Gromov-Hausdorff limits of flat Riemannian surfaces and non-Archimedean geometry

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Abstract

We study Gromov-Hausdorff limits of complex curves with singular flat metrics and show that the limit of a maximally degenerate family of such curves is a metric graph which is homeomorphic to a quotient of a skeleton of the associated Berkovich analytic space with respect to a semi-stable vertex set.

Contents

1 Introduction

This note deals with a problem inspired by the approach to the SYZ conjecture due to Kontsevich and Soibelman. We consider Gromov-Hausdorff limits of complex projective curves with flat Kähler metrics, relaxing the assumption on the triviality of the tangent bundle and allowing conical singularities, and relate them with subsets of the Berkovich analytification of the degeneration which is described in terms of a weight function on the non-Archimedean analytic space. This function is associated to the metric, and it was defined in [13] and further studied in [18] and [23]. This study serves as a proof of concept for a certain new approach to the study of Gromov-Hausdorff limits of Kähler manifolds in the collapsed case that it would be desirable to extend to higher dimensions.

We find it convenient to use the language of model theory to facilitate our construction of the limit. It is used to convert information related to the

version of February 11, 2018

geometry of the non-Archimedean analytic space associated to the degeneration into estimates on the asymptotic length of geodesics. One relies on the comparison of definable sets in Cartesian powers of the algebraic closure of a certain expansion of an infinite non-principal ultrapower of the field of real numbers with a convex subring (the so-called Robinson asymptotic field). One considers the algebraic closure of this field from two points of view: as an algebraically closed valued field and as a Cartesian square of a real closed field, with two collections of definable sets arising. This comparison leads to methods with the flavour of non-standard analysis (though not in the sense of the book of Robinson [22]). As it turns out, in the setting of the considered problem in order to understand the Gromov-Hausdorff limit it is only important to distinguish between polynomial and logarithmic growth of distances between points.

The Robinson's asymptotic field has already been used by Kramer and Tent [14] to study asymptotic cones of non-compact Lie groups. This is yet another instance of a relationship between ultraproducts and Gromov-Hausdorff limits first studied by van den Dries and Wilkie [26]. In fact, there exists considerable literature on Gromov-Hausdorff or Hausdorff limits in the tame (o-minimal or subanalytic) setting, which although it does not make use of the Robinson asymptotic field, relies on the "tameness" of geometry of the metric spaces involved in order to deduce the necessary estimates and construct ε-isometries; the principal technical tool here is Lipschitz stratifications [8].

Hausdorff limits of families of real semi-algebraic subsets of a Euclidean space, or of subsets definable in an o-minimal expansion of reals, were considered by van den Dries [24], Lion and Speisseger [17], and Kocel-Cynk, Pawłucki, and Vallete [12]. Gromov-Hausdorff limits of real semi-algebraic subsets of Euclidean space with the induced inner metric were considered by Bernig and Lytchak [2, 3]. Gromov-Hausdorff limits of families of definable metric spaces were considered by Walsberg [27]. The inner metric on subanalytic sets has been studied by Kurdyka and Orro [15]. Let us finally remark that glimpses of a relationship between Berkovich spaces and Gromov-Hausdorff limits of complex varieties can be seen in the work of Odaka [20, 21].

In the context of the SYZ conjecture, let us mention the work of Mustaţa and Nicaise [18], Nicaise and Xu [19], Temkin [23], and Boucksom and Jonsson [5] on asymptotic behaviour of volume, which glues in Berkovich spaces as "singular fibres" of a degeneration. Though the latter work concerns itself with weak limits of volume forms, the expectation (predicted in the original conjectures of Konstevich and Soibelman) is that the limit metric behaviour of Kähler manifolds is reflected in the associated Berkovich space as well. The collapsed case have been studied in the case of K3 surfaces and Abelian fibred varieties: [9], [10], [11].

Let us say a few words about the main steps of our strategy. We work with a family $\mathfrak{X}$ of curves fibred over $\mathbb{C}\setminus\{0\}$, and we consider the associated base change of $\mathfrak{X}$, a curve over the filed of Laurent series $X = \mathfrak{X} \otimes \mathbb{C}((t))$. We are given a relative holomorphic 1-form Ω and endow the fibres $\mathfrak{X}_s$ with the Kähler metric $\Omega_s \wedge \bar{\Omega}_s$. We consider families that satisfy the property of maximal degeneration near 0, in particular, this property implies that the diameter of $\mathfrak{X}_s$ explodes logarithmically. The metrics $\Omega_s \wedge \bar{\Omega}_s$ are flat away from the zeroes of Ω. In order to study the Gromov-Hausdorff limits we, following Kontsevich and Soibelman, normalize the metric so that the diameter of the fibres $\mathfrak{X}_s$ is constantly 1.

Recall that a semi-stable vertex set in a projective Berkovich curve is a set of points such that the connected components of its complement are finitely many open annuli and open balls; semi-stable vertex sets are in one-to-one correspondence with models of the curve over the value ring such that the special fibre is a strict normal crossings divisor. Given a form Ω we pick a semi-stable vertex set and refine it so that it acquires a certain property that we call "being monomial with respect to Ω", it means that on the finitely many open annuli U_i of the decomposition the form Ω can be written down as fdx where x is a coordinate function of the annulus and f is such that $|f|= |x|^a$ for some integer a.

The formalism of Robinson fields allows us to pass from a cover of a curve X by affinoid sets to a cover of $\mathfrak{X}_s$ with real semi-algebraic sets for s sufficiently close to 0. We, therefore, find a cover of X by affinoid sets $\bar{U}_i$ such there for each i exists a regular function y_i and $|y_i|< r_i$ defines either an open annulus U_i inside $\bar{U}_i$ or an open ball B_j from the semi-stable decomposition; to the sets $\bar{U}_i$ we associate real semi-algebraic subsets $\bar{\mathcal{U}}_i$ such that $(\bar{\mathcal{U}}_i)_s$ cover $\mathfrak{X}_s$ for s close to 0. This decomosition facilitates the computaition of the weight function on $\bar{U}_i$ (its minimum locus will be a subset of the skeleton associated to the semi-stable decomposition), as well as the computation of the asymptotic bounds on the lengths of shortest geodesics in $\mathfrak{X}_s$ by breaking them down into pieces that lie entirely in $(\mathcal{U}_i)_s$.

We further define for each sufficiently small $\alpha \in \mathbb{Q}$ semi-algebraic sets $\mathcal{D}_{\leq\alpha}, \mathcal{D}_{\geq\alpha} \subset \mathfrak{X} \times_{\mathbb{C}\setminus\{0\}} \mathfrak{X}$ so that the length of a shortest geodesic (in the normalized metric) with endpoints in $\mathcal{D}_{\leq\alpha}$, resp. $\mathcal{D}_{\geq\alpha}$ is asymptotically bounded above, resp. below, by α. The intersection of $(\mathcal{D}_{\leq\alpha})_t$ is an ACVF-definable equivalence relation such that the quotient $X_t/\!\sim$ is the Gromov-Hausdorff limit. It is a metric graph which can be alternatively described as a quotient of the minimality locus of the weight function.

The main result of this paper is Theorem 3.26, which is preceded by the study of the asymptotic behaviour of the geodesic metric in terms of the geometry of the associated Berkovich space in Section 3. Section 2 summarizes well-known results on the structure of Berkovich curves and Robinson fields that are used further on.

Acknowledgements. I would like to thank Ehud Hrushovski, Grisha Papayanov, Jérôme Poineau, Matthew Stevenson, Misha Verbitsky and Eric

Walsberg for helpful discussions. I also would like to thank I.H.É.S. for hospitality in January and February 2018 when this paper was completed.

2 Preliminaries

2.1 Structure of Berkovich curves

I refer to Section 5 of [1] for results on the structure of non-Archimedean analytic curves; we recall briefly some of the statements that will be used further.

Recall that if x is a point on a Berkovich analytic space then the induced valuation $|\cdot|_x\colon A \to \mathbb{R}$ on an affinoid neighbourhood $\mathcal{M}(A)$ descends to the completion of the fraction field of the quotient of A by the kernel of $|\cdot|_x$, the *completed residiue field* $\mathcal{H}(x)$ of x.

