20], Proposition I.2.4).

Definition 3.3.11 (Étale morphism)**.** Let $f : Y \to X$ be a morphism of schemes. It is called *unramified at a point* $y \in Y$ if $\mathfrak{m}_y = \mathfrak{m}_{f(y)}\mathcal{O}_{Y,y}$ and $\kappa(y)$ is a finite separable extension of $\kappa(x)$ (where $\kappa(y)$ and $\kappa(x)$ are respective residue fields). A morphism is called *unramified* if it is unramified at all points of X. A morphism is called *étale* if it is flat and unramified.

The condition on residue fields is superfluous for closed points in case X and Y are schemes over an algebraically closed field.

Example 3.3.12. Let X be the nodal cubic $y^2 = x^3 + x^2$. The map $\tilde{X} \to X$ where $\tilde{X}$ is the normalisation is finite, surjective and unramified. However, this cover is not étale, since its fibres consist of one point everywhere except the singularity, where two points of the normalisation are projected to one on X.

Example 3.3.13. Let k be a field and let $X = \mathbb{A}^1_k - \{0\}$. The map $f : X \to X$ defined by the formula $f(x) = x^m$ is étale.

Example 3.3.14. Let $Y = \operatorname{Spec} B$ and let $A = B[x]/(p(x))$ where $p(x) \in B[x]$ is a monic polynomial such that the image of $p'(x)$ in A is a unit. Then the morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is étale. One can show that any étale morphism is of this form in some affine neighbourhood (Milne [20], Theorem I.3.14).

Example 3.3.15. A morphism $\operatorname{Spec} K \to \operatorname{Spec} k$ is étale if and only the field extension K/k is separable.

Étale morphisms form a class that satisfies the requirements of the Definition 3.3.3.

Definition 3.3.16 (Étale site)**.** Let X be a scheme. A *(small) étale site* $X_{\text{ét}}$ is defined to be the full subcategory of the category of schemes over X such that:

- it is closed under fibre products;
- for any scheme Y in $X_{\text{ét}}$ with the structure morphism f and any étale morphism $U \xrightarrow{g} Y$, the composition $g \circ f$ is in $X_{\text{ét}}$;

A cover is defined to be a family of morphisms $(U_i \xrightarrow{f_i} Y)$ such that Y is the union of images of g_i-s.

As a consequence of the following Proposition, restricting our attention to reduced schemes (say, varieties) does not leave out any étale morphisms.

Proposition 3.3.17 (Milne [20], Remark I.3.17)**.** *If $f : X \to Y$ is an étale morphism and Y is reduced then X is reduced too.*

Under some mild assumptions the Čech cohomology coincides with the derived functor of the global sections functor on the category of sheaves over an étale site.

Theorem 3.3.18 (Milne [20], Theorem III.2.17)**.** *Let X be a quasi-compact scheme such that every finite subset of X is contained in an affine open (e.g. X is a quasi-projective variety). Then the functors $\check{H}^k(X_{\text{ét}}, -)$ are right derived functors of $\Gamma(X, -)$ for all $k > 0$. In particular, for every short exact sequence*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

one has the long exact sequence

$$\begin{aligned} 0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B}) &\to \Gamma(X, \mathcal{C}) \to \check{H}^1(X, \mathcal{A}) \to \\ &\to \check{H}^1(X, \mathcal{B}) \to \check{H}^1(X, \mathcal{C}) \to \check{H}^2(X, \mathcal{A}) \to \dots \end{aligned}$$

A similar statement about non-Abelian $\check{H}^1$ is also true, once we define what it means for sheaves of non-Abelian groups to form a short exact sequence.

Definition 3.3.19 (Short exact sequence of sheaves of groups)**.** Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be sheaves of (multiplicative) groups on a site X_E. We say that

$$1 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 1$$

is a short exact sequence if for any $U \in X_E$, $\mathcal{A}(U)$ is the kernel of $g|_U : \mathcal{B}(U) \to \mathcal{C}(U)$ and if for any section $s \in \mathcal{C}(U)$ there exists a covering $\{U_i \to U\}$ and sections $s_i \in \mathcal{B}(U_i)$ such that $g(s_i) = s|_{U_i} \in \mathcal{C}(U_i)$.

Proposition 3.3.20 (Proposition III.4.5, Milne [20]). *To any exact sequence of sheaves of (not necessarily Abelian) groups*

$$1 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 1$$

on a site X_E corresponds an exact sequence of pointed sets

$$\begin{aligned} 1 \to \mathcal{A}(X) \to \mathcal{B}(X) &\to \mathcal{C}(X) \to \\ &\to \check{H}^1(X,\mathcal{A}) \to \check{H}^1(X,\mathcal{B}) \to \check{H}^1(X,\mathcal{C}) \end{aligned}$$

Galois coverings

Example 3.3.15 illustrates the idea that an étale morphism is a relative generalisation of a separable field extension. The notion of a Galois extension of a field also has its relative counterpart.

Definition 3.3.21 (Galois covering, cf. [SGA 1], Exposé V). A faithfully flat morphism $f : X \to Y$ is called a *Galois covering* (or a *Galois morphism*, or a *principal covering*) if there is a finite group G acting on X on the left such that the morphism

$$\begin{gathered} X \times G \to X \times_Y X \\ (x, \sigma) \mapsto (x, \sigma(x)) \end{gathered}$$

is an isomorphism (where $X \times G$ is to be understood as the disjoint union of $|G|$ copies of X indexed by elements of the group). The group G is called the *Galois group* of the morphism f.

To any two points x, y from the same fibre corresponds a point (x, y) in the fibre product which belongs to a graph of a unique automorphism σ of X over Y, $\sigma(x) = y$. Therefore the Galois group G of f acts acts freely and transitively on the fibres of f (this can be taken as an equivalent definition of a Galois covering). Since f is unramified, it is thus étale (see [SGA 1], Exposé V, Proposition 2.6).

If a finite group G acts *admissibly* on X (i.e. the every orbit is contained in an affine open set) then there exists a scheme X/G and a finite morphism $p : X \to X/G$ such the fibres of p are the orbits of the action of G ([SGA 1], Exposé V, Proposition 1.3). In particular, if the action of G is free, then $X \to X/G$ is a Galois covering.

If $X \to Y$ is a Galois covering with a Galois group G, one can consider quotients of X by subgroups $H \subset G$; for each such quotient $X' = X/H$ the morphism $X \to X'$ is Galois again. Reciprocally, any étale cover is a quotient of a Galois cover (compare this to the existence of the splitting field of a polynomial in Galois theory of fields).

Proposition 3.3.22 (Serre [31], Section 1.5)**.** *For every étale morphism $f : X \to Y$ there exists a Galois morphism $\tilde{f} : \tilde{Y} \to X$ such that $\tilde{f}$ factors through f.*

Group schemes

Definition 3.3.23. Let G be a scheme over a scheme S, and let $m : G \times_S G \to G$, $e : S \to G$, $i : G \to G$ be a morphism . The tuple (G, m, e, i) is called a group scheme if m, e and i, viewed as multiplication, identity and inverse, satisfy the axioms of groups: associativity of m, identity axiom for e and m, and the inverse axiom for i. In particular, the set of morphisms $\mathrm{Hom}_S(X, G)$ is a group for any scheme X over S.

Example 3.3.24. Let S be a variety over a field k. Then $\mathbb{G}_m(S)$ is defined to be $S \times (\mathbb{A}^1_k \setminus \{0\})$ with multiplication given my the morphism $((s, x), (s, y)) \mapsto (s, xy)$.

Example 3.3.25. In general, let G be a group scheme over a field k and let S be a variety over k. Then $S \times G$ has a natural structure of a group scheme (with fibrewise multiplication).

The group scheme $\mathrm{GL}_n(S)$ can be regarded as the group of linear automorphisms of the trivial rank n vector bundle over S, and the group scheme $\mathrm{PGL}_n(S)$ can be regarded as the group of projective linear automorphisms of the trivial bundle.

Example 3.3.26. Let S be connected and let E be a locally free rank n vector bundle over S which is free over open sets U_i that cover S. Then there is a group scheme $\mathrm{GL}(E)$ is the group scheme which locally over U_i looks like $\mathrm{GL}_n(U_i)$. We leave out the details of the construction.

Group schemes are a vast source of sheaves of groups on an étale site. Let G be a group scheme over S. Then the following correspondence defines a functor from the category of schemes over S to the category of groups

$$\mathcal{G}(U) = \mathrm{Hom}_S(U, G)$$

Proposition 3.3.27 (Milne [20], Corollary II.1.7)**.** *The presheaf* $\mathcal{G}$ *is a sheaf on* $S_{\text{ét}}$*.*

Relationship between Čech and Galois cohomology

It will be convenient to write out in explicit form what Čech cocycles look like in the case of a covering $\mathscr{U}$ that consists of a single Galois morphism $U \to X$ with Galois group Γ.

Recall that being a Galois morphism by definition means that the following map is an isomorphism

$$\Gamma \times U \xrightarrow{\sim} U \times_X U \qquad (\sigma, u) \mapsto (u, \sigma \cdot u)$$

By applying this isomorphism twice to various factors of $\Gamma \times \Gamma \times U$ we obtain the following isomorphism

$$\Gamma \times \Gamma \times U \xrightarrow{\sim} U \times_X U \times_X U \qquad (\sigma, \tau, u) \mapsto (u, \sigma \cdot u, (\sigma\tau) \cdot u)$$

Using these isomorphisms we can rewrite the three projections $U \times_X U \times_X U \to U \times_X U$ as follows:

$$\begin{aligned} \pi_{12}(\sigma, \tau, u) &= (\sigma, u) \\ \pi_{23}(\sigma, \tau, u) &= (\tau, \sigma \cdot u) \\ \pi_{13}(\sigma, \tau, u) &= (\sigma\tau, u) \end{aligned}$$

Proposition 3.3.28. *Let* $\mathscr{U} = (U \to X)$ *be a covering of a scheme* X *such that* $U \to X$ *is Galois with Galois group* Γ*. Let* $\mathcal{F}$ *be a sheaf of groups on* $X_{\text{ét}}$*. Then* Γ *naturally acts on* $\mathcal{F}(U)$ *and*

$$\check{H}^1(X_{\text{ét}}, \mathcal{F}) \cong H^1(\Gamma, \mathcal{F}(U))$$

Proof. A cocycle in $\check{H}^1(X_{\text{ét}}, \mathcal{F})$ is a section $h \in \mathcal{F}(U \times_X U)$ subject to a cocycle condition $(\pi_{13})^* h = (\pi_{23})^* h \cdot (\pi_{12})^* h$. Define the Galois cocycle $\{\bar{h}_\sigma \in \mathcal{F}(U)\}_{\sigma \in Gamma}$ as follows:

$$\bar{h}_\sigma = h|_{\{\sigma\} \times U}$$

The pullbacks by π_{ij} of h correspond to sections of $\mathcal{F}(\Gamma \times \Gamma \times U)$ which, taking into account the discussion before the theorem, can be written down as follows

$$\begin{aligned} (\pi_{12}^* h)_{\sigma,\tau} &= h_\sigma \\ (\pi_{23}^* h)_{\sigma,\tau} &= \sigma(h_\tau) \\ (\pi_{13}^* h)_{\sigma,\tau} &= h_{\sigma\tau} \end{aligned}$$

and therefore the cocycle condition for Čech cohomology rewrites as the cocycle condition for Galois cohomology: $h_{\sigma\tau} = h_\sigma \cdot \sigma(h_\tau)$.

□

Using a similar strategy one can show that for a sheaf of Abelian groups all Čech cohomology groups are isomorphism to Galois cohomology groups.

