Geometric model theory

Abstract
This is lecture notes of a mini-course held at Maths Department of Higher School of Economics, Moscow, in November-December 2013.

Contents

Lecture 0. Model-theoretic terminology

Recommended references: [1] (Chapters 6-7), [2].

Definition 1 (Definable sets and theories)**.** A structure is a set M together with some distinguished relations (called *atomic*) on M^n, and constants — 0-ary relations. A collection of *definable subsets of* M^n (for all n) is the collection of sets that includes the diagonals — $x_i = x_j$ — and is closed under:

1. Boolean combinations
2. projections
3. Cartesian products

A definable set can be described by a formula. A set of formulas is called a theory. The set of all closed formulas that hold in a struture is *the theory of the structure*. Two structures are called *elementary equivalent* if they have the same theory. A theory is called *complete* if any two models of it are elementarily equivalent.

One is often interested in sets definable over parameters, i.e. one temporarily enlarges the language with constants that name elements of a model. The subsets of M^n definable with parameters in A are denoted $\mathrm{Def}_n(A)$.

We will further consider only complete theories. This