Points of Berkovich curves are classified according to the transcendence degree e_x of the residue field $\widetilde{\mathcal{H}}(x)$ over $\tilde{k}$, and rational rank f_x of the Abelian group $|\mathcal{H}(x)^\times|/|k^\times|\otimes_{\mathbb{Z}}\mathbb{Q}$. The points belong to one of 4 types as follows

- type 1: $\mathcal{H}(x) \subset k^{alg}$;
- type 2: $e_x = 1$, $f_x = 0$;
- type 3: $e_x = 0$, $f_x = 1$;
- type 4: $e_x = 0$, $f_x = 0$ and $\mathcal{H}(x)$ is not contained in k^{alg}.

The union of type 2 and type 3 points of X is denoted $\mathbb{H}(X)$.

We will use the following notation for spaces isomorphic to domains in the affine line

$$\begin{aligned} A(r,r') &:= \{\ x \in (\mathbb{A}^1)^{\mathrm{an}} \ \mid\ r \leq |x| \leq r'\ \} \\ A^-(r,r') &:= \{\ x \in (\mathbb{A}^1)^{\mathrm{an}} \ \mid\ r < |x| < r'\ \} \\ B^-(r) &:= \{\ x \in (\mathbb{A}^1)^{\mathrm{an}} \ \mid\ |x| < r\ \} \end{aligned}$$

where r or s can be $+\infty$ or $-\infty$, and call $A(c,r,s)$ *closed annulus* if r and s are finite, call $A-(c,r,s)$*open annulus* if either r or s finite; sets $A(0,r)$ and $B^-(r)$ are called *closed* and *open balls*. The pullback of the analytic function x is called a *coordinate function* on a closed or open annulus, or a ball, respectively.

A point of an analytic space X belongs to its *interior* if it has a neighbourhood isomorphic to an open polydisc; the *boundary* of X, denoted ∂X is the complement of the interior. The skeleton $\mathrm{Sk}(A)$ of an open annulus A is the interior of the interval that connects its boundary points. There exists a retraction map $\mathfrak{r}_A : A \to \mathrm{Sk}(A)$ which is constant on the connected components of $A \setminus \mathrm{Sk}(A)$.

Fact 2.1. *An invertible function on an annulus with coordinate function x is of the form*

$$cx^b(1+u)$$

where $|u|< 1$, $c \in K^\times, b \in \mathbb{Q}$.

Each point x of a closed ball $A(0,s)$ is the boundary point of a smallest open ball $B_x = A(0, r(x)) \subset A(0,s)$, the map $x \mapsto r(x)$ is called the *radius function.* For two points $x, y \in A(0,r)$ there exists the smallest open ball $B_z = A(0, r_z) \subset$ such that $B_x \cong A(0, r(x)) \subset B_z \cong A(0, r_z)$ and $B_y \cong A(0, r(y)) \subset B_z \cong$. The boundary point of B_z is denoted $x \vee y$. Put

$$d(x,y) = (r(x \vee y) - r(x)) + (r(x \vee y) - r(y))$$

A *geodesic segment* on a curve between points x, y is an interval in X joining x and y of length $d(x,y)$ in the metric defined above.

Call two geodesic segments $\gamma, \gamma' : [0,1] \to X$ starting at $x \in \mathbb{H}$ equivalent if they agree on a neighbourhood of 0. Define *tangent direction* at a point $x \in \mathbb{H}(X)$ to be the equivalence classes of nontrivial geodesic segments starting at x. There is a bijective correspondence beweet tangent directions and valuations on the field $\widetilde{\mathcal{H}}(x)$; we will denote the valuation corresponding to the tangent direction v as ord_v.

A sufficiently small neighbourhood of type 1 point is an interval, of type 2 point — a union of intervals starting at the point (and indexed by points of the projective $\tilde{k}$-curve with the function field $\widetilde{\mathcal{H}}(X)$), a small neighbourhood of a type 3 point is an interval containing the point.

A function $F : X \to \mathbb{R}$ is *piecewise linear* if for any geodesic segment $\gamma : [0,1] \to \mathbb{H}(X)$ the pullback $F \circ \gamma : [0,1] \to \mathbb{R}$ is piecewise linear. The *outgoing slope* of a piecewise linear function F at a point $x \in H(X)$ along a tangent direction v is defined to be $d_v F(x) = \lim_{\tau \to 0} (F \circ \gamma(\tau))$ where γ is a path in the tangent direction v

Lemma 2.2 (Slope formula)**.** *Let f be an analytic function and x be a type 2 point. The function $-\log|f|$ is piecewise linear in a neighbourhood of x and*

$$\mathrm{ord}_v \tilde{f} = d_v(-\log|f|)$$

Definition 2.3 (Semi-stable vertex set)**.** *Let X be a projective analytic curve.* A semi-stable vertex set of X *is a finite set of type 2 points* $\Sigma = \{ x_1, \ldots, x_n \} \subset X^{an}$*, such that* $X \setminus \Sigma$ *is a disjoint union of finitely many open annuli and open balls.*

Definition 2.4 (Embedded graph)**.** *By an embedded graph in X^{an} we will understand the following data: a collection of points* $V = \{ v_1, \ldots, v_n \} \subset X^{an}$*, and a collection of paths* $\gamma_{i,j} : [0,1] \to X^{an}$ *for some pairs of integers* $i, j \subset \{ 1, \ldots, n \}$ *such that* $\gamma_{i,j}(0) = v_i, \gamma_{i,j}(1) = v_j$ *and* $v_{i,j}(x) = v_{j,i}(1-x)$.

Definition 2.5 (Skeleton of the semi-stable vertex set)**.** *The* skeleton $Sk(\Sigma)$ of X with respect to the semi-stable vertex set Σ *is the embedded graph that has Σ as the vertices and skeleta of the open rings as edges.*

Proposition 2.6 ([1])**.** *Let U be a Zariski dense subset of a projective curve X. Then there exists a semi-stable vertex set Σ of U. Moreover, there exists a retraction map $\mathfrak{r}_\Sigma : U^{an} \to \mathrm{Sk}(\Sigma)$ which is constant on connected components of $X \setminus \mathrm{Sk}(X, \Sigma)$ and moreover for any function f invertible on U*

$$|f| \circ \mathfrak{r}_\Sigma = |f|$$

Fact 2.7 (Proposition 1.11 of [7])**.** *Let $x \in X^{an}$ be a point, let f be a function, and let E be the set of connected components of the complement $X^{an} \setminus \{x\}$. Then $|f|$ takes values strictly less than and strictly greater than $|f(x)|$ on finitely many components in E.*

Recall that the *multiplicity* of a map $f : X \to Y$ at any point $x \in X^{an}$ is defined to be the length of the $\mathcal{O}_{Y,y}$-module $\mathcal{O}_{X,x} \otimes \mathcal{H}(y)$ where $y = f(x)$. We will denote it as $m_f(x)$. The set of points x such that $m_f(x) > 1$ is called *ramification locus R_f* .

Fact 2.8. *The ramification locus of an analytification of a map between algebraic curves is a graph that has finitely many connected components and finitely many vertices.*

Fact 2.9 (Baker and Rumely, Theorem 9.45)**.** *The image of an annulus under a morphism f is an annulus if and only if m_f is constant on its skeleton and not increasing off it.*

2.2 The set up

Let (X, ω) be a *Kähler manifold*, that is, a complex manifold X endowed with a closed $(1,1)$-form ω. The Kähler metric induces a Hermitian metric on the canonical bundle Ω^n_X: if α, α' are smooth non-vanishing sections of Ω^n_X then

$$(\alpha, \alpha') = \frac{\alpha \wedge \bar{\alpha}'}{\omega^n}$$

There exists a unique connection ∇ compatible with the holomorphic structure on K_X with respect to which this metric is parallel, the Chern connection. If for some open $U \subset X$ the section $s \in H^0(U, \Omega^n_X)$ is non-vanishing then the curvature form F_∇ of the Chern connection over U can be computed as

$$F_\nabla = \partial\bar{\partial} \log ||s||_h$$

A Kähler metric is called *Ricci-flat* if F_∇ vanishes.