Proposition 3.3.29. *Let $\mathcal{F}$ be a sheaf of Abelian groups on $X_{\text{ét}}$, and let $\mathscr{U} = (U \to X)$ be a covering consisting of a single Galois morphism with the Galois group Γ. Then*

$$\check{H}^i(X_{\text{ét}}, \mathcal{F}) \cong H^i(\Gamma, \mathcal{F}(U))$$

Let $1 \to A \to B \to C \to 1$ be a sequence of non-commutative G-groups such that A is central in B. Then there exists a connecting morphism that extends the long exact sequence of cohomology to H^2 term. Take a 1-cocycle $\{h_\sigma\}$ with the values in C. Lift it to $\{b_\sigma\}$ in B and define:

$$a_{\sigma,\tau} = b_\sigma {}^\sigma b_\tau b_{\sigma\tau}^{-1}$$

By Serre [32], Section I.5.6 $\{a_{\sigma,\tau}\}$ is a cocycle in $H^2(\Gamma, A)$, and this formula defines a map $\delta : H^1(\Gamma, C) \to H^2(\Gamma, A)$.

Proposition 3.3.30 ([32], Proposition 43)**.** *The sequence*

$$\ldots \to H^1(\Gamma, A) \to H^1(\Gamma, B) \to H^1(\Gamma, C) \xrightarrow{\delta} H^2(\Gamma, A)$$

is exact.

This induces a connecting morphism in non-Abelian Čech étale cohomology via the isomorphisms constructed in Propositions 3.3.29 and 3.3.28.

$$\ldots \to \check{H}^1(\mathscr{U}, \mathcal{A}) \to \check{H}^1(\mathscr{U}, \mathcal{B}) \to \check{H}^1(\mathscr{U}, \mathcal{C}) \xrightarrow{\delta} \check{H}^2(\mathscr{U}, \mathcal{A})$$

Flat and Galois descent

Detailed treatment of flat descent can be found in Bosch et al. [1], Chapter 6, Milne [20], Chapter I, §2 and Knus and Ojanguren [17], Chapter 2.

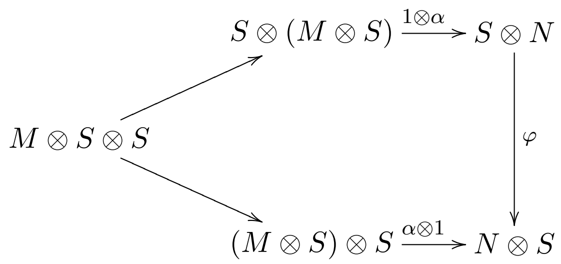
Let $p : Y \to X$ be a morphism. Suppose we are given a quasi-coherent sheaf $\mathcal{F}$ over Y. Descent is a tool that allows us to find out if $\mathcal{F}$ is of the form $p^*\mathcal{G}$ where $\mathcal{G}$ is some sheaf on X, and calculate the sheaf $\mathcal{G}$, using a piece of information called *descent datum*.

Suppose for simplicity that X and Y are affine, $X = \operatorname{Spec} R, Y = \operatorname{Spec} S$, and so the sheaf of modules $\mathcal{F}$ corresponds to an S-module N. Consider a map $\varphi : S \otimes_R N \to N \otimes_R S$ of $S \otimes_R S$-modules. This morphism induces three morphisms of $S \otimes S \otimes S$-modules:

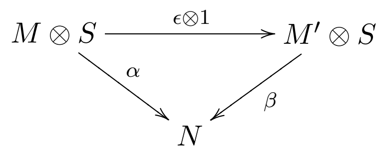
$$\varphi_{12} : S \otimes N \otimes S \to N \otimes S \otimes S$$
$$\varphi_{23} : S \otimes S \otimes N \to S \otimes N \otimes S$$
$$\varphi_{13} : S \otimes S \otimes N \to N \otimes S \otimes S$$

Definition 3.3.31 (Descent datum for modules)**.** The map φ is called a *descent datum* for a module N if $\varphi_{23} \circ \varphi_{12} = \varphi_{13}$.

Theorem 3.3.32 (Knus and Ojanguren [17], Theoréme II.3.2)**.** *Let S be a faithfully flat R-algebra, and let φ be a descent datum for an S-module N. Then there exists an R-module M and an isomorphism $\alpha : M \otimes S \to N$ such that the following diagram commutes:*



and whenever there is an R-module M' and a map $\beta : M' \otimes S \to N$ that satisfies the above diagram, there is an isomorphism $\epsilon : M \to M'$ that makes the following diagram commute:



The theorem can be generalized to algebras as well:

Theorem 3.3.33 (Knus and Ojanguren [17], Theorème II.3.4). *If N is an S-algebra (not necessarily commutative) and the descent datum $\varphi : S \otimes N \to N \otimes S$ is an isomorphism of $S \otimes S$-algebras, then the module M from the Theorem 3.3.33 can be endowed with a structure of an algebra such that the isomorphism $\alpha : M \otimes S \to N$ is an isomorphism of S-algebras.*

This can be further generalized to a non-affine base, but we will stick to the affine case for simplicity.

When $p : Y = \operatorname{Spec} S \to X = \operatorname{Spec} R$ is a Galois morphism, a descent datum has a particularly simple form. Recall that by the definition of a Galois morphism $\sqcup_{\sigma \in G} Y_\sigma \xrightarrow{\sim} Y \times_X Y$ where Y_σ is a copy of Y, $G = \operatorname{Gal}(Y/X)$ and the isomorphism is given by the map $y \mapsto (y, \sigma(y))$ for $y \in Y_\sigma$. Dually, it means that the map $S \otimes_R S \to \oplus_{\sigma \in G} S_\sigma, s \otimes t \mapsto s \cdot \sigma(t)$ is an isomorphism; here again S_σ are just copies of S indexed by elements of the Galois group.

Let $\varphi : S \otimes_R A \to A \otimes_R S$ be a descent datum for A. The isomorphism between $\oplus S_\sigma$ and $S \otimes_R S$ is given by the formula:

$$(s_e, \ldots, s_\sigma, \ldots, s_\tau) \mapsto s_e \otimes s_e + \ldots + s_\sigma \otimes \sigma(s_\sigma) + \ldots + s_\tau \otimes \tau(s_\tau)$$

Taking into account the isomorphisms

$$(S \otimes_R S) \otimes_S A \cong S \otimes_R A \qquad A \otimes_S (S \otimes_R S) \cong A \otimes_R S$$

we get the following commutative diagram:

$$\begin{array}{ccc} (\oplus S_\sigma) \otimes_S A & \xrightarrow{\varphi} & A \otimes_S (\oplus S_\sigma) \\ \uparrow{\scriptstyle id \otimes 1} & & \uparrow{\scriptstyle 1 \otimes id} \\ \oplus S_\sigma & = & \oplus S_\sigma \end{array}$$

where two instances of $\oplus S_\sigma$ in the upper row have two different structures of an S-module: when s acts as $s \otimes 1$ and when s acts as $1 \otimes s$. Then notice that the following diagram commutes:

$$\begin{array}{ccc} A \otimes_S S_\sigma & \xrightarrow{\psi_\sigma} & A \\ \uparrow & & \uparrow \\ S & \xrightarrow{\sigma} & S \end{array}$$

where $\psi(a \otimes s) = \sigma(s)a$. Note that $S_\sigma \otimes_S A$ is isomorphic to A over S. We obtain a map $h_\sigma = \psi_\sigma \circ \varphi$ of R-modeles

$$\begin{array}{ccc} A & \xrightarrow{h_\sigma} & A \\ \uparrow & & \uparrow \\ S & \xrightarrow{\sigma} & S \end{array} \qquad (*)$$

The collection of maps h_σ is enough to get back the descent datum φ.

Theorem 3.3.34. *Let S be an R-algebra such that $\operatorname{Spec} S \to \operatorname{Spec} R$ is a Galois group with the Galois group G. Let A be an S-algebra and let $\{h_\sigma\}$ be the descent datum for Galois descent, i.e. a collection of maps $h_\sigma : A \to A$ such that the diagram $(*)$ commutes. Then there exists an algebra B and a morphism $\eta : B \otimes_R S \to A$ such that the following diagram commutes*

$$\begin{array}{ccc} B \otimes S & \xrightarrow{\eta} & A \\ \downarrow{\scriptstyle id\otimes\sigma} & & \downarrow{\scriptstyle h_\sigma} \\ B \otimes S & \xrightarrow{\eta} & A \end{array}$$

for any $\sigma \in G$.

Proof. To define a descent datum $\varphi : S \otimes A \to A \otimes S$ it suffices to define the morphisms $S_\sigma \otimes_S A \to A \otimes_S S_\sigma$ of algebras over S_σ:

$$s \otimes a \mapsto h_\sigma(a)\sigma^{-1}(s) \otimes s$$

and then take a direct sum of them.

The cocycle condition $\varphi_{13} = \varphi_{23} \circ \varphi_{12}$ amounts to the condition $h_{\sigma\tau} = h_\tau \circ h_\sigma$.

We have observed that a collection of maps h_σ such that each map makes the diagram $(*)$ commute and such that $h_{\sigma\tau} = h_\tau \circ h_\sigma$ defines a descent datum. □

A particular case of Theorem 3.3.32 (for affine schemes instead of algebras, but this amounts to the same due to the duality between the two) when f is a morphism $f : \operatorname{Spec} K \to \operatorname{Spec} k$ corresponding to the Galois extension of fields K/k has been discussed in Section 2.2, before the statement of Theorem 2.2.11.

We now state, for later use, a theorem on descent of morphism of algebras.

Theorem 3.3.35 (Knus and Ojanguren [17], Theoréme II.5.3). *Let $R \hookrightarrow S$ be an extension of algebras corresponding to a Galois morphism with the Galois group G, and let A, B be two S algebras. Let $\{h_\sigma\}, \{h'_\sigma\}$ be descent data for A and B respectively. Let $f : A \to B$ be a moprhism of algebras. If f is equivariant with respect to the action of G on A and B defined by the descent data then there exists a map $\bar{f} : \bar{A} \to \bar{B}$ between the descended R-algebras, such that $f = \bar{f} \otimes 1$.*

Twisted forms and their cohomological description

Definition 3.3.36 (Twisted form). Let X be a scheme. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-algebras. A sheaf of $\mathcal{O}_X$-algebras $\mathcal{F}'$ is called a *twisted form* (in some Grothendieck topology E) of $\mathcal{F}$ if there is a cover $\{f_i : U_i \to X\}$ of the site X_E and $\varphi_i : \mathcal{F} \otimes_X U_i \xrightarrow{\sim} \mathcal{F}' \otimes_X U_i$ for any i.

Let $\mathcal{F}$, $\mathcal{F}'$, f_i, U_i be as in the definition, and let the morphisms $U_i \to X$ be flat. For any pair of indices i, j denote the projections $p_1^{ij} : U_i \times_X U_j \to U_i, p_2^{ij} : U_i \times_X U_j \to U_j$. Then, as pulling back along these projections is a functor, we have natural maps:

$$\begin{array}{ccc} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_j} & \xrightarrow{(p_1^{ij})^* \varphi_i} & \mathcal{F}' \times_{\mathcal{O}_X} \mathcal{O}_{U_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_j} \\ \mathcal{F} \times_{\mathcal{O}_X} \mathcal{O}_{U_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_j} & \xrightarrow{(p_2^{ij})^* \varphi_j} & \mathcal{F}' \times_{\mathcal{O}_X} \mathcal{O}_{U_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_j} \end{array}$$

Then define a collection of maps

$$\begin{array}{l} h_{ij} : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_j} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_j} \\ h_{ij} = (p_2^{ij})^* \varphi_j^{-1} \circ (p_1^{ij})^* \varphi_i \end{array}$$

Consider the sheaf $\mathcal{A}ut(\mathcal{F})$ defined by the correspondence $U \mapsto \mathrm{Aut}_U(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U)$. Now note that the cocycle condition for elements of $\check{H}^1(\mathscr{U}, \mathcal{A}ut(\mathcal{F}))$

$$(h_{ij}|_{U_{ijk}}) \cdot (h_{jk}|_{U_{ijk}}) = h_{ik}|_{U_{ijk}}$$

can be written down as follows. Let $Y = \sqcup U_i$, and let p_{ab} be the projections from $Y \otimes_X Y \otimes_X Y$ onto fibre products of a-th and b-th factors of the triple fibre product. If $\{h_{ij}\}$ is regarded as a map $h : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ (over X) then the cocycle condition can be expressed as:

$$(p_{23}^* h) \circ (p_{12}^* h) = (p_{13}^* h)$$

A routine check similar to that in the Theorem 2.2.12 shows that $\{h_{ij}\}$ is a 1-cocycle, and that two twisted forms $\mathcal{F}'$ and $\mathcal{F}''$ are isomorphic if and only if the corresponding cocycles are cohomologous.