In particular, if $n = 1$ and $\omega = \Omega \wedge \bar{\Omega}$, then tautologically $F_\nabla = \partial\bar{\partial} \log \dfrac{\Omega \wedge \bar{\Omega}}{\Omega \wedge \bar{\Omega}} = 0$ over the complement to the zeroes of Ω. It is metrics of this form that will further consdier; they play a prominent role in the study of moduli spaces of curves and dynamics, see for example [28] and references therein. Note that the inner metric corresponding to such ω is complete.

The *Hausdorff distance* between two metric subspaces X and Y of a third metric space is the infimum of real numbers $\varepsilon > 0$ such that $B_\varepsilon(X) \subset Y$ and $B_\varepsilon(Y) \subset X$, where $B_\varepsilon(A)$ denotes the ε-neighbourhood of a set A

$$B_\varepsilon(A) = \{ \ x \in X \ \mid \ \exists a \in A \ d(x, a) < \varepsilon \ \}$$

The *Gromov-Hausdorff distance* between two metric spaces (X, d) and (Y, d') is the infimum of Hausdorff distances between X and Y with respect to all possible isometric embeddings of X and Y into some metric space. Note that finite metric spaces are dense in the space of (isometry classes of) compact metric spaces with the Gromov-Hausdorff metric.

Let $\mathfrak{X} \to \mathbb{D}^\circ$ be a family of projective complex manifolds over a punctured disc. Let Ω be a holomorphic volume form on X, then according to [13], as $s \to 0$,

$$\int_{X_s} \Omega_s \wedge \bar{\Omega}_s = C(\log|s|)^m |s|^{2l}(1 + o(1))$$

where $m \leq n = \dim X$. Kontsevich and Soibelman [13] distinguish a class of such degenerations: if $m = n$ then the family in question is called *maximally degenerate*.

Assume ω_s is the Ricci-flat metric on X_s in the cohomology class of $\Omega_s \wedge \bar{\Omega}_s$. Then it is conjectured in [13] that the limit of metric spaces (X_s, d_s), where d_s is the inner metric associated to ω_s, normalised so that $\operatorname{diam}(X_s) = 1$, exists, and is isometric to a Riemannian manifold B with the metric satisfying the real Monge-Ampere equation, and having affine structure outside a codimension 2 set.

We will consider families of curves $\mathfrak{X} \to \mathbb{C} \setminus \{0\}$, of genus $g \geq 1$, endowed with a relative differential form $\Omega \in H^0(\mathfrak{X}, \Omega_{\mathfrak{X}/\mathbb{A}^1})$. We are interested in the study of the metric behaviour of $\mathfrak{X} \to \mathbb{C} \setminus \{0\}$ as $s \to 0$. To this end we consider the variety $X := \mathfrak{X} \otimes \mathbb{C}((t))$ obtained via base change with respect to the natural inclusion $\mathbb{C}[t, t^{-1}] \hookrightarrow \mathbb{C}((t))$, and its Berkovich analytification which we will denote X^{an}.

2.3 Robinson's asymptotic field

Let $\mathcal{R}$ be a real closed field which contains $\mathbb{R}$, for example, an infinite non-principal ultrapower of $\mathbb{R}$. Let $\mathcal{O} \subset \mathcal{R}$ be the convex hull of $\mathbb{R}$, it is a convex value ring with the residue field isomorphic to $\mathbb{R}$. Let $\Gamma = \mathcal{R}^\times / \mathcal{O}^\times$ be the value group. It is possible to choose $\mathcal{R}$ in such a way that $\Gamma \cong \mathbb{R}$ though the isomorphism is not canonical. Pick an infinite number ρ and define the isomorphism so that the valuation map maps ρ to 1. We will denote ${}^\rho\mathbb{R}$ the valued field with Γ identified in this way with $\mathcal{R}$. If $\mathcal{R}$ is an ultrapower of $\mathbb{R}$ then the valuation v can be described as

$$v : {}^\rho\mathbb{R} \to \mathbb{R}, v([x]) = -\operatorname{st}\lim_U \log_\rho |x|$$

For more details on this construction see [16]; a version of it was later developed by van den Dries, where the residue field can be an arbitrary o-minimal expansion of $\mathbb{R}$ (see [25]).

Recall that the collection of definable sets in a structure M is a collection containing some subcollection of *atomic relations*, which always include the diagonals

$$\Delta_{ij} := \{ (x_1, \dots, x_n) \in M^n \mid x_i = x_j \}$$

and is closed under the following operations

- Boolean operations;
- taking images of projections $M^{n+k} \to M^n$ for $n \in \mathbb{N}, k \in \mathbb{Z}_{\geq 0}$ of definable sets;
- taking fibres of definable sets with respect to these projections.

Regarding the last clause let us remark that if one allows to take a fibre over a tuple of elemetns of M that belong to a set A, then the collection of definable sets is called *definable with parameters in A*.

The structure we will be primarily interested in further is the algebraic closure of ${}^\rho\mathbb{R}$, denoted ${}^\rho\mathbb{C}$. The atomic relations on ${}^\rho\mathbb{C}$ regarded as a valued field are the graphs of addition and multiplication, and the subset of ${}^\rho\mathbb{C}$ picking out the elements of the value ring. By quantifier elimination, definable subsets of ${}^\rho\mathbb{C}$ are semi-algebraic: they are Boolean combinations of solution sets to finite systems of polynomial equalities and valuational inequalities.

We will consider two collections of definable sets simultaneously: definable sets when ${}^\rho\mathbb{C}$ is treated like an algebraically closed valued field, and definable sets when ${}^\rho\mathbb{C}$ is treated as the Cartesian square of the real closed field ${}^\rho\mathbb{R}$. In the first case, we will refer to definable sets as ${}^\rho\mathbb{C}$-*definable*, or *non-Archimedean semi-algebraic*, and in the second case we will refer to definable sets as $\mathbb{R}$-*definable*, or *real semi-algebraic*.

Now we assume that $p : \mathfrak{X} \to \mathbb{C} \setminus \{0\}$ is a family of projective manifolds, and we regard $\mathfrak{X}_t$, where t is a point $t \in {}^\rho\mathbb{C}$ such that $v(t) = 1$, as a ${}^\rho\mathbb{C}$-definable set that has both $\mathbb{R}$-definable and ${}^\rho\mathbb{C}$-definable subsets (and similarly for its Cartesian powers).

Proposition 2.10. *Let $\mathcal{A} \subset \mathfrak{X}$ be an $\mathbb{R}$-definable set such that $p(\mathcal{A})$ contains a punctured neighbourhood of 0, and let f, g be definable functions on $\mathcal{A}$. Assume on of the following*

1. $v(\frac{f(x,t)}{g(x,t)} - 1) > 0$

2. $v(f(x,t)) \geq 0$

3. $v(f(x,t)) = 0$

Then respectively

1. $|f(x,s)| \sim |g(x,s)|$

2. there exists a constant $C_1 > 0$ *such that* $\sup|f(x,s)| \geq C_1$

3. there exist constants $C_1, C_2 > 0$ *such that* $C_1 \leq |f(x,s)| \leq C_2$

uniformly on $\mathcal{A}_s$ *as* $s \to 0$*.*

Proof. Indeed, for $\varepsilon \in \mathbb{R}$ consider the formula

$$\varphi_\varepsilon(s) := \forall\ s \in A_s\ |\frac{f(x,s)}{g(x,s)} - 1| < \varepsilon$$

Then for all positive ε, ${}^\rho\mathbb{C} \models \varphi_\varepsilon(t)$ which means that there is an $\mathbb{R}$-definable set O_ϵ containing t such that ${}^\rho\mathbb{C} \models \varphi(s)$ for all $s \in O_\epsilon$.

Since t and $t \cdot e^{i\varphi}$ have the same type in the theory of algebraically closed valued fields, the same statement is true for all $t \cdot e^{i\varphi}$ and by logical compactness theorem there are finitely many $\mathbb{R}$-definable sets $O_{\varepsilon,1}, \ldots, O_{\varepsilon,n}$ covering the circle of radius $|t|$ and such that ${}^\rho\mathbb{C} \models \varphi_\varepsilon(s)$ for all $s \in O_{\varepsilon,i}$. Then there exists s_ε such that $\{\ |s| \leq s_\varepsilon\ \} \subset \cup_{i=1}^n O_{\varepsilon,i}$, and it follows that $|\frac{f(x,s)}{g(x,s)} - 1| \to 0$ as $s \to 0$, so we conclude.