Conversely, given a cocycle $\{h_{ij}\} \in \check{H}^1(\mathscr{U}, \mathcal{A}ut(\mathcal{F}))$ consider the morphism $p : Y = \sqcup U_i \to X$, and the corresponding map $h : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_Y$. We want to construct a descent datum for $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ and then apply the Theorem 3.3.33 to get $\mathcal{F}'(X)$, so let us assume that $Y = \operatorname{Spec} S = \oplus S_i$ and $X = \operatorname{Spec} R$, and denote $A = \mathcal{F}(X)$. We have a map

$$h : A \otimes_R S \otimes_R S \to A \otimes_R S \otimes_R S$$

of $S \otimes S$-modules. By composing it with the map

$$A \otimes_R S \otimes_R S \to S \otimes_R A \otimes_R S, \qquad a \otimes s \otimes t \mapsto s \otimes a \otimes t$$

we get a map $\varphi : (A \otimes_R S) \otimes_R S \to S \otimes_R (A \otimes_R S)$, which is a descent datum thanks to cocycle conditions imposed on h. By applying Theorem 3.3.33 to $A \otimes_R S$ and φ, we get an algebra B such that $A = B \otimes_R S$, and since B fits into the diagram from the statemnent of the Theorem 3.3.32, the cocycle associated to B is h.

This correspondence looks much simpler (and resembles the situation where the base is the field, see Section 2.2) if the twisted form trivialises after a base change that is a Galois morphism. Let $U \to X$ be a Galois morphism with the Galois group Γ and let there be an isomorphism $\varphi : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U \to \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{O}_U$. The corresponding Čech cocycle is

$$h : \mathcal{F} \otimes_{\mathcal{O}_X} (\oplus_{\sigma \in \Gamma} \mathcal{O}_{U_\sigma}) \to \mathcal{F} \otimes_{\mathcal{O}_X} (\oplus_{\sigma \in \Gamma} \mathcal{O}_{U_\sigma})$$
$$h = p_2^* \varphi^{-1} \circ p_1^* \varphi$$

Once we identify the components of $\Gamma \times U$ with U via the projection on the first factor, the projection of on the second factor becomes twisting by the corresponding element of the Galois group: $\{\sigma\} \times U \xrightarrow{\sigma} U$.

We can view h as a collection of maps $h_\sigma : \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U$ where $h_\sigma = \sigma^* \varphi_\sigma^{-1} \circ \varphi_\sigma$. This defines the same Galois cocycle in $H^1(\Gamma, \mathcal{F}(U))$ as the one obtained from the cocycle $h \in \check{H}^1(X_{\text{ét}}, \mathcal{A}ut(\mathcal{F}))$ via Theorem 3.3.28.

Conversely, given a Galois cocycle $\{h_\sigma\}$, a Galois decent datum is constructed as follows. For any $\sigma \in \Gamma$ let $\tilde{h}_\sigma = \sigma \circ h_\sigma$.

Clearly,

$$\begin{array}{ccc} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U & \xrightarrow{\tilde{h}_\sigma} & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U \\ \uparrow & & \uparrow \\ \mathcal{O}_U & \xrightarrow{\sigma} & \mathcal{O}_U \end{array}$$

commutes (this follows from the cocycle condition). The descended sheaf is $\mathcal{F}'$ and

$$\begin{array}{ccc} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U & \xrightarrow{\varphi} & \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{O}_U \\ \downarrow{\scriptstyle \tilde{h}_\sigma} & & \downarrow{\scriptstyle id\otimes\sigma} \\ \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U & \xrightarrow{\varphi} & \mathcal{F}' \otimes_{\mathcal{O}_X} \mathcal{O}_U \end{array}$$

Brauer group of a variety

We have encountered twisted forms of matrix algebras over a field , central simple algebras, in Section 3.2. In fact, in the proof of Theorem 3.2.9 we have seen that the generic fibre of the interpretation of the Structure 4 carries a faithful representation of a certain algebra, and the non-definability of the structure is a consequence of the fact that this algebra is not isomorphic to the matrix algebra.

We will now recall some properties of the relative counterparts of central simple algebras and their cohomological interpretation (references and proofs of the statements below can be found in Milne [20], Chapter IV).

Definition 3.3.37 (Azumaya algebra over a local ring)**.** Let R be a local ring. An R-algebra A is called an *Azumaya algebra* if there exists an étale R-algebra S such that $A \otimes_R S \cong M_n(S)$.

Proposition 3.3.38. *$M_n(R)$ is an Azumaya algera.*

Theorem 3.3.39. *Let R be a local ring with the maximal ideal $\mathfrak{m}$ and let A be an R-algebra. The following are equivalent:*

- *An R-algebra A is Azumaya;*

- *A is free as an R-module and $A \otimes (R/\mathfrak{m})$ is a central simple algebra over $R/\mathfrak{m}$;*

Definition 3.3.40 (Azumaya algebra over a variety)**.** Let X be a variety over an algebraically closed field k. A sheaf of algebras $\mathcal{F}$ is called an *Azumaya algebra* if it is a twisted form of the sheaf $M_n(\mathcal{O}_X)$ (defined by $U \mapsto M_n(\mathcal{O}_X(U))$) in the flat topology.

Theorem 3.3.41. *Let X be a variety over an algebraically closed field k. Then the following are equivalent*

- *$\mathcal{F}$ is an Azumaya algebra;*
- *$\mathcal{F}$ is a coherent $\mathcal{O}_X$-module and for all $x \in X$, $\mathcal{F}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$;*
- *$\mathcal{F}$ is locally free in Zariski topology and for any point x, $\mathcal{F}_x \otimes (\mathcal{O}_{X,x}/\mathfrak{m}_x) \cong M_n(k)$;*
- *there is a Zariski open covering $\cup O_i$ of X and a collection of finite étale surjective morphisms $U_i \to O_i$ such that $\mathcal{F} \otimes_{\mathcal{O}_{O_i}} \mathcal{O}_{U_i} \cong M_n(\mathcal{O}_{U_i})$*
- *$\mathcal{F}$ is a twisted form of the sheaf $M_n(\mathcal{O}_X)$ in the flat topology.*

Theorem 3.3.42. *Let $\mathcal{F}$ be an Azumaya algebra over X. Any automorphism α of $\mathcal{F}$ is locally inner, i.e. for any $x \in X$ there exists a Zariski open $U \ni x$ and an invertible element $u \in \mathcal{F}(U)$ such that $\alpha(x) = u \cdot x \cdot u^{-1}$ on U. In particular, $\mathcal{A}ut(M_n(X)) = \mathrm{PGL}_n(X)$.*

By the previous subsection it follows that isomorphism classes of Azumaya algebras on X are in bijective correspondence with the set $\check{H}^1(X_{\text{ét}}, \mathrm{PGL}_n)$.

Theorem 3.3.43. *Let $\mathcal{F}$, $\mathcal{G}$ be Azumaya algebras over X. Then $\mathcal{F}\otimes\mathcal{G}$ is an Azumaya algebra too.*

Definition 3.3.44 (Brauer group)**.** Let $\mathcal{F}, \mathcal{G}$ be Azumaya algebras on X. They are called *equivalent* if there are locally free sheaves E, E' on X (in Zariski topology) such that

$$\mathcal{F} \otimes \mathcal{E}nd(E) \cong \mathcal{G} \otimes \mathcal{E}nd(E')$$

where $\mathcal{E}nd(E)$ is the sheaf defined by the correspondence $U \mapsto \mathrm{End}(E(U))$. The *Brauer group* of X is the group of classes of equivalent of Azumaya algebras with the tensor product the group operation. The class of $\mathrm{End}(\mathcal{O}_X)$ is the identity, and the inverse of a class $[\mathcal{F}]$ is the class of $\mathcal{F}^{\mathrm{op}}$, the sheaf of algebras, opposite to $\mathcal{F}$. It is denoted $Br(X)$.

Proposition 3.3.45. *Let X be a variety. The following is a short exact sequence of sheaves on $X_{\text{ét}}$ (associated to corresponding group schemes) in the sense of Definition 3.3.19*

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

Theorem 3.3.46. *The connecting morphism $\check{H}^1(X_{\text{ét}}, \mathrm{PGL}_n) \to \check{H}^2(X_{\text{ét}}, \mathbb{G}_m)$ induces an injective homomorphism*

$$Br(X) \hookrightarrow \check{H}^2(X_{\text{ét}}, \mathbb{G}_m)$$

The image of this map is torsion.

Theorem 3.3.47. *Let X be an integral smooth variety and let K be its field of functions. The restriction to generic point $Br(X) \to Br(K)$ is an injective homomorphism.*

3.4 Quantum Zariski geometries

We will consider the class of structures defined by Zilber and formulate the necessary and sufficient condition that a structure in this class be definable in an algebraically closed field.

Definition 3.4.1 (Input data)**.** The following is the input data for the construction of Zariski geometries in Zilber [37].

- a non-commutative algebra A finitely generated over an algebraically closed field F with a centre $Z(A)$ which is an algebra without zero divisors, also finitely generated over F. Denote by X the irreducible affine variety over F which has the coordinate algebra $Z(A)$;

- a collection of n-dimensional (n fixed) A-modules M_x indexed by points of X;
- a surjective map $p : Y \to X$ with finite fibres;
- a finite group Γ and a partial map $g : \Gamma \times X \to \mathrm{GL}_n(F)$.

These data are to satisfy the following requirements:

1. for any point $x \in X$, the vanishing ideal $I(x) \subset Z(A)$ annihilates the module M_x;
2. for each subgroup $\Gamma' \subset \Gamma$, $g|_{\Gamma' \times X}$ is defined on an open subset $U \subset X$, and Γ' acts freely and transitively on the fibres of Y over U;
3. in each module M_x there is a distinguished base $\{e_1^\alpha, \ldots, e_n^\alpha\}$ associated to every point α of the fibre $p^{-1}(x)$. There is a map $\iota : Y \to X \times \mathbb{A}_F^{d \cdot n^2}$ such that

$$\begin{array}{ccc} Y & \longrightarrow & X \times \mathbb{A}_F^{d\cdot n^2} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & X & \end{array}$$

   and such that the coordinates of each point $\iota(y)$ in $\mathbb{A}_F^{d \cdot n^2}$ describe the matrices of the generators $\mathbf{u}_1, \ldots, \mathbf{u}_d$ of A in the basis $\{e_1^y, \ldots, e_n^y\}$, in other words the matrices have entries that algebraically depend on the point of the base variety X;
4. the distinguished bases are related as follows. For any pair of points $y, \sigma \cdot y$ such that $p(y) = x$ there exists a non-zero scalar $\lambda \in F^\times$ such that

$$e_i^{\sigma \cdot y} = \lambda \cdot g(\sigma, x) e_i^y$$

The structure associated to this data is the following:

Definition 3.4.2 (Quantum Zariski geometry)**.** Let $V = \bigsqcup M_x$ and let $p : V \to X$ be the natural projection. A *quantum Zariski geometry* is the abstract linear space (V, X, F, p) with the following additional structure:

- the graphs of the action of the generators $\mathbf{u}_1, \ldots, \mathbf{u}_d$ on individual modules M_x and hence on the whole V;

- the set $B \subset V^m$ is the set of distinguished bases;

The language of the structure includes the language of an abstract linear space (V, X, F, p), the predicates for the graphs of the action of generators $\mathbf{u}_1, \ldots, u_d$ and the predicate B.

The structure with the predicate B omitted is called a structure in the "weak language".

Theorem 3.4.3 (Zilber [37], Section 4)**.** *For any input data as described in Definition 3.4.1 the structure described in Definition 3.4.2 is a Zariski geometry.*

Quantum Zariski geometries as twisted forms

We will now describe a correspondence between a class of certain quantum Zariski geometries and Azumaya algebras. Given the input data as in Definition 3.4.1, we are going to associate to it an Azumaya algebra that will determine whether the corresponding quantum Zariski geometry is definable in an algebraically closed field or not. The original quantum Zariski geometry can be recovered from this Azumaya algebra.

Consider the input data as in Definition 3.4.1, and let Y and X have coordinate rings S and R respectively (recall that X and Y are affine). The map $\iota : Y \hookrightarrow X \times \mathbb{A}_F^{d \cdot n^2}$ specifies d matrices of size $n \times n$ with coefficients in S. These d matrices $\mathbf{U}_1, \ldots, \mathbf{U}_d$ generate a subalgebra of the algebra of endomorphisms of the trivial bundle $Y \times \mathbb{A}_F^n$. A fibre of this trivial bundle over a point y is an A-module isomorphic to M_x, $x = p(y)$, via the map that sends the standard basis in $N_y = (S^n) \otimes (S/\mathfrak{m}_y)$ to $\{e_1^y, \ldots, e_n^y\}$. This defines a map of R-modules $A \to \mathrm{End}_S(S^n)$; extend this map to $A \otimes S$ to obtain a map of S-modules $\rho : A \otimes S \to \mathrm{End}_S(S^n)$.

Theorem 3.4.4. *Suppose that Y is a Galois cover of X with the Galois group Γ. Then there exists an R-algebra B such that $B \otimes_R S \cong \mathrm{End}(S^n)$ and a map $\bar{\rho} : A \to B$*

of R-algebras such that $\rho = \bar{\rho} \otimes 1$. For each $x \in X$ there exists such an isomorphism $\eta_x : B \otimes R/\mathfrak{m}_x \to End_F(M_x)$ that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{\rho}\otimes R/m_x} & B \otimes R/\mathfrak{m}_x \\ & \searrow^{\theta_x} & \downarrow{\eta_x} \\ & & \mathrm{End}_F(M_x) \end{array}$$

commutes (where θ_x is the structure morphism of the module M_x).

Proof. Construct a Galois descent datum for the algebra $\mathrm{End}(S^n)$; for that denote $g_\sigma \in \mathrm{GL}_n(R)$ the matrix corresponding to the map $g(\sigma, -) : X \to \mathrm{GL}_n$, and define:

$$h_\sigma : \mathrm{End}(S^n) \to \mathrm{End}(S^n) \qquad h_\sigma(u) = \sigma \circ (g_\sigma^{-1} \otimes 1) \circ u \circ (g_\sigma \otimes 1)$$

where σ denotes the map $S^n \to S^n$ induced by σ. Note that $h_\sigma(s \cdot u) = \sigma(s)h_\sigma(u)$, and therefore $\{h_\sigma\}$ indeed is a Galois descent datum. Now apply Theorem 3.3.33 and let B be the descended algebra.

Let us check that the map ρ is equivariant with respect to the action of Γ on $A \otimes S$ and $\mathrm{End}(S^n)$, that is

$$\rho(a \otimes \sigma(s)) = h_\sigma(\rho(a \otimes s))$$

We will check this equality fibrewise, i.e. we will show that the images of the left and right part in $\mathrm{End}(S^n) \otimes (S/\mathfrak{m}_y)$ coincide for every $y \in Y$. Since $\mathrm{End}(S^n)$ is a free module over a reduced ring, it will follow that the equality holds.

Let us adopt the following notation: $s(y)$ or $U_i(y)$ mean the image of s or U_i or any other object in the fibre over y, i.e. $S/\mathfrak{m}_y$ or $\mathrm{End}(S^n) \otimes (S/\mathfrak{m}_y)$ etc., where y is a point of Y and $\mathfrak{m}_y$ is the corresponding maximal ideal of S.

It suffices to check the equality for the generators of A, so suppose $a = \mathbf{u}_k$ for some k. Then $\rho(a \otimes \sigma(s))$ is represented by the matrix $s(\sigma \cdot y)\mathbf{U}_k(y)$ in the fibre over y, where $\mathbf{U}_k(y)$ is the matrix that represents the action of $\mathbf{u}_k$ in the basis $\{e_1^y, \ldots, e_n^y\}$ is M_x.

By definition of g_σ there exists a scalar λ such that $e_i^{\sigma y} = \lambda(g_\sigma \otimes 1)(y)e_i^y$, therefore conjugation by $g_\sigma \otimes 1$ maps $\mathbf{U}_k(y)$ to $\mathbf{U}_k(\sigma^{-1}y)$ and

$$h_\sigma(\rho(a \otimes s)) = \sigma(s(y)\mathbf{U}_k(\sigma^{-1}y)) = s(\sigma y)\mathbf{U}_k(y)$$

which is equal to $\rho(\mathbf{u}_k \otimes \sigma(s))(y)$.

We have shown that ρ is equivariant, and by Theorem 3.3.35, there exists a map $\bar{\rho} : A \to B$ such that $\bar{\rho} \otimes 1 = \rho : A \otimes S \to \mathrm{End}(S^n)$. It is left to prove the last commuting diagram in the statement of the theorem.
By descent we get the following diagram

$$\begin{array}{ccc} A \otimes S & \xrightarrow{\rho} & B \otimes S \\ {\scriptstyle id\otimes 1}\uparrow & & \uparrow{\scriptstyle id\otimes 1} \\ A & \xrightarrow{\bar{\rho}} & B \end{array}$$

Since $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is an unramified map, $S/\mathfrak{m}_x S \cong \oplus_{y \in p^{-1}(x)} S/\mathfrak{m}_y$ and moreover there exists a collection of isomorphisms $i_y : R/\mathfrak{m}_x \xrightarrow{\sim} S/\mathfrak{m}_y$. Since $B \otimes S \cong \mathrm{End}(S^n)$ the following isomorphisms of fibres over x exist:

$$\mathrm{End}(S^n) \otimes S/\mathfrak{m}_x S \cong B \otimes S/\mathfrak{m}_x S \cong B \otimes (\oplus_{y \in p^{-1}(x)} S/\mathfrak{m}_y S)$$

and taking into account the isomorphisms $\epsilon_y : \mathrm{End}(S^n) \to \mathrm{End}_F(M_x)$

$$\begin{array}{ccccc} A \otimes S & \xrightarrow{\rho\otimes R/\mathfrak{m}_x} & B \otimes (\oplus S/\mathfrak{m}_y) & \xrightarrow{\epsilon} & \oplus \mathrm{End}(M_x) \\ \uparrow & & \uparrow{\scriptstyle i} & & \\ A & \xrightarrow{\bar{\rho}\otimes R/\mathfrak{m}_x} & B \otimes R/\mathfrak{m}_x & & \end{array}$$

where $i = (i_y, i_{y'}, \ldots)$, $\epsilon = (\epsilon_y, \epsilon_{y'}, \ldots)$. The composition of morphisms $id \otimes 1 : A \to A \otimes S$, $\rho_y : A \otimes S \to \mathrm{End}(S^n) \otimes S/m_y$ and $\epsilon_y : \mathrm{End}(S^n) \otimes S/m_y \to \mathrm{End}_F(M_x)$ is the structure morphism θ_x by definition of ρ and ϵ_y. The desired morphism $\eta_x : B \otimes R/\mathfrak{m}_x \to \mathrm{End}_F(M_x)$ can be taken to be any of the compositions $i_y \circ \epsilon_y$ for $y \in p^{-1}$. □

Remark 3.4.5. The construction above does not depend on the fact that the modules M_x are irreducible. Note that by Burnside's theorem (see, for example, [19]) if a subalgebra of $M_n(k)$, where k is algebraically closed, does not have a proper invariant subspace then it is the entire $M_n(k)$. So if M_x are irreducible then $\mathrm{Im}\, \theta_x = \mathrm{End}_F(M_x)$.

Corollary 3.4.6. *Suppose the variety X is reduced. If θ_x induces an isomorphism between $A \otimes R/\mathfrak{m}_x$ and $\mathrm{End}_F(M_x)$ for all $x \in X$ then $\bar{\rho}$ is an isomorphism, and consequently A is an Azumaya algebra.*

Proof. By previous theorem the morphism $\bar{\rho}$ induces isomorphisms of fibres. Since B is an Azumaya algebra, it is a locally free R-module. Since X is reduced and the rank of $A \otimes R/\mathfrak{m}_x$ is constant, A is locally free as a module.

To show that $\bar{\rho}$ is an isomorphism it suffices to do it locally over Zariski open sets of X. Choose an open cover of X that trivialises both A and B. We have reduced the question to showing that a morphism of free modules over a reduced ring is an isomorphism if it induces isomorphisms on fibres, which is clearly true. □

Theorem 3.4.7. *Let X and Y be affine varieties over an algebraically closed field k with coordinate algebras R and S respectively, $Y \to X$ a Galois morphism with Galois group Γ and let $\{h_\sigma\}$ be a cocycle representing a Galois cohomology class from $H^1(\Gamma, PGL_n(R))$. Let A be the Azumaya algebra corresponding to the cocycle h (see Section 3.3), in particular $\mathcal{F} \otimes \mathcal{O}_Y \cong M_n(S)$. Then there exists a quantum Zariski geometry (V, X, k, p) corresponding to the input data consisting of the algebra A, the morphism $p : Y \to X$ and a map $g : \Gamma \times X \to GL_n$ (which will be constructed in the proof). Moreover, for any $x \in X$ there exists an isomorphism $\eta_x : A \otimes R/m_x \to$ $\mathrm{End}(M_x)$ compatible with the natural map $A \otimes R/\mathfrak{m}_x$ and the structure maps of the modules M_x, θ_x:*

$$\begin{array}{ccc} A & \longrightarrow & A \otimes R/\mathfrak{m}_x \\ & \overset{\theta_x}{\searrow} & \downarrow \eta_x \\ & & \mathrm{End}_k(M_x) \end{array}$$

Proof. To complete the proof we only need to describe the set of modules M_x, the matrices for a set of generators of A (via the map $\iota : Y \to X \times \mathbb{A}_k^{d \cdot n^2}$ where d is the number of generators) and the map $g : \Gamma \times X \to \mathrm{GL}_n$ such that all the data satisfy the requirements of Definition 3.4.1

Construct the map $\iota : Y \to X \times \mathbb{A}_k^{d \cdot n^2}$ as follows. Pick some generators $\mathbf{u}_1, \ldots, \mathbf{u}_d$ of A and consider their images in $A \otimes S \cong \mathrm{End}(S^n)$. These are $n \times n$ matrices

$\mathbf{U}_1, \dots, \mathbf{U}_d$ with the values in S, and define a map

$$y \mapsto ((\mathbf{U}_1)_{11}(y), \dots, (\mathbf{U}_1)_{ij}(y), \dots, (\mathbf{U}_d)_{nn}(y))$$

from Y to $X \times \mathbb{A}_k^{d \cdot n^2}$.

Let us now construct the modules M_x. Pick $x \in X$ and pick some y_0 in the fibre $p^{-1}(x)$. We let $M_x = k^n$ and the standard basis in k^n will be the distinguished basis e^{y_0}. Define the action of $\mathbf{u}_1, \dots, \mathbf{u}_d$ in this basis by the matrices $\mathbf{U}_1(y_0), \dots, \mathbf{U}_d(y_0)$ where the said matrices are the images of $\mathbf{U}_1, \dots, \mathbf{U}_d$ in $\mathrm{End}(S^n) \otimes S/m_{y_0}$ which is isomorphic to $\mathrm{End}(k^n)$.