The third clause follows from the second, so let us prove it. Suppose for the sake of contradiction that, for example, $|f(x, te^{i\varphi})| \geq C_2$ for some φ all C_2. But that would imply $v(f(x, te^{i\varphi})) > 0$, a contradiction. □

Lemma 2.11. *Let* A_t *be a non-Archimedean semi-algebraic subset of the fibre* $\mathfrak{X}_t$ *defined with parameters in the field* $\mathbb{C}(t) \subset {}^\rho\mathbb{C}$ *by finitely many inequalities of the form* $v(y_{i,j}) \geq a$*,* $a_i \in \mathbb{Q}$ *, for* y_i *a regular function on a Zariski open neighbourhood of* A*. Then there exist infinitely many semi-algebraic subsets* $\mathcal{A}_i$ *of* $\mathfrak{X}$ *such that* $A_{t'} = \bigcup_{i=1}^{\infty} (\mathcal{A}_i)_{t'}$ *for all* $t' = e^{i\varphi}t$*.*

Proof. Let

$$\mathcal{B}_i^n = \bigcup_{n=1}^{\infty} \{\ x \in \mathfrak{X}\ \mid\ (|y_i(x)| \leq n|t(x)|^{a_i}\ \}$$

It follows from the definition of Robinson field that

$$A_{t'} = \bigcap_{i=1}^{l_i} \bigcup_{n=1}^{\infty} (\mathcal{B}_i^n)_{t'}$$

Therefore

$$A_{t'} = \bigcup_{\substack{k_1=1 \\ \dots \\ k_{l_i}=1}}^{\infty} (\mathcal{B}_1^{k_1} \cap \mathcal{B}_2^{k_2} \cap \ldots \cap \mathcal{B}_{k_{l_i}}^{l_i})_{t'}$$

□

Remark. By convention, we will denote non-Archimedean semi-algebraic sets with letters of straight script $(A, D, U \ldots)$ and real semi-algebraic sets with letters of calligraphic script $(\mathcal{U}, \mathcal{D}, \mathcal{U} \ldots)$.

Lemma 2.12. *Let* $X = \bigcup_{i=1}^{n} U_i$ *be a cover of* X *by closed non-Archimedean semi-algebraic sets. By Lemma 2.11* $U_i = \bigcup \mathcal{U}_i^j$ *where* $\mathcal{U}_i^j$ *are closed real semi-algebraic sets, and therefore by compactness theorem* $\mathfrak{X}_t = (\bigcup \mathcal{V}_i)_t$ *where* $\mathcal{V}_i = \mathcal{U}_i^{j_i}$ *for some indices* j_i.

Assume $U_i \cap U_j \neq \emptyset$*, then there exists a sequence* $k_1 = i, k_2, \ldots, k_n = j$ *such that* $\mathcal{U}_{k_n} \cap \mathcal{U}_{k_{n+1}} \neq \emptyset$.

Proof. If $U_i \cap U_j = \emptyset$ then $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$. Therefore each $\mathcal{V}_i$ must intersect with some $\mathcal{V}_l$, $l \neq i$, such that $U_i \cap U_l \neq \emptyset$, or otherwise $U_i \sqcup (\cup_{l \neq i} \mathcal{V}_l) = X$ is a disjoint partition of X into two closed sets, and this contradicts connectedness of X.

Since there are finitely many sets $\mathcal{V}_i$, repeating the above argument finitely many times will yield the required sequence. □

3 Main results

3.1 Semi-stable vertex sets and weight function

The estimates on the asymptotic behaviour of the inner metric associated to the Kähler form ω will be reduced to few "model situations" by decomposing the curve X first into non-Archimedean semi-algebraic and then into real semi-algebraic sets.

Definition 3.1 (Kähler semi-norm and weight, [23])**.** *A map of semi-normed modules* $f : M \to N$ *is called* non-expansive *if* $|f(x)| \leq |x|$.

A Kähler semi-norm *on the module of differentials* $\Omega_{B/A}$ *is the maximal semi-norm that makes the differentiation map non-expansive.*

Let ω *be a regular 1-form on a curve* X*. The* weight function on X associated to Ω $\mathrm{wt}_\Omega : X^{an} \to \mathbb{R}$ *is defined as* $\mathrm{wt}_\Omega(x) = 1 - \log||\Omega||_{\mathcal{H}(x)}$ *where* $||\cdot||$ *is the Kähler semi-norm.*

We denote by $\mathrm{wt}_\Omega(X)$ *the supremum of* wt_Ω *on* X.

Fact 3.2 ([23])**.** *The Kähler semi-norm exists and satisfies*

$$||\Omega|| = \inf_{\Omega = \sum_i a_i db_i} \max_i |a_i||b_i|$$

Let $\Omega = fdx$ be a differental form on an annulus A with coordinate x, then $\mathrm{wt}_{dx}(x) = 1 - \log r(x)$ and so,

$$\mathrm{wt}_\Omega(x) = -\log|f|+1 - \log r(x)$$

This function reaches its minimum on $\Sigma(A)$ if $\omega = fdx$ and $v(f) = -v(x)$ on A.

Definition 3.3 (Semi-stable vertex set monomial with respect to a form)**.** *A semi-stable vertex set Σ is called* monomial with respect to Ω *such that on each open annulus U, with coordinate function x, which is a connected component of the complement $X \setminus \Sigma$ there exists an analytic function f such that $\Omega = fdx$ on U, and $|f|= |x|^\alpha$ for a rational number α on U.*

Let U be an open subset which admits a dominant map $\pi : U \to \mathbb{A}^1$ so that $\omega = fdx$ where dx is the generator of $\Omega_{\mathbb{A}^1}$.

Proposition 3.4. *Let X be projective curve with a regular 1-form Ω on it, and let Σ be a semi-stable vertex set. Then there exists a semi-stable vertex set $\Sigma' \supset \Sigma$ on X monomial with respect to Ω.*

Proof. Assume that $X \setminus \Sigma = (\bigcup_{i=1}^{n} A_i) \cup (\bigcup_{j=1}^{m} B_j)$ where A_i are isomorphic to $A_i^-(r_i, 1)$ and B_j are isomorphic to $B^-(r'_j)$.

Let $\cup_{k=1}^{l} W_k = X$ be a covering of X by Zariski open sets, and such that for each W_k a dominant unramified projection to $\mathbb{A}^1$ is chosen. Then $\Omega = f_k dx_k$ on each W_k, where dx_k is the pullback of the generator of $\Omega_{\mathbb{A}^1}$ with respect to the chosen projection of W_k to $\mathbb{A}^1$, and f_k is a regular function on W_k. We will denote $m_k(x)$ the multiplicity of a point $x \in W_k^{an}$ with respect to the projection of W_k on $\mathbb{A}^1$.

For each W_k and each A_i define the set of undesirable points $Z_{ik} \subset \mathrm{Sk}(A_i)$ consisting of

- the images in A_i under the retraction map $\mathfrak{r}_\Sigma$ of poles of f_k
- points of $\mathrm{Sk}(A_i)$ that belong to the boundary of the set $\mathfrak{r}_\Sigma(\{\ x \in A_i \ \mid m_k(x) > \inf_{x \in \mathrm{Sk}(a_i)} m_k(x)\ \}$

Each Z_{ik} is finite because the sets belonging to the first category are obviously finite, and because the ramification locus of the projection of W_k to $\mathbb{A}^1$ is a finite graph by Fact 2.8.

Let Σ' be the union of Σ and for each A_i of a set Z_{ik} for some k if none of Z_{ik} is empty. Then clearly Σ' is a semi-stable vertex set.