To give the map $g : \Gamma \times X \to \mathrm{GL}_n$ is the same as specify a family of elements $g_\sigma \in \mathrm{GL}_n(S)$. Define g_σ to be the image of some lifting of h_σ to $GL_n(R)$. Note that $\{h_\sigma\}$ can be considered as a cocycle with values in $\mathrm{PGL}_n(S)$ under the natural inclusion $\mathrm{PGL}_n(R) \hookrightarrow \mathrm{PGL}_n(S)$.

Now for any other point in the fibre $p^{-1}(x)$, which is of the form $\sigma \cdot y_0$ for some $\sigma \in \Gamma$, define the distinguished basis to be

$$e_i^{\sigma y_0} = g_\sigma(x) \cdot e_i^{y_0}$$

We need to show that the action of $\mathbf{u}_1, \dots, \mathbf{u}_d$ in this basis will actually be represented by the matrices $\mathbf{U}_1(\sigma \cdot y_0) \otimes 1, \dots, \mathbf{U}_d(\sigma \cdot y_0)$. In other words we need to prove that

$$\mathbf{U}_i(\sigma \cdot y_0) = g_\sigma(x) \mathbf{U}_i(y_0) g_\sigma^{-1}(x)$$

The basic property of the Galois descent (see Section 3.3) is that the following diagram is commutative

$$\begin{array}{ccc} A \otimes S & \xrightarrow{\sim} & \mathrm{End}(S^n) \\ \downarrow{\scriptstyle id \otimes \sigma} & & \downarrow{\scriptstyle \sigma \circ h_\sigma} \\ A \otimes S & \xrightarrow{\sim} & \mathrm{End}(S^n) \end{array}$$

The elements of A — such as $\mathbf{u}$ — are invariant under Γ so their images in different fibres $\mathrm{End}(S^n) \otimes S/m_y$ are related by maps $\sigma \circ h_\sigma$ restricted to the fibre $\mathrm{End}(S^n) \otimes R/m_x$. But h_σ by definition acts on $\mathrm{End}(S^n)$ by conjugation with $g_\sigma(x)$.

Further, we need to check that for any two points $\sigma \cdot y_0$ and $\tau \cdot y_0$ there exists a scalar λ such that

$$e_i^{\sigma \cdot y_0} = \lambda g_{\sigma\tau^{-1}}(x) \cdot e_i^{\tau \cdot y_0}$$

This follows from cocycle condition for h:

$$h_\sigma = h_{\sigma\tau^{-1}}{}^{\sigma\tau^{-1}} h_\tau$$

The Galois action on h_τ is trivial since h takes its values in $PGL_n(R)$, and the λ multiplier arises from the fact g_σ is a lifting of h_σ to $GL_n(R)$.

Finally, to construct isomorphisms $\eta_x : A \otimes R/\mathfrak{m}_x$ consider the isomorphism $\mathrm{End}(M_x) \cong \mathrm{End}(S^n) \otimes S/\mathfrak{m}_{y_0}$, and note that by properties of descent $\mathrm{End}(S^n) \otimes S/\mathfrak{m}_{y_0} \cong A \otimes R/\mathfrak{m}_x$ (cf. the end of the proof of Theorem 3.4.4). □

Remark 3.4.8. It is clear from the construction that the algebra B associated in the proof of Theorem 3.4.4 to the quantum Zariski geometry constructed in Theorem 3.4.7 is isomorphic to A.

Remark 3.4.9. The restriction that the cocycle h takes its values in $\mathrm{PGL}_n(X)$ is an artifact of the Definition 3.4.2. The construction goes through if we consider a map $g : \Gamma \times Y \to GL_n$, i.e. the cocycle h takes its values in $\mathrm{PGL}_n(Y)$. The distinguished bases are then related by the equation

$$e_i^{\sigma y} = \lambda g_\sigma(y) \cdot e_i^y$$

Remark 3.4.10. The choice of a particular set of generators of A and the matrices that represent them in $\mathrm{End}(S^n)$ is a superfluous piece of data. It suffices to specify the maps g_σ and the map $\bar{\rho} : A \to B$ in order to define a structure with the same definable sets. One considers all elements of A as a part of the language and defines their graphs fibrewise, on each M_x, using the maps θ_x.

Remark 3.4.11. By Theorem 3.3.41 any Azumaya algebra locally splits after an étale base extension, and hence after a Galois extension. The Theorem 3.4.7 can be extended to construct a quantum Zariski geometry for an arbitrary Azumaya algebra by first constructing the structures corresponding to local splitting (Galois) morphisms and then gluing the resulting structures. The glued structure will have the characteristic property: the structure maps $A \to \mathrm{End}(M_x)$ will fit the diagram as in the Theorem 3.4.7.

Definability criterion

The correspondence established in the previous section allows us to formulate necessary and sufficient conditions that a quantum Zariski geometry corresponding to a certain Azumaya algebra is definable in an algebraically closed field.

Theorem 3.4.12. *Let X be a variety over an algebraically closed field. Let (V, X, F, p) be a quantum Zariski geometry and let B be the Azumaya algebra that corresponds to it by Theorem 3.4.4. If the class of $B \otimes F(X)$ in $Br(F(X))$ where $F(X)$ is the field of rational functions over X is non-trivial, then (V, X, F, p) is not definable in an algebraically closed field.*

Proof. Suppose (V, X, F, p) is interpretable in an algebraically closed field K. By Theorem 3.1.4 we conclude that K is definably isomorphic to $F(K)$, and $X(F(K))$ is definably isomorphic to $X(K)$ (reminder: $F(K)$ means the definable set that interprets F in K).

By Corollary 3.1.8 there exists an open $U \subset X(K)$ such that $V(K)$ restricted to U is the trivial rank n bundle and the linear space operations are interpreted as the natural addition and scalar multiplication on $U \times \mathbb{A}^n_K$. Without loss of generality we can assume that U is affine with coordinate algebra T. The interpretation of the action of an element a of A is an endomorphism $\epsilon(a)$ of the trivial bundle over U, and so by Theorem 3.4.4 the following diagram commutes for any $x \in X(K)$

$$\begin{array}{ccccc} A \otimes T & \xrightarrow{\bar{\rho} \otimes T} & B \otimes T & \longrightarrow & B \otimes R_K/\mathfrak{m}_x \\ & \searrow^{\epsilon} & & & \downarrow{\scriptstyle \eta_x} \\ & & \mathrm{End}(T^n) & \longrightarrow & \mathrm{End}(T^n) \otimes R_K/m_x \end{array}$$

where $R_K = R \otimes K$, and the composition of ϵ with the projection onto the fibre $\mathrm{End}(T^n) \to \mathrm{End}(T^n) \otimes R/m_x$ is θ_x, the structure map of the module M_x. Since, $B \otimes T$ and $\mathrm{End}(T^n)$ are two locally free modules over a reduced ring that are fibrewise isomorphic, they are actually isomorphic. Therefore $B \otimes K(X)$, the generic fibre of B, is isomorphic to $\mathrm{End}(K(X))$, and its class in $\mathrm{Br}(K(X))$ is trivial, a contradiction. □

Theorem 3.4.13. *Let X be a smooth reduced affine variety over an algebraically closed field. Let (V, X, F, p) be a quantum Zariski geometry and let B be the Azumaya algebra that corresponds to it by Theorem 3.4.4. Then (V, X, F, p) is definable in an algebraically closed field if and only if the class of B in $Br(F(X))$, where $F(X)$ is the field of rational functions on X, is trivial.*

Proof. The left to right part is Theorem 3.4.12.

Suppose that the class of $B \otimes F(X)$ in $\mathrm{Br}(F(X))$ is trivial. In order to define (V, X, F, p) in a field K, we need to specify a vector bundle $E \to X$ that interprets the linear space V and construct a map $A \to \mathrm{End}(E)$ that induces the structure of modules M_x on its fibres. The addition and multiplication by scalar maps on E are defined using local trivialisations: given that $p^{-1}(U)$ is isomorphic to $U \times \mathbb{A}^n_K$ one defines the addition and multiplication on $p^{-1}(U)$ as the images on the corresponding operations on $U \times \mathbb{A}^n_K$ regarded as a (trivial) family of K-vector spaces.

By Theorem 3.3.47 the class of B in $\mathrm{Br}(X)$ is trivial, i.e. B is isomorphic to $\mathrm{End}(E)$ for some vector bundle $E \to X$. Define V to be the total space E of the vector bundle. By Theorem 3.4.4 we have a morphism $\bar{\rho} : A \to B$. For each element of $a \in A$ define the action of a on E to be $\rho(a)$. This defines an interpretation of the Zariski geometry (V, X, F, p). □

3.5 Examples of non-algebraic Zariski geometries

The criteria for definability of quantum Zariski geometries in algebraically closed fields that we have obtained in Section 3.4 provide a link between the model theory of quantum Zariski geometries and central simple algebras. In this section we will illustrate this connection with some examples, taken from Zilber [37].

Cyclic algebras

Certain central simple algebras have a particularly simple cocycle description which makes it easier to test whether their class in the Brauer group is trivial. We will use the following well-known results on the so-called cyclic algebras in further sections to decide the definability of quantum Zariski structures.

Definition 3.5.1. Let K/k be a Galois extension of fields with the Galois group G cyclic of order n with generator α. Let b be a element of $k^\times$. Define a class in $H^1(G, \mathrm{PGL}_n)$ by putting

$$h_\alpha = \begin{pmatrix} 0 & 0 & \dots & 0 & b \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

and letting $h_{\alpha^n} = h_\alpha^n$. Since the Galois action on the matrix that defines h_α is trivial and since $h_\alpha^n = 1$, as an element of PGL_n, this is a well-defined cocycle. The central simple algebra that corresponds to this cocycle is called *cyclic*.

Proposition 3.5.2 (Gille and Szamuely [6], Corollary 4.7.5)**.** *The class of a cyclic algebra that corresponds to an element $b \in k^\times$ is trivial in $Br(k)$ if and only if b equals $N_{K/k}(a)$, the norm of an element $a \in K^\times$, i.e. is of the form $\prod_{\sigma\in G}\sigma(a)$ for some $a \in K^\times$.*

The quantum plane

A structure associated with the "quantum plane" algebra

$$A = \langle\, \mathbf{u}, \mathbf{v}, \mathbf{u}^{-1}, \mathbf{v}^{-1} \mid \mathbf{uv} = q\mathbf{vu}, \rangle_{\mathbb{C}}$$

where $q^n = 1$ for some n, is considered in Zilber [37], Example 2.3.4.

The centre of A is generated by u^n and v^n, and A is an algebra over $k[x, y]$. The corresponding base variety for a quantum Zariski geometry would be an affine variety with the coordinate algebra $k[x, y]$, that is, $X = \mathbb{A}^2_k$.

It is claimed in [37] that one can define two quantum Zariski geometries, each over one of two opens that cover X, U_0, the plane without the line $x = 0$, and U_1, the plane without the line $y = 0$. Over U_0 the trivialising cover is $Y_0 \cong U_0 \to U_0$, $(x, y) \mapsto (x^n, y)$ and over U_1 it is $Y_1 \cong U_1 \to U_1$, $(x, y) \mapsto (x, y^n)$.

Over U_0, the matrices of the generators $\mathbf{u}, \mathbf{v}$ in a distinguished basis are:

$$\mathbf{u} = \begin{pmatrix} \mu & 0 & 0 & \dots & 0 \\ 0 & \zeta \cdot \mu & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \zeta^{n-1} \cdot \mu \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0 & 0 & \dots & 0 & y \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \qquad (*)$$

where $\mu^n = x$.

Over U_1, the matrices of the generators $\mathbf{u}, \mathbf{v}$ in a distinguished basis are:

$$\mathbf{u} = \begin{pmatrix} 0 & 0 & \dots & 0 & x \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} \nu & 0 & 0 & \dots & 0 \\ 0 & \zeta \cdot \nu & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \zeta^{n-1} \cdot \nu \end{pmatrix} \qquad (**)$$

where $\nu^n = x$.