Let $X \setminus \Sigma' = (\cup U_i) \cup (\cup B'_j)$ be the new decomposition. One notices that due to the way points in Σ' were chosen, and by upper semi-continuity of m_k-s there exists $W_k \supset U_i$ such that m_k is constant on $\mathrm{Sk}(U_i)$ and is non-increasing outside of $\mathrm{Sk}(U_i)$.

Let y_i be the coordinate function of U_i. By Fact 2.9, the image $p(U_i) \subset \mathbb{A}^1$ is an annulus, isomorphic, say, to $A^-(r,1)$ for some $r \in \mathbb{R}$, with coordinate function x_i. By Fact 2.1, $y^n = c(1+u)x$ where $|u|< 1$, and so

$$ny_i^{n-1}dy_i = (cu'x_i + c(1+u))dx_i$$

One can verify by considering the expansion of u into series in x_i that $|u'|< 1$ as well. We have then that $|cu'x_i + c(1+u)|= 1$. It follows that $\Omega = g_i\, dy_i$ where $g_i = \dfrac{ny^{n-1}}{cu'x_i + c(1+u)}$, and clearly $|g_i|= |y_i|^{n-1}$. This shows that Σ' is indeed monomial with respect to Ω. □

Proposition 3.5. *Let X be an affine algebraic variety and let $A \subset X$ be an affinoid domain in X, then A is cut out by finitely many inequalities of the form $|f_i(x)|\leq r_i$ where f_i are regular functions on X.*

Proof. Let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/I$ and let $r_1, \ldots, r_n$ be a multiradius such that A is contained in the affinoid subset B of X defined by inequalities $|x_i|\leq r_i$. By Gerritzen-Grauert theorem, A is cut out by finitely many inequalities $|g_i|\leq s_i$ where $g_i \in H^0(B, \mathcal{O}_B)$. It follows from the definition of Tate algebra that the image of the natural map $k[x_1, \ldots, x_n] \to H^0(B, \mathcal{O}_B)$ is dense, therefore there exist functions $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ such that $|f_i|= |g_i|$ on B, which proves the claim. □

Proposition 3.6. *Let Σ be a semi-stable vertex set and let A be a connected component in $X \setminus \Sigma$ contained in an affine open U. Then there exists an affinoid $B \supset A$ and a regular function f on U such that*

- *if A is isomorphic to an open annulus then $A = \{\ x \in U \ \mid\ r < |f(x)|< r'\ \}$ and $B = \{\ x \in U \ \mid\ r \leq |f(x)|\leq r'\ \}$ for some $r, r' \in \mathbb{R}$,*
- *if A is isomorphic to an open ball then $A = \{\ x \in U \ \mid\ |f(x)|< r'\ \}$ and $B = \{\ x \in U \ \mid\ |f(x)|\leq r'\ \}$ for some $r \in \mathbb{R}$.*

Proof. By [1, Theorem 5.38] there exists a semi-stable model of X with reduction map $\pi : X^{an} \to \tilde{X}$ such that Σ is the set of preimages under π of generic points of irreducible components of $\tilde{X}$, and by Theorem 5.34 *loc.cit.* $A = \pi^{-1}(\tilde{x})$ for a double point $\tilde{x}$. The theorem then follows from Proposition 3.5 and [4, Proposition 2.3]. □

Lemma 3.7. *Let X be a curve and let Σ be a semi-stable vertex set of X^{an} which is monomial with respect to a form Ω. Then there exists a cover of U by affinoid sets $\bar{U}_i$ so that for each i there exist an open annulus or an open ball $U_i \subset \bar{U}_i$ with coordinate function y_i so that U_i is the subset of $\bar{U}_i$ defined by inequalities $r_i < |y_i|< r_i'$ or $|y_i|< r_i'$, and $|y_i|$ is constant on $\bar{U}_i \setminus U_i$.*

Proof. Let $U_1, \ldots, U_n$ be the finitely many open annuli in the complement of Σ.

By Proposition 3.6 for each i there exists an affinoid set $B_i \supset U_i$ and a function y_i such that $U_i = \{ x \in B_i \mid r_i < |y_i| < r'_i \}$. Define

$$A_i := \{ x \in B_i \mid r_i < |y_i| \leq r'_i \}$$

Put

$$\bar{U}_i = \bar{A}_i \setminus \bigcup_{\partial A_i \cap \partial A_l \neq \emptyset} U_l$$

If $X \setminus \cup \bar{U}_i$ is non-empty then its connected components belong to finitely open balls $V_1, \ldots, V_m \subset X \setminus \Sigma \setminus (\cup U_i)$ (the finiteness follows from Lemma 2.2). By Proposition 3.6 $V_j \subset C_j$ for some affinoid sets C_j and there exist analytic functions z_j such that $V_j := \{ x \in C_j \mid |x| < r''_j \}$. Define

$$\bar{V}_j := \{ x \in C_j \mid |x| < r''_j \}$$

Now similarly to the above put

$$\bar{U}_{n+l} = \bar{V}_l \setminus \bigcup_{\partial V_j \cap \partial V_l \neq \emptyset} V_j$$

It follows naturally that $X = \cup_{i=1}^{n+m} \bar{U}_i$. □

3.2 Semi-algebraic asymptotic estimates for a family of metrics

Definition 3.8 (ε-isometries)**.** *Let $f : X \to Y$ be a map (not necessarily continuous) between two metric spaces (X, d) and (Y, d'). The* distortion *of f is defined as $\sup_{x_1, x_2 \in X} |d(x_1, x_2) - d'(f(x_1), f(x_2)|$. A map $f : X \to Y$ is an* ε-isometry *if its distortion is less than 2ε and $B_\varepsilon(f(X)) = Y$.*

Existence of an ε-isometry between two metric spaces implies a bound on the Gromov-Hausdorff distance between them.

Lemma 3.9 ([6])**.** *Let $f : X \to Y$ be an ε-isometry. Then*

$$d_{GH}(X, Y) < \varepsilon$$

Let $\mathfrak{X} \to T$ be a family of metric spaces, with X and T semi-algebraic in $\mathbb{R}$ (the metric on each of the fibres X_t is not assumed to be definable).

Definition 3.10 (Semi-algebraic asymptotic estimates)**.** *We say that a family of metrics d_t* admits semi-algebraic asymptotic estimates *if there exists a positive $d \in \mathbb{Q}$ such that for any rational $\alpha \in (0, d]$ there exist semi-algebraic subsets $D_{\leq\alpha}, D_{\geq\alpha}$ of $X \times_T X$ such that*

- *for* $(x, y, s) \in D_{\leq\alpha}$*, resp.* $(x, y, s) \in D_{\geq\alpha}$*,* $\inf d_s(x, y) \to \alpha$*, resp.* $\sup d_s(x, y) \to \alpha$*, as* $s \to 0$*;*
- *there exists* s_α *such that* $(D_{\leq\alpha} \cup D_{\geq\alpha}) \cap X_s = X_s$ *for* $|s| < s_\alpha$*.*

Define the equivalence relation $\sim$ on points of X_t where $|t|$ is infinitesimal to be $(\cap_{\varepsilon>0} D_{\leq\varepsilon}) \cap X_t$. Define $\Delta = X_t/\sim$ and for two points $x, y \in X_t$ define $d_\Delta(x, y) = a$ if $(x, y, t) \in D_{\leq a+\varepsilon} \cap D_{\geq a-\varepsilon}$ for all rational $\varepsilon > 0$.

Proposition 3.11. *This is an equivalence relation, and* d_Δ *defines a metric on* Δ*.*

Proof. Straightforward. □

Proposition 3.12. *If a family of metrics* d_s *on* X *admits a semi-algebraic asymptotic estimates as* $s \to 0$ *then* X_s *tends to* $\Delta := X_t/R$ *regarded as a metric space with distance* d_Δ *in the Gromov-Hausdorff metric.*

Proof. Pick $\varepsilon > 0$ and pick an $1/2\varepsilon$-net in Δ, call it $\{x_1, \ldots, x_n\}$, and consider it as a finite metric space, call it F.

If we could find an $1/2\varepsilon$-isometry from F to a fiber (X_s, d_s) then by Lemma 3.9 and triangle inequality

$$d_{GH}(\Delta, X_s) < d_{GH}(\Delta, F) + d_{GH}(F, X_s) = \varepsilon$$

Finding an $1/2\varepsilon$-isometry from F to X_s amounts to checking validity of the following statement in free variable s

$$\exists y_1, \ldots, y_n \in X_s \text{ such that } \quad \begin{array}{l} \forall i, j \quad |d_\Delta(x_i, x_j) - d_s(y_i, y_j)| < 1/4\varepsilon - \delta \text{ and} \\ \cup_i B_{1/4\varepsilon-\delta}(y_i) = X_s \end{array} \tag{1}$$

for a small $\delta > 0$. We can impose a condition on $y_1, \ldots, y_n$ that implies the above condition, in terms of $D_{\leq\alpha}$, $D_{\geq\alpha}$: if we find points $y_1, \ldots, y_n \in X_s$ such that

$$\begin{array}{l} \forall i, j \quad (y_i, y_j, s) \in D_{\leq d_\Delta(x_i,x_j)+1/2\varepsilon} \cap D_{\geq d_\Delta(x_i,x_j)-1/2\varepsilon} \\ \cup_i \{\ z \in X_s \ \mid \ (y_i, z, s) \in D_{\leq 1/4\varepsilon}\ \} = X_s \end{array} \tag{2}$$

then (1) is satisfied.

The latter condition is a first-order formula and so it makes sense to evaluate it on non-standard points such as t. Clearly, $x_1, \ldots, x_n$ satisfy (2) for $s = t$, therefore the set of standard parameters s for which this condition can be satisfied is non-empty. □

3.3 Distance estimates

Let Ω be a holomorphic 1-form on X and let $\omega = \Omega\wedge\bar{\Omega}$. Then the symmetric tensor g_Ω defined by $g_\Omega(\xi,\zeta) = \omega(I\xi,\zeta)$ is a pseudo-Riemannian metric.

Let $y = y_1 + iy_2$, the metric g_Ω is defined by the metric tensor $|f|^2(dy_1 \otimes dy_1 + dy_2 \otimes dy_2)$ in the corresponding local frame; and in the polar coordinates $\rho = |y|, \varphi = \operatorname{Arg} y$, the metric tensor g_Ω will have the form $|f|^2(d\rho \otimes d\rho + \rho\, d\varphi \otimes d\varphi)$.

The length of a shortest path $\gamma : [a,b] \to \mathfrak{X}_s$ will be denoted $l(\gamma)$.

Lemma 3.13. *Let A be a semi-algebraic subset of $\mathfrak{X}$ and let y be a function on $\mathfrak{X}$ regular in a neighbourhood of A.*

Assume that

$$|f| \sim c \cdot |y|^\alpha |s|^\beta$$

where $\alpha, \beta \in \mathbb{Z}$, uniformly on A_s as $s \to 0$, and that whenever $a_s, b_s \in A_s$, the set $\{\ x \in X \ \mid\ \rho(a_s) \le \rho(x) \le \rho(b_s)\ \}$ is also contained in A.

Let γ be the shortest path between points $a_s, b_s \in A$, $\rho(b_s) > \rho(a_s)$. Then

$$l(\gamma_s) - c\log\frac{\rho(b_s)}{\rho(a_s)}|s|^\beta = o(c\log\frac{\rho(b_s)}{\rho(a_s)}|s|^\beta) + o(\log|s|)$$

if $\alpha = -1$ and

$$l(\gamma_s) = o(c\log\frac{\rho(b_s)}{\rho(a_s)}) + o(\log|s|)$$

otherwise.

Proof. Bound $l(\gamma_s)$ from below by parametrizing $\gamma : [\rho(a_s), \rho(b_s)] \to A$ so that $\partial/\partial\rho\gamma \equiv 1$, then

$$l(\gamma) \ge \int_{\rho(a_s)}^{\rho(b_s)} \sqrt{g_\Omega(\dot\gamma(\tau),\dot\gamma(\tau)}d\tau = \int_{\rho(a_s)}^{\rho(b_s)} \sqrt{|f|^2(\frac{d}{d\tau}\rho(\gamma) + \rho\frac{d}{d\tau}\varphi(\gamma))}d\tau \ge$$
$$\ge \int_{\rho(a_s)}^{\rho(b_s)} |f(\gamma(\tau))|d\tau \ge \int_{\rho(a_s)}^{\rho(b_s)} c(1-\varepsilon_s)\rho^\alpha|s|^\beta d\rho$$

where $\varepsilon_s \ge 0$ and $\epsilon_s = o(1)$.

The length of a shortest path is bounded from above by the length of any curve that connects a and b. So let $\gamma_1 : [\rho(a_s), \rho(b_s)] \to A$ be the curve starting at a such that $\partial/\partial r\ \gamma_1 \equiv 1$ (i.e. tangent to the gradient field of $|y|$), and let $c = \gamma_1(\rho(b_s))$. Let $\gamma_2 : [\varphi(c), \varphi(b)] \to A$ be a curve such that $\partial/\partial\varphi\ \gamma_2 \equiv 1$. Assume that C is a constant such that $|f|$ is bounded by $C|y|^\alpha|s|^\beta$ on A. Then

$$l(\gamma) \le l(\gamma_1) + l(\gamma_2) = \int_{\rho(a_s)}^{\rho(b_s)} |f(\gamma_1(r)|d\rho + \int_{\varphi(c)}^{\varphi(b_s)} |\rho(b_s)||f(\gamma_2(\varphi)|d\varphi \le$$
$$\le \int_{\rho(a_s)}^{\rho(b_s)} c(1+\varepsilon_s)\rho^\alpha|s|^\beta d\rho + \int_{\varphi(c_s)}^{\varphi(b_s)} c(1+\varepsilon_s')\rho(b_s)^{\alpha+1}|s|^\beta d\varphi$$

where $\varepsilon_s, \epsilon'_s \geq 0$, $\varepsilon_s = o(1), \epsilon'_s = o(1)$ as $s \to 0$.

The first term is bounded by

$$c(1+\varepsilon_s)\log\frac{\rho(b_s)}{\rho(a_s)}|s|^\beta$$

if $\alpha = -1$ and by

$$c(1+\varepsilon_s)\frac{\rho(b_s)^{\alpha+1} - \rho(a_s)^{\alpha+1}}{\alpha+1} = o(\log\frac{\rho(b_s)}{\rho(a_s)})$$

otherwise. The second term is bounded by

$$2\pi c(1+\varepsilon'_s)\rho(b_s)^{\alpha+1}|s|^\beta = o(\log|s|)$$

The asymptotics of $l(\gamma)$ is thus established. □

Let X be a curve and let Σ be a semi-stable vertex set of X^{an} which is monomial with respect to a form Ω. Let $X = \cup \bar{U}_i$ be the cover as in Lemma 3.7. By Proposition 3.5 we may assume that all inequalities that define $\bar{U}_i$ involve functions regular on an affine superset, and that all coordinate functions of annuli and balls U_i are restrictions of regular funcitons y_i.

Lemma 3.14. *Then there exists a collection of semi-algebraic subsets $\bar{\mathcal{U}}_i \subset \mathfrak{X}$ such that for all $t' = e^{i\varphi}t$, where $\varphi \in [0, 2\pi)$,*

1. *$(\bar{\mathcal{U}}_i)_{t'} \subset \bar{U}_i$ and $\mathfrak{X}_{t'} = \cup \bar{\mathcal{U}}_i$*
2. *over each $\bar{\mathcal{U}}_i$ the form Ω can be written down as $f_i dy_i$ so that $|f_i| \sim c_i \cdot |y|^\alpha |s|^\beta$ as $s \to 0$,*
3. *if U_i is an open ball then $\alpha \neq -1$ in the expression above.*

Proof. The first statement follows from Lemma 2.11.

The second statement follows from Proposition 2.10 and Fact 2.1.

The last statement follows from the fact that By [23, Theorem 8.2.4], the minimality locus of wt_Ω is contained in $\mathrm{Sk}(X, \Sigma)$, which is the union of skeleta of annuli in the semi-stable decomposition of X, therefore, the weight function is not constant on any interval inside U_i if it is a ball. □

Lemma 3.15. *Let a_s, b_s be in $\bar{\mathcal{U}}_i$, and let γ_s be a shortest path between a_s and b_s. Then the same estimates on $l(\gamma_s)$ as in Lemma 3.13 hold.*

Proof. The reasoning for the lower bound is not changed.