However, it is impossible to define the map g (part of the data needed to define a quantum Zariski geometry) even over these opens. The matrix of $\mathbf{v}$ over U_0 degenerates to

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

when $y = 0$. It is clear that the only invertible martices this matrix commutes with are the scalar multiples of the identity matrix, and therefore it is impossible to specify a base change matrix T such that $T \cdot \mathbf{u} \cdot T^{-1}$ equals the matrix $\mathbf{u}$ with μ replaced with one of its Galois conjugates.

Still, it is possible to define a quantum Zariski geometry with the base variety $U_0 \cap U_1$, taking the restrictions of either $(*)$ or $(**)$ as matrices of the generators.

Consider the trivialising cover $Y' \cong U_0 \cap U_1 \to U_0 \cap U_1, (x, y) \mapsto (x^n, y)$ and the corresponding Galois cocycle with values in PGL_n

$$g_{\alpha^n} = \begin{pmatrix} 0 & 0 & \dots & 0 & y \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}^n$$

where α is the generator of the Galois group $G = C_n$. Let B be the Azumaya algebra corresponding to this cocycle.

The cocycle $\{g_\sigma\}$ describes the central simple algebra $B \otimes k(x,y)$. This central simple algebra is cyclic, corresponding to the element $y \in k(\mu, y)$. Therefore by Proposition 3.5.2, $B \otimes k(x,y)$ is split if and only if y is the norm of an element $a \in k(\mu, y)$.

The basis of K as a k-vector space consists of vectors $1, \mu, \mu^2, \ldots, \mu^n$. Suppose $y = N_{K/k}(a)$. The element a can be written in the form $a_0 + a_1\mu + a_2\mu^2 + \ldots + a_{n-1}\mu^{n-1}$ where $a_i \in k(x,y)$. Expanding $N_{K/k}(a) = \prod_{\sigma\in G}(a)$ we find that it is of the form $a_0^n + b_0 \cdot x + b_1\mu + \ldots$ where $b_i \in k(x,y)$ and therefore $N_{K/k}(a)$ cannot be y. We conclude that $B \otimes k(x,y)$ is not split. It is left to apply Theorem 3.4.12.

Proposition 3.5.3. *The "quantum plane" structure is not definable in an algebraically closed field.*

Quantum torus structure revisited

Structure 4 ("quantum torus") is an instance of the construction given by Theorem 3.4.7.

Indeed, as follows from the proof of Theorem 3.2.9, every fibre of the algebra A, $A \otimes R/\mathfrak{m}_x$ is the full matrix algebra, and by Corollary 3.4.6 A is an Azumaya algebra.

We will now prove that the Structure 4 is not interpretable in an algebraically closed field by applying the Theorem 3.4.12. The map $g : \Gamma \times X \to \mathrm{GL}_n$ from the definition of Structure 4 defines a cocycle in $H^1(\Gamma, \mathrm{PGL}_n)$ that defines A as a twisted form.We need to show that $A \otimes F(X)$ is non-trivial in the Brauer group of the field $F(X)$.

Recall the input data of Structure 4. The base variety is $X = (\mathbb{A}^1_F \setminus \{0\})^2$ and the trivialising cover is $Y = X \to X, (x,y) \mapsto (x^n, y^n)$, with Galois group $\Gamma = C_n \times C_n$. Let α and β be the two generators of Γ. The cocycle is given by the following elements of PGL_n that correspond to two generators:

$$g_\alpha = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \qquad g_\beta = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \zeta & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & \zeta^{n-1} \end{pmatrix}$$

where $\zeta^n = 1$, and the other values of the cocycle are defined by the formula $g_{\sigma\tau} = g_\sigma g_\tau$ (as elements in PGL_n).

Theorem 3.5.4. *The image of the cocycle $\{g_\sigma\}$ in $H^2(G, k(\mu, \nu))$ (where $\mu^n = x, \nu^n = x$) is non-trivial. Consequently, Structure 4 is not definable in an algebraically closed field.*

Proof. Let h be the following matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & \nu^{n-1} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & \nu & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & \nu^{n-2} & 0 \end{pmatrix}$$

The cocycle $\{g_\sigma\}$ is cohomologous, by definition, to the cocycle $\{g'_\sigma\}$ defined as $g'_\sigma = h^{-1} g_\sigma{}^\sigma h$.

Let as compute g'_α and g'_β, where α and β are generators of $\Gamma = C_n \times C_n$, and the value of g' on other elements Γ will follow from cocycle condition:

$$g'_\alpha = \begin{pmatrix} 0 & 0 & \dots & y \\ 1 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and g'_β is the identity matrix.

Now we will need a basic fact about Galois cohomology (see, for example, Serre [32], I.5.8). Let H be a normal subgroup of G. A cocycle representing a class in $H^1(G/H, A^H)$ (A not necessary commutative) is naturally a cocycle representing a class in $H^1(G, A)$. Then the induced map on cohomology classes is injective: $H^1(G/H, A^H) \hookrightarrow H^1(G, A)$.

Let H be the subgroup $e \times C_n$ in $\Gamma = C_n \times C_n$. The cocycle g'_α defined above is constant on cosets of H and takes values in A^H, therefore it belongs to the image of the inclusion and there is a cocycle g''_σ representing a class in $H^1(C_n, \mathrm{PGL}_n(k(\mu, y))$ such that g' is the image of g''.

But the cocycle g'' is exactly the cocycle considered in the previous section, so it's image in $H^2(C_n, k^\times(\mu, y))$ is non-trivial. $\square$

Chapter 4

Complex analytic interpretations

In the previous chapter we have considered interpretability of certain Zariski geometries in algebraically closed fields. A Zariski geometry is a geometric object that has properties similar to that of algebraic varieties (over algebraically closed fields) and also to compact complex manifolds. It is therefore natural to try to generalize the question of interpetability of a Zariski geometry to a complex analytic setting. The formal framework that immediately comes to mind is the structure $\mathcal{A}$ (see Section 2.3).

Looking at (non-)definability proofs in Chapter 3 one can notice that they all translate a problem about constructible (definable) objects into a problem about algebro-geometric objects: varieties, sheaves. For example, Proposition 3.1.7 says that after restricting to an open on the base and up to definable isomorphism, an interpretation of an abstract linear space in an algebraically closed field is just a trivial vector bundle. Similarly, Theorem 3.3.41 tells us that in quantum Zariski geometry the structure on top of this trivial vector bundle can be interpreted as an Azumaya algebra.

It is much harder to give such translations in the structure $\mathcal{A}$, for reasons that will be discussed in Section 4.2. In this Chapter we will define what it means for a quantum Zariski geometry to have a "complex analytic model", a notion weaker than interpretability in $\mathcal{A}$. To establish if a quantum Zariski geometry has a complex analytic model one does not have to cope with subtle issues of translating constructible objects into complex-analytic objects.

4.1 Complex analytic models

An algebraic variety X over $\mathbb{C}$ has an analytification naturally associated to it, a complex analytic space. Let X_i be the affine varieties X is glued of, then every X_i is the zero set of some finite number of polynomials and since polynomial functions are holomorphic, X_i are complex spaces. Since the transition functions between X_i-s are regular, they are also holomorphic, therefore X_i^{an} glue into a complex space X^{an} with the structure sheaf $\mathcal{O}_X^{\mathrm{an}}$ and the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X^{\mathrm{an}}$. If $\mathcal{F}$ is a sheaf (in Zariski topology) of $\mathcal{O}_X$-modules, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\mathrm{an}}$ is a sheaf of $\mathcal{O}_X^{\mathrm{an}}$-modules, and we call it the analytification of $\mathcal{F}$, $\mathcal{F}^{\mathrm{an}}$; same for a sheaf of algebras.

Definition 4.1.1 (Complex analytic model)**.** Let $M = (V, X, \mathbb{C}, p)$ be a quantum Zariski geometry, and let us keep the notation for input data as in Definition 3.4.1.

We say that M has a *complex analytic model* if there exists the following data:

- a connected complex analytic space $\overline{V}$ with a projection map $\overline{p} : \overline{V} \to X^{\mathrm{an}}$;
- holomorphic functions $\overline{a} : \overline{V} \times \overline{V} \to \overline{V}, \overline{m} : \mathbb{C} \times \overline{V} \to \overline{V}$, such that every fibre $\overline{V}_x = \overline{p}^{-1}(x), x \in X^{\mathrm{an}}$ is a vector space with addition and multiplication given by the restrictions of $\overline{a}, \overline{m}$ to $\overline{V}_x$;
- for every element of A there is an endomorphism of $\overline{V}$ that is linear on the fibres of $\overline{V}$, these endomorphisms turn each $\overline{V}_x$ into an A-module isomorphic to M_x;

Holomorphic linear spaces

A holomorphic linear space is a family of vector spaces fibered over a base complex space, with vector space operations varying "complex analytically" over the base. A holomorphic linear space is a part of a complex analytic model.

Definition 4.1.2 (Holomorphic linear space)**.** Let $p : X \to S$ be a map of complex spaces and let $a : X \times X \to X$, $m : \mathbb{C} \times X \to X$ and $0 : S \to X$ be holomorphic maps such that the following diagrams commute

$$\begin{array}{ccc} X \times_S X & \xrightarrow{a} & X \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & S & \end{array} \qquad \begin{array}{ccc} (\mathbb{C} \times S) \times_S X & \xrightarrow{m} & X \\ & {\scriptstyle \pi_2}\searrow \quad \swarrow{\scriptstyle p} & \\ & S & \end{array}$$

and that satisfy the axioms of a $(\mathbb{C} \times S)$-module in the category of complex spaces over S. Note that $(\mathbb{C} \times S) \times_S X \cong \mathbb{C} \times X$ so the second diagram can be changed accordingly.

These data define a *holomorphic linear space* over S (the standard terminology is just "linear space", but we will use this term to avoid confusion with abstract linear spaces).

Definition 4.1.3 (Morphisms of linear spaces)**.** Let (X, a, m) and (Y, a', m') be two holomorphic linear spaces over S. Then a map $f : X \to Y$ is called a morphism if the following diagrams commute:

$$\begin{CD} X \times_S X @>{f\times f}>> Y \times_S Y \\ @V{a}VV @VV{a'}V \\ X @>{f}>> Y \end{CD} \qquad \begin{CD} \mathbb{C} \times_S X @>{f\times f}>> \mathbb{C} \times_S Y \\ @V{m}VV @VV{m'}V \\ X @>{f}>> Y \end{CD}$$

By a result of Prill [27] any holomorphic linear space over S locally, over a neighbourhood U of every point of S, embeds into $S \times \mathbb{C}^n$ for some n. Further, by a result of Fischer [4], any holomorphic linear space is locally a kernel of a morphism of holomorphic linear spaces $U \times \mathbb{C}^n \to U \times \mathbb{C}^m$.

One can associate to a holomorphic linear space X over S an analytic sheaf $\mathcal{F}_X$ on S defined by

$$\mathcal{F}_X(U) = \mathrm{Hom}(X, S \times \mathbb{C})(U)$$

where Hom means the set of morphisms of holomorphic linear spaces. It follows from the results of Fischer and Prill quoted above that this sheaf is coherent.

Conversely, let $\mathcal{F}$ be a coherent sheaf over Y. By definition, it can be presented locally as a quotient: $\mathcal{O}_U^n \xrightarrow{f} \mathcal{O}_U^m \to \mathcal{F}|_U \to 0$, where f can be written down as an $m \times n$ matrix with entries in $\mathcal{O}_S(U)$. Consider the dual map of holomorphic linear spaces $U \times \mathbb{C}^m \to U \times \mathbb{C}^n$ given by the $n \times m$ matrix f^*, adjoint to f. Define $X_U^{\mathcal{F}}$ to be the complex subspace of $Y \times \mathbb{C}^m$ defined by the equations $\sum_j f_{ji}x_i = 0, 1 \leq j \leq n$. Cover Y with opens U_i and glue $X_{\mathcal{F}}$ from U_i. This defines a correspondence:

$$\{\text{holomorphic linear spaces over } S\} \Longleftrightarrow \{\text{coherent sheaves over } S\}$$

Note that in general $X_{\mathcal{F}}$ is not reduced.