For the upper bound notice that the path γ_2 as it was defined in Lemma 3.13 might run into finitely many holes in $\bar{\mathcal{U}}_i$, each of which can be contoured by the path δ_1 that goes in the radial direction, the path δ_2 that goes in the angular direction and the path δ_3 going back in the radial direction.

Since δ_1 and δ_3 lie entirely in $\mathcal{U}_i \cap \mathcal{U}_j$, $\log \rho(\delta_k(1))/\rho(\delta_k(0) = o(\log|s|)$, and therefore $l(\delta_k) = o(\log|s|)$, $k = 1, 3$. The length of the arc of a circle, δ_2 of radius bounded by $C|s|^{r'_i}$ is bounded by $2\pi C|s|^{r'_i}$ and so $l(\delta_2) = o(\log|s|)$. This concludes the estimate. □

Corollary 3.16. *Let a_s, b_s be in $\bar{\mathcal{U}_i} \cap \bar{\mathcal{U}_j}$, in the notation of the Lemma 3.7, let*

$$\rho_i(a_s) = \Theta(|s|^{r'_i}) \qquad \rho_j(b_s) = \Theta(|s|^{r_j})$$

as $s \to 0$ and let γ_s be a shortest path between a_s and b_s. Then $l(\gamma_s) = o(\log|s|)$.

Corollary 3.17. *Let a_s, b_s be in $\bar{\mathcal{U}_i}$ such that U_i is a ball, and let γ_s be a shortest geodesic between a_s and b_s. Then $l(\gamma_s) = o(\log|s|)$.*

3.4 The structure of the limit

For this section let us fix a semi-stable vertex set Σ monomial with respect to Ω, and let $X = \bigcup \bar{U}_i$ be the cover with the properties stipulated in the statement of Proposition 3.7. By Proposition 3.6 $\bar{U}_i$ are cut out by finitely many inequalities involving functions regular on affine neighbourhoods of U_i-s; also, the coordinate functions of annuli and balls U_i are restrictions of functions regular on an affine open that contains U_i, and $U_i = \{\ x \ \in \bar{U}_i \ \mid \ r_i \leq |y_i| \leq r'_i\ \}$. Moreover, $\Omega = f_i dy_i$ on $\bar{U}_i$ and $f_i = c_i y_i^{\alpha_i} t^{\beta_i}$, where $|c_i| = 1$.

Let $\bar{\mathcal{U}_i} \subset \mathfrak{X}$ be the collection of semi-algebraic sets that exist by Proposition 3.14.

Define an equivalence relation on the points of the Berkovich analytification of X: put $x \sim y$ if and only if there exists a path $\gamma : [0,1] \to X^{an}$ joining x and y such that the set $\{\ \tau \in [0,1] \ \mid \ \mathrm{wt}_\Omega(\gamma(\tau)) = \mathrm{wt}_\Omega(X)\ \}$ is discrete or empty. Denote the quotient $X/\sim$ as Δ.

Proposition 3.18. *The natural map $\mathrm{Sk}(X, \Sigma)/\sim \to \Delta$ induced by inclusion $\mathrm{Sk}(X, \Sigma) \hookrightarrow X$ is a bijection, and so is the map of quotients induced by the inclusion $\mathfrak{X}_t({}^\rho\mathbb{C}) \hookrightarrow X$.*

Proof. By [23, Theorem 8.2.4], or by direct observation which is easy to carry out in the case of curves, the minimality locus of wt_Ω is a subset of any skeleton associated to a semi-stable vertex set. Then we can conclude by observing that equivalence classes of $\sim$ are preimages of subsets of $\mathrm{Sk}(X, \Sigma)$ under the retraction map $\mathfrak{r}_\Sigma$. □

Define

$$d_i = \begin{cases} \pi(c_i) & \text{if } f_i = c_i y_i^{-1} s^{\beta_i} \text{ and } \beta_i = -\log \mathrm{wt}_\Omega(X) \\ 0 & \text{otherwise} \end{cases}$$

where $\pi : \mathbb{C}((t))^\times \to \mathbb{C}^\times$ is the reduction map.

Note that $d_i = 0$ if and only if all points $\mathrm{Sk}(U_i)$ are equivalent to each other with respect to $\sim$. Put a pseudo-metric $\mathrm{Sk}(X, \Sigma)$ considered as a metric graph, with $\mathrm{Sk}(U_i)$ of length $d_i \cdot |r_i' - r_i|$. Denote the projection $X \to \Delta$ as q, and the projection $\mathrm{Sk}(X, \Sigma) \to \Delta$ as p. Define the metric d_Δ on Δ such that $d_\Delta([x], [y])$ is the length of a shortest path between x and y on $\mathrm{Sk}(X, \Sigma)$, in view of the above remark, this is a well-defined metric.

For any rational $\alpha \in (0, \operatorname{diam} \Delta]$ define

$$\begin{aligned} D_{\leq\alpha} &= q^{-1}(\{\ (x, y) \in \Delta^2 \ \mid\ d_\Delta(x, y) \leq \alpha\ \}) \\ D_{\geq\alpha} &= q^{-1}(\{\ (x, y) \in \Delta^2 \ \mid\ d_\Delta(x, y) \geq \alpha\ \}) \end{aligned}$$

Let us study the sets $D_{\leq\alpha}, D_{\geq\alpha}$.

Consider the equivalence relation on the set of paths in $\mathrm{Sk}(X, \Sigma)$ that identifies paths γ_1 and γ_2 if they both start in U_i and end in U_j and coincide outside U_i and U_j. We will denote classes of paths under this equivalence relation as $[\gamma]$, and the set of classes starting in U_i and ending in U_j as P_{ij}.

Definition 3.19. *Define $E_{[\gamma],\leq\alpha} \subset U_i \times U_j$, resp. $E_{[\gamma],\leq\alpha} \subset U_i \times U_j$ to be subsets of pairs of endpoints of paths in P_{ij} of length $\leq \alpha$, resp. $\geq \alpha$.*

Proposition 3.20. *The sets $E_{i,j,[\gamma],\leq\alpha}, E_{i,j,[\gamma],\geq\alpha}$ are semi-algebraic.*

Proof. For $[\gamma] \in P_{ij}$ denote the length of the part of γ outside of $\mathrm{Sk}(U_i) \cup \mathrm{Sk}(U_j)$ as $e_{[\gamma]}$, and let $\kappa_{i,[\gamma]} = (\operatorname{sgn} \dfrac{\partial |y_i(\gamma(t))|}{\partial t} + 1)/2$, $\lambda_{i,[\gamma]} = (1 - \operatorname{sgn} \dfrac{\partial |y_i(\gamma(t))|}{\partial t})/2$.

One easily observes that

$$\begin{aligned} E_{[\gamma],\leq\alpha} = \{x \in U_i \times U_j \mid \kappa_{j,[\gamma]} \log \frac{|y_j(x)|}{r_j} + \lambda_{j,[\gamma]} \log \frac{r_j'}{|y_j(x)|} \\ + \lambda_{i,[\gamma]} \log \frac{|y_i(x)|}{r_i} + \kappa_{i,[\gamma]} \log \frac{r_i'}{|y_i(x)|} \leq \alpha - e_{[\gamma]}\} \end{aligned}$$

$$\begin{aligned} E_{[\gamma],\geq\alpha} = \{x \in U_i \times U_j \mid \kappa_{j,[\gamma]} \log \frac{|y_j(x)|}{r_j} + \lambda_{j,[\gamma]} \log \frac{r_j'}{|y_j(x)} \\ + \kappa_{i,[\gamma]} \log \frac{|y_i(x)|}{r_i} + \lambda_{i,[\gamma]} \log \frac{r_i'}{|y_i(x)|} \geq \alpha - e_{[\gamma]}\} \end{aligned}$$

□

Corollary 3.21. *The sets $D_{\leq\alpha}, D_{\geq\alpha}$ are semi-algebaric.*

Proof.