A trivial vector bundle of rank n inherits holomorphic linear space structure from the vector space $\mathbb{C}^n$. A locally trivial vector bundle X has a natural structure of a holomorphic linear space on each trivialisation, and these glue together into a holomorphic linear space structure on the whole of X.

A *fibre* of a holomorphic linear space X over a point s is defined as $X_s = X \times_S \{s\}$ and has a natural structure of a holomorphic linear space over s. By a theorem of Cartier (see Fischer [5], 1.4 for details) X_s is reduced, therefore there is a biholomorphic map to $\mathbb{C}^n$ that turn a and m into the standard addition and scalar multiplication operations on $\mathbb{C}^n$. A morphism of holomorphic linear spaces induces a linear map between the fibres.

Recall that the universal property of reduction is that given a space X its reduction is a space X^{red} endowed with a morphism $r : X^{red} \to X$ such that any morphism $Z \to X$ from a reduced space Z factors through r. The product in the category of *reduced* complex spaces is the reduction of the product in the category of all complex spaces: indeed, given any pair of morphisms $Z \to X, Z \to Y$ over S there exists a morphism $Z \to X \times_S Y$ and by universal property of the reduction it factors through $(X \times_S Y)^{red}$, so $(X \times_S Y)^{red}$ satisfies the universal property of a product.

Definition 4.1.4 (Reduced holomorphic linear space)**.** A *reduced holomorphic linear space* is a holomorphic linear space in the category of reduced complex spaces, i.e. it has the same definition as a holomorphic linear space with all spaces considered reduced and fibre products understood as reduced fibre products.

A *morphism* of reduced holomorphic linear spaces is a holomorphic map that induces linear maps between the fibres.

While a reduction of a holomorphic linear space is a reduced holomorphic linear space, an arbitrary reduced linear space (X, a, m) does not have a canonical structure of a holomorphic linear space: there might be no lifting of $a : (X \times_S X)^{red} \to X$ to $X \times_S X$ if the latter is non-reduced — this is why we have introduced this distinct notion. The definition of morphisms of holomorphic linear spaces in the category of reduced complex spaces simplifies to the one stated above. If X and Y are non-reduced holomorphic linear spaces then a map $f : X \to Y$ can be linear on fibres but do something non-linear with the nilpotent elements of the structure sheaf of X, that is why one has to define a morphism non-reduced linear spaces diagrammatically.

However, if the rank of the fibres is constant, a smooth reduced holomorphic linear space over a smooth base is indeed a holomorphic linear space.

Proposition 4.1.5. *Let X be a reduced holomorphic linear space over S, of constant fibre rank, let X and S be a complex manifolds and let X be connected. Then X is a holomorphic linear space. Moreover, X is a holomorphic vector bundle.*

Proof. By Theorem 2.3.29 p is flat. The morphism $X \times_S X \to S$ is flat as a fibre product of two flat morphisms.

We now use two facts: first, that flat maps are open (Fischer [5], Corollary 3.20), second, that if there is an open map $f : X \to Y$ to a smooth manifold with reduced fibres then X reduced (Proposition in the paragraph 3.20, loc.cit.).

Recall that fibres of $p : X \to S$ are reduced, then so are the fibres of $X \times_S X$. It follows that $X \times_S X$ is reduced, and the map $a : (X \times_S X)^{\mathrm{red}} \to X$ obviously lifts to $X \times_S X$. Therefore, X is a holomorphic linear space. □

Proposition 4.1.6 (Fischer [5],Theorem 1.8)**.** *A holomorphic linear space X over a reduced space S is a vector bundle if and only if the function $s \mapsto \dim_{\mathbb{C}} X_s$ is locally constant.*

With these results in hand we can show that complex analytic models over smooth varieties are vector bundles with some extra structure.

Proposition 4.1.7. *Let X be a smooth reduced algebraic variety, $M = (V, X, \mathbb{C}, p)$ be a quantum Zariski geometry and suppose it has a complex analytic model $\overline{V}$. Then $\overline{V}$ is a holomorphic vector bundle over X^{an}.*

Proof. It follows from the definition of a complex analytic model that $(\overline{V}, \overline{a}, \overline{m})$ is a reduced holomorphic linear space with constant fibre rank. By Proposition 4.1.5 it is a holomorphic linear space and by Theorem 4.1.6 it is a vector bundle. □

A famous result of Serre says that over projective varieties there is no difference between algebraic and holomorphic sheaves.

Theorem 4.1.8 (Serre [30],Théorème 3)**.** *Any coherent analytic sheaf over a projective variety is isomorphic to an analytification of an algebraic coherent sheaf.*

The definitions of a holomorphic linear space and reduced holomorphic linear space have natural analogues in the category of algebraic varieties over $\mathbb{C}$. As a corollary of the correspondence between coherent sheaves and holomorphic linear spaces and Theorem 4.1.8 we have:

Corollary 4.1.9. *Any holomorphic linear space of constant rank over a projective variety is biholomorphically equivalent to an algebraic linear space.*

Complex-analytic Brauer group

In [11], introducing cohomological methods for the study of Brauer groups, Grothendieck gave the following abstract definition of an Azumaya algebra.

Definition 4.1.10 (Azumaya algebra over a topological space)**.** Let X be a topological space and let $\mathcal{O}_X$ be a sheaf of continuous functions from X to $\mathbb{C}$. An *Azumaya algebra* over X is a sheaf of $\mathcal{O}_X$-algebras which is locally trivial as a sheaf of modules and with fibres isomorphic to matrix algebras $M_n(\mathbb{C})$.

In particular, one can consider a complex analytic space X with its structure sheaf. Then the theory of a Brauer group of a variety (see Section 3.3) can be repeated almost verbatim. Two Azumaya algebras A and A' are considered equivalent, if there are locally free sheaves E and E' such that $A \otimes \mathcal{E}nd(E) \cong A' \otimes \mathcal{E}nd(E')$.

Definition 4.1.11 (Brauer group of a complex analytic space)**.** The complex-analytic Brauer group $\mathrm{Br}^{\mathrm{an}}(X)$ of a complex analytic space X is the group of equivalence classes of complex-analytic Azumaya algebras with the group operation the tensor product.

Azumaya algebras are described by cohomology classes of analytic Čech cohomology group $\check{H}^1(X, \mathrm{PGL}_n)$, and similarly to the algebraic case, there exists a natural map from the complex-analytic Brauer group into $\check{H}^2(X, \mathcal{O}_X^\times)$ via the connecting homomorphism δ:

$$\ldots \to \check{H}^1(X, \mathrm{GL}_n) \to \check{H}^1(X, \mathrm{PGL}_n) \xrightarrow{\delta} H^2(X, \mathcal{O}_X^\times)$$

Grothendieck [11] (Proposition 1.4) shows that the image of this morphism is torsion and that the morphism $\mathrm{Br}^{\mathrm{an}}(X) \to H^2(X, \mathcal{O}_X^\times)$ is injective.

Theorem 4.1.12. *Let X be a smooth algebraic variety over $\mathbb{C}$. Let $M = (V, X, \mathbb{C}, p)$ be a quantum Zariski geometry and let B be the Azumaya algebra that corresponds to it by Theorem 3.4.4. The structure M has a complex analytic model if and only if the class of B in $\mathrm{Br}^{\mathrm{an}}(X)$ is trivial.*

Proof. Since X is smooth, we can apply Theorem 4.1.7 and infer that any complex analytic model of any quantum Zariski geometry over X is a holomorphic vector bundle.

The right to left implication is almost immediate: since B is trivial in the complex-analytic Brauer group of X, it is of the form $\mathcal{E}nd(W)$. The action of elements of A on W is given by the map ρ from Theorem 3.4.4, and by this theorem W constitutes the complex analytic model of M.

For the left to right direction, suppose that B is non-trivial in $\mathrm{Br}(X)$ but there is a vector bundle W, a map $\rho' : A \to \mathcal{E}nd(W)$ and isomorphisms η'_x for any $x \in X$ that make the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\rho'} & \mathcal{E}nd(W)_x \\ & \searrow^{\theta_x} & \downarrow{\scriptstyle \eta'_x} \\ & & \mathrm{End}_F(M_x) \end{array}$$

(η'_x induce isomoprhisms of A-modules). Since there is a similar diagram for B and B_x-s, the sheaves of modules B and $\mathcal{E}nd(W)$ are fibrewise isomorphic. Since these are locally free modules over a reduced complex space, they are isomorphic. But that contradicts the non-triviality of the class of B in the complex-analytic Brauer group of X. □

Quantum torus has a complex analytic model

Consider the Structure 3.2, which has been already mentioned in Sections 3.2 and 3.5. Theorems 3.4.4 and 3.4.11 associate to it an Azumaya algebra over a variety X over an algebraically closed field F. Let F be $\mathbb{C}$, we will show that the analytification of the Azumaya algebra A is of the form $\mathcal{E}nd(E)$ for some vector bundle on the analytification of X.

Definition 4.1.13 (Stein space)**.** A *Stein space* is a complex analytic space biholomorphic to an analytic subset of $\mathbb{C}^n$. In particular, an analytification of an affine variety is Stein.

Proposition 4.1.14 ([13])**.** *The homotopy type of an n-dimensional Stein variety contains only cells of dimension $\leq n$. In particular, cohomology groups with constant coefficients $H^i(X, C)$ vanish for $i > n$.*

Theorem 4.1.15 (Cartan's Theorem B, Grauert and Remmert [8], Chapter I,§4.6)**.** *Let X be a Stein (in particular, affine algebraic) complex space and let $\mathcal{F}$ be an analytic coherent sheaf over X. Then*

$$H^q(X, \mathcal{F}) = 0, \text{ for all } q > 0.$$

Theorem 4.1.16. *Any quantum Zariski geometry over a complex Stein surface, and in particular the Structure 4, has a complex analytic model.*

Proof. The following argument is from Section 2 of Schroeer [29]. Consider the exponential sequence of sheaves on X

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 1$$

and the corresponding long cohomology sequence

$$\ldots \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^\times) \to H^3(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X)$$

The first term vanishes by Theorem 4.1.15 and the last term vanishes by Proposition 4.1.14, therefore $H^2(X, \mathcal{O}_X^\times) = 0$, and in particular the class of the Azumaya algebra $\mathcal{A}$ is trivial. By Theorem 4.1.12 we get the desired. $\square$

This result is a bit surprising when compared to the algebraic case (Section 3.5), where the "algebraic" Azumaya algebra corresponding to the Structure 4 is not of the form $\mathcal{E}nd(E)$, and the quantum Zariski geometry is not interpretable in an algebraically closed field.

4.2 Structure $\mathcal{A}$

This section contains a partial result on interpretation of abstract linear spaces in the structure of compact complex spaces. Understanding how abstract linear spaces can be interpreted in the structure $\mathcal{A}$ is essential to getting any results about interpretability of structures of the kind that have been considered in Chapter 3. We also discuss the difficulties one encounters trying to formulate (non-)definability results for $\mathcal{A}$.

Interpretations of abstract linear spaces in $\mathcal{A}$

We now prove that an interpretation of a linear space of constant rank in the structure $\mathcal{A}$ can be locally trivialised in complex topology in a weak sense: there is piecewise biholomorphic map between a restriction of the holomorphic linear space to a neighbourhood of a point and the trivial vector bundle over this neighbourhood. This statement is analogous to Proposition 4.1.5.

Lemma 4.2.1. *Let O be a set dense in $\mathbb{C}^n$ in Zariski topology. Then there exists a basis $e_1, \ldots, e_n$ such that every e_i lies in O.*

Proof. Proof by induction. The $n = 1$ case is clear. Suppose the statement is proven for $n = k$ and let $O \subset \mathbb{C}^{k+1}$. Pick a k-dimensional subspace $V \subset \mathbb{C}^{k+1}$ with a basis $e_1, \ldots, e_k$. Since O is dense in $\mathbb{C}^{k+1}$, $O \setminus V$ is non-empty. Hence we can pick a vector $e_{k+1} \in O \setminus V$. The set $e_1, \ldots, e_{k+1}$ is a basis of $\mathbb{C}^{k+1}$ that lies in O. □

Lemma 4.2.2. *Let $f : X \to Y$ be a bijective holomorphic map of reduced complex spaces. Then X and Y are "piecewise biholomorphic", i.e. there is a decomposition $X = \sqcup X_i, Y = \sqcup Y_i$ such that X_i and Y_i are biholomorphic.*

Proof. Recall that a morphism $f : X \to Y$ which is a submersion and 0-mersion at $x \in X$ is a biholomorphic map in some neighbourhood of x (by the definition of submersion). By Theorem 2.3.28 the necessary and sufficient condition for that is that $f : X \to Y$ is flat at x (the condition for the fibre to be non-singular being vacuous as the fibre consists of a single reduced point). By Theorem 2.3.30 the non-flat locus of a morphism is an analytic set.

We proceed by Noetherian induction. By what has been said above, f is a local biholomorphism outside the non-flat locus X^{nf}. Since it is bijective, its restriction to $X \setminus X^{\mathrm{nf}}$ is just a biholomorphic map. Put $X_0 = X \setminus X^{\mathrm{nf}}$ and $Y_0 = f(X \setminus X^{\mathrm{nf}})$. Then apply the procedure inductively to X^{nf} and $f(X^{\mathrm{nf}})$. □

Lemma 4.2.3 (Fischer [5], Proposition 3.4)**.** *Let $f : X \to Y$ be a holomorphic map. Then every point $p \in X$ has a neighbourhood U such that for all $x \in U$, $\dim_x X_{f(x)} \leq n$ where $n = \dim_p X_{f(p)}$.*

Theorem 4.2.4. *Let $M = (X, S, \mathbb{C}, p)$ be an abstract linear space where S is an irreducible algebraic variety and the dimension of all the fibres is n. Consider an interpretation of M in the structure of compact complex manifolds $\mathcal{A}$. Then $S(\mathcal{A})$ is definably isomorphic to S and there exists an open in Zariski topology set $U \subset S(\mathcal{A})$ such that for any point $s \in U$ there exists an open dense (in Zariski topology) $U \subset S(\mathcal{A})$ such that for any point $s \in U$ there exists an open neighbourhood $U_s \subset U$ (in complex topology) such that $p^{-1}(U_s)$ is piecewise biholomorphic to a trivial vector bundle over $U_s \times \mathbb{C}^n$.*

Proof. Since S is an ample pre-smooth Zariski geometry (after possibly removing the singular points), by Theorem 2.1.11 it interprets a field F. It is shown in Pillay [24] (see discussion after Remark 3.10) that F is definably isomorphic to $\mathbb{C}$, where by the latter we understand a definable subset of the complex manifold $\mathbb{P}^1$ in $\mathcal{A}$, equipped with definable field operations. It follows that $S(\mathcal{A})$ is definably isomorphic to itself as a definable subset in $\mathbb{C}^n$.

Let X^{nf} be the non-flat locus of p which is an analytic subset of X by Theorem 2.3.30. The set $p(X \setminus X^{\mathrm{nf}})$ consists of points s of S such that $p^{-1}(s)$ does not belong entirely to X^{nf}. This set is dense in S in Zariski topology as a continuous image of the dense set $X \setminus X^{\mathrm{nf}}$. By quantifier elimination in the structure $\mathcal{A}$, $p(X \setminus X^{\mathrm{nf}})$ is constructible, and since it is also dense, it contains a Zariski open U. By Theorem 2.3.28 and Theorem 2.3.33 for any $s \in U$, there is a dense open subset O of the fibre $p^{-1}(s)$ such that for any $x \in O$ the morphism p is a submersion at x.

We are going to construct for every point $s \in U$ a neighbourhood U_s and a bijection $p^{-1}(U_s) \cong U_s \times \mathbb{C}^n$. Take a point s, and using Lemma 4.2.1 pick a basis $e_1, \ldots, e_n$ in $X_s = p^{-1}(s)$ so that p is flat and X_s is non-singular in $e_1, \ldots, e_n$.

By Theorem 2.3.28 p is a submersion at all points of a dense subset of the fibre $p^{-1}(s)$. It admits local sections $f_1, \ldots, f_n$ passing through $e_1, \ldots, e_n$, defined over neighbourhoods U_i of s. Let $U' = \cap U_i$. Consider the map

$$h : U' \times \mathbb{C}^n \to p^{-1}(U') \qquad (u, a_1, \ldots, a_n) \mapsto \sum a_i \cdot f_i(u)$$

Depending on a point $u_0 \in U'$ either the restriction of this map to the fibre $\{u_0\} \times \mathbb{C}^n$ is a bijection (when $f_1(u_0), \ldots, f_n(u_0)$ is a basis in $p^{-1}(u)$) or any fibre $h^{-1}(y), y \in p^{-1}(u_0)$ is a linear subspace of $\mathbb{C}^n$ of positive dimension. Pick a point $x \in \{s\} \times \mathbb{C}^n$. By Lemma 4.2.3 there is an open $U'' \ni x$ such that restriction of h to U'' is a bijection, and by above considerations h is a bijection over $\pi^{-1}(\pi(U''))$ too.

Let $U_s = \pi(U'')$. By Lemma 4.2.2, $U_s \times \mathbb{C}^n$ and $p^{-1}(U_s)$ are piecewise biholomorphic.

□

Discussion

If a quantum Zariski geometry $M = (V, X, \mathbb{C}, p)$ has a complex analytic model W it doesn't necessarily imply that M is definable in $\mathcal{A}$. Indeed, a definable set in $\mathcal{A}$ is a constructible subset of a compact complex space and a definable map is a piecewise meromorphic map. It might well turn out that for any compactification of W the projection from W to X is not meromorphic. The classification of compactifications of a holomorphic linear space is a hard problem: a much simpler particular case of this question, finding compactifications of $\mathbb{C}^n$, is only resolved for $n = 1, 2, 3$ (see Brenton and Morrow [2]).

Question: Let E be an analytic vector bundle over X. Is there always a compactification E' of E such that the projection $E \to X$ is a meromorphic function on E'? If not, how does one find out if such a compactification exists?

In any case, in order to prove something about interpretations of structures that have an abstract linear space as a reduct one needs to understand the interpretations of abstract linear spaces in $\mathcal{A}$.

Theorem 4.2.4 constructs piecewise biholomorphic trivialisations $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^n$. The transition functions $\varphi_j \circ \varphi_i^{-1}$ are not holomorphic maps from $(U_i \cap U_j)$ to GL_n in general, only piecewise holomorphic. However, it might be possible that one can choose the trivialisations carefully, so that the transition functions actually are holomorphic. This defines a Čech cocycle corresponding to a vector bundle, and the intepretation is piecewise biholomorphic (via trivialisiations) to this vector bundle.

Conjecture: Let $M = (X, S, \mathbb{C}, p)$ be an abstract linear space where S is algebraic and irreducible and the dimension of all the fibres is n. Then an interpretation of M in the structure $\mathcal{A}$ is definably isomorphic to a rank n holomorphic linear space over $S(\mathcal{A})$.

If this conjecture holds when S is a curve, we have reduced the situation to the algebraic one, since any complex analytic bundle E over a curve S is biholomorphic to an algebraic one. Indeed, if S is compact then by Riemann existence theorem it is projective and by GAGA principle (Theorem 4.1.8), E is isomorphic to an algebraic bundle. If S is not compact then by a result of Röhrl [28] any holomorphic vector bundle over S is trivial.

Note that in this case we do not have to worry about compactifications and the projection function being meromorphic, because algebraic varieties in $\mathcal{A}$ are definable in a canonical compact space ($(\mathbb{P}^1)^n$) and regular functions are automatically meromorphic.

However, starting from dimension 2 there are plenty of non-algebraic vector bundles even over an algebraic base. The standard example is $X = \mathbb{C}^2 \setminus \{0\}$. The line bundles over X are in bijective correspondence with Čech cohomology classes in $\check{H}^1(X, \mathcal{O}_X^\times)$. Using the exponential sheaf sequence we get the following long exact sequence:

$$\ldots \to \check{H}^1(X,\mathbb{Z}) \to \check{H}^1(X,\mathcal{O}_X) \xrightarrow{\exp} \check{H}^1(X,\mathcal{O}_X^\times) \to \check{H}^2(X,\mathbb{Z}) \to \ldots$$

Since $\mathbb{C}^2 \setminus \{0\}$ is homotopy equivalent to the 3-sphere, the cohomology groups $\check{H}^1(X,\mathbb{Z})$ and $\check{H}^2(X,\mathbb{Z})$ vanish, so $\check{H}^1(X,\mathcal{O}_X) \cong \check{H}^1(X,\mathcal{O}_X^\times)$. To compute $\check{H}^1(X,\mathcal{O}_X)$ cover X with two affine open sets $U_1 = \mathbb{C}^\times \times \mathbb{C}$ and $U_2 = \mathbb{C} \times \mathbb{C}^\times$. The cocycles with values in $\mathcal{O}_X$ form a vector space with the basis given by monomials $x^i y^j, i, j \in \mathbb{Z}$ and coboundaries is the subspace generated by monomials $x^i y^j, i, j \geq 0$ where x, y are coordinates on $\mathbb{C}^2$. Gluing trivial bundles on U_1 and U_2 using transition functions $e^{x^i y^j}$ defines distinct line bundles for distinct pairs of numbers $i, j < 0$. On the other hand, as is well-known, removing a codimension 2 closed set does not change $\check{H}^1_{\text{Zar}}(X, \mathcal{O}_X^\times)$ (see Hartshorne [14], Proposition II.6.5) and $\check{H}^1_{\text{Zar}}(\mathbb{C}^2, \mathcal{O}_X^\times) = 0$ by Quillen-Suslin theorem (Lang [18], XXI, §3, Theorem 3.5), so $\check{H}^1_{\text{Zar}}(X, \mathcal{O}_X^\times) = 0$ and all algebraic line bundles over X are trivial.

Let us finally discuss the problems specific to interpretations of quantum Zariski geometries. The Brauer group of a topological or complex analytic space can be equivalently defined as the group of equivalence classes of principal topological or holomorphic PGL_n-bundles. Such bundles, just like Azumaya algebras, are classified by elements of the Čech cohomology group $\check{H}^1(X, \mathrm{PGL}_n)$ where PGL_n is the sheaf of germs of continuous or holomorphic functions with values in $\mathrm{PGL}_n(\mathbb{C})$. The operation

$$\mathrm{PGL}_n(\mathbb{C}) \times \mathrm{PGL}_m(\mathbb{C}) \to \mathrm{PGL}_{nm}(\mathbb{C}) \qquad (A, B) \mapsto A \otimes B$$

gives rise to the group operation on principal PGL_n-bundles, and bundles E_1 and E_2 are considered equivalent if there are vector bundles F_1 and F_2 with projectivisations $\mathbb{P}(F_1)$ and $\mathbb{P}(F_2)$ such that $E_1 \otimes \mathbb{P}(F_1) \cong E_2 \otimes \mathbb{P}(F_2)$.

A result of Grauert [7] says that over a Stein manifold X there is no difference between complex analytic and topological fibre bundles with fibre complex Lie group. More precisely, any two such analytic fibre bundles are isomorphic analytically if they are isomorphic topologically and for every such topological fibre bundle there exists an analytical fibre bundle topologically isomorphic to it.

Now back to definability in $\mathcal{A}$. Even if we suppose that the **Conjecture** is true, and even if we do not care about compactifications for a moment, checking that a quantum Zariski geometry over a variety X with an associated Azumaya algebra B is definable amounts to checking whether there exists a decomposition of X in definable parts $X = \cup X_i$ such that restrictions B_i of B to X_i are trivial in $\mathrm{Br}(X_i)$. Even if we only consider such partitions that all X_i are Stein, the Brauer group of X_i-s depends on the *topology* of X_i-s, which makes the problem quite challenging.

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