$$\begin{aligned} D_{\leq\alpha} &= \bigcup_{i,j} \bigcup_{[\gamma] \in P_{ij}} E_{[\gamma],\leq\alpha} \\ D_{\geq\alpha} &= \bigcup_{i,j} \bigcap_{[\gamma] \in P_{ij}} E_{[\gamma],\geq\alpha} \end{aligned}$$

□

Definition 3.22 (Subordinate path)**.** *Let γ be a path in* $\mathrm{Sk}(X,\Sigma)$ *that consecutively passes through $U_{k_1},\dots,U_{k_n}$. We call a path $\gamma'_s \in \mathfrak{X}_s$* subordinate *to γ if there exists a set of points $\xi_1,\eta_1,\dots,\xi_n,\eta_n \in \gamma_s$, ordered by the direction of the path, and such that ξ_1 is the starting paint, η_n is the ending point, and*

- *$\xi_i,\eta_i \in \mathcal{U}_i$ for all i*
- *$\xi_i \in \mathcal{U}_{k_i} \cap \mathcal{U}_{l_i}$, for some $l_i \neq k_i$ for $i > 1$*
- *$\eta_i \in \mathcal{U}_{k_i} \cap \mathcal{U}_{m_i}$, for some $m_i \neq k_i$ for $i > 1$*

Proposition 3.23. *Let $\gamma \in P_{ij}$, $\mathcal{A} \subset \mathfrak{X}$ be a semi-algebraic set such that $\mathcal{A}_t \subset E_{[\gamma],\leq\alpha}$ (resp. $\mathcal{A}_t \subset E_{[\gamma],\geq\alpha}$) and*

$$\sup \kappa_{j,[\gamma]} v(\frac{|y_j(x)|}{r_j}) + \lambda_{j,[\gamma]} v(\frac{r'_j}{|y_j(x)|}) + \lambda_{i,[\gamma]} v(\frac{|y_i(x)|}{r_i}) + \kappa_{i,[\gamma]} v(\frac{r'_i}{|y_i(x)|}) = \alpha - e_{[\gamma]}$$

respectively,

$$\inf \kappa_{j,[\gamma]} v(\frac{|y_j(x)|}{r_j}) + \lambda_{j,[\gamma]} v(\frac{r'_j}{|y_j(x)}) + \kappa_{i,[\gamma]} v(\frac{|y_i(x)|}{r_i}) + \lambda_{i,[\gamma]} v(\frac{r'_i}{|y_i(x)|}) = \alpha - e_{[\gamma]}$$

Let $(a_s, b_s) \in \mathcal{A}$, and let $w = \min_{x\in\gamma} \mathrm{wt}_\Omega(x) - 1$. Then

$$\sup l(\gamma'_s) = \alpha \cdot \log|s| \cdot |s|^w (1 + o(1)),$$

respectively,

$$\inf l(\gamma'_s) = \alpha \cdot \log|s| \cdot |s|^w (1 + o(1)),$$

where the supremum and infimum are taken over all shortest paths γ'_s between a_s and b_s subordinate to γ.

Proof. Denote an interval between two points $a, b \in \gamma'_s$ as $[a,b]$.

By Lemma 2.12, Corollary 3.16 and Corollary 3.17, for $1 < i < n$

$$l([\eta_i, \xi_{i+1}]) = o(\log|s|)$$

By Lemma 3.13 for $1 < i < n$

$$l([\xi_i, \eta_i]) = c_i |r'_i - r_i| \cdot \log|s| \cdot |s|^\beta (1 + o(1))$$

and if $\beta_i > \min_l \beta_{k_l} = w$ then $l([\xi_i, \eta_i]) = o(\log|s| \cdot |s|^w$. Therefore

$$\sum_{l=2}^{n-1} l([\xi_l, \eta_l]) = \sum_{l=1}^{n-1} d_l |r'_l - r_l| (\log|s|) |s|^{\mathrm{wt}_\Omega(X)} (1 + o(1)) = e_{[\gamma]} \log|s| \cdot |s|^w$$

Now let us estimate the asymptotics of $l([\xi_1, \eta 1]) + l([\xi_n, \eta_n])$ assuming $\kappa_i = \kappa_j = 1$ (the estimates for other combinations of values of κ_i and κ_j are carried out similarly). By Lemma 3.13

$$\begin{aligned} l([\xi_1, \eta_1]) + l([\xi_n, \eta_n]) &= \\ &= (c_i \log \frac{\rho_i(\eta_i)}{\rho(\xi_i)|} |s|^{\beta_i} + c_j \log \frac{\rho_j(\eta_j)}{\rho_j(\xi_j)} |s|^{\beta_j})(1 + o(1)) + o(\log|s|) \\ &= (d_i \log \frac{\rho_i(\eta_i)}{\rho(\xi_i)} + d_j \log \frac{\rho_j(\eta_j)}{\rho_j(\xi_j)} |s|^w)(1 + o(1)) + o(\log|s|) \end{aligned}$$

Then by Proposition 2.10 and the assumption on y_i, y_j

$$\sup \frac{\rho_i(\eta_i)}{\rho(\xi_i)} \frac{\rho_j(\eta_j)}{\rho_j(\xi_j)} \cdot |s|^{e_{[\gamma]}-\alpha} = C$$

for some positive constant C. Therefore,

$$\sup l([\xi_1, \eta 1]) + l([\xi_n, \eta_n]) = (\alpha - e_{[\gamma]}) \log|s| \cdot |s|^w)(1 + o(1))$$

Finally,

$$\begin{aligned} \sup\ l(\gamma'_s) &= \sup\ \left(l([\xi_1, \eta_1] + \sum_{i=1}^{n-1} l([\eta_i, \xi_{i+1}]) + [\xi_{i+1}, \eta_{i+1}] \right) \\ &= \alpha \cdot \log|s| \cdot |s|^w (1 + o(1)) \end{aligned}$$

The estimate for shortest paths in $E_{[\gamma], \geq \alpha}$ is obtained similarly. □

Corollary 3.24. *If $\mathfrak{X}$ is maximally degenerate then the diameter of $\mathfrak{X}_s$ with respect to the Kähler metric $\Omega \wedge \bar{\Omega}$ has the asymptotics*

$$\operatorname{diam} \mathfrak{X}_s = \operatorname{diam} \Delta \cdot (\log|s|)|s|^{\mathrm{wt}_\Omega(X)-1})(1 + o(1))$$

as $s \to 0$.

Proof. By [5, Theorem A] (which requires that Ω be a holomorphic volume form, but the proof goes through even if it is degenerate in finitely many points) if $\mathfrak{X}$ is maximally degenerate then the minimality locus of wt_Ω contains an interval.

Now apply Proposition 3.23 to all path classes in all P_{ij} in the decomposition of $D_{\leq \alpha}$ in Corollary 3.21. □

Corollary 3.25. *There exist semi-algebraic asymptotic estimates (in the sense of Definition 3.10) of the normalized metric on $\mathfrak{X}_s$ as $s \to 0$.*

Proof. By construction, for each α, $D_{\leq \alpha} \cup D_{\geq \alpha} = X^2$, and both $D_{\leq \alpha}$ and $D_{\geq \alpha}$ are infinite unions of semi-algebraic subsets of $\mathfrak{X}_t$. Therefore, by Lemma 2.11 there exist semi-algebraic $\mathcal{D}_{\leq \alpha}, \mathcal{D}_{\geq \alpha} \subset \mathfrak{X}$ such that $(\mathcal{D}_{\leq \alpha})_{t'} \subset \mathcal{D}_{\leq \alpha}$, $(\mathcal{D}_{\geq \alpha})_{t'} \subset \mathcal{D}_{\geq \alpha}$ for all $t' = e^{i\varphi} t$.

The sets $\mathcal{D}_{\leq \alpha}, \mathcal{D}_{\geq \alpha}$ satisfy the first condition of Definition 3.10 by Proposition 3.23 and Corollary 3.24. □

Theorem 3.26. *As s tends towards 0, the fiber X_s with the normalized metric converges to the space Δ with the metric $d_\Delta/\operatorname{diam}\Delta$.*

Proof. Follows from Propositions 3.12 and Corollary 3.25. □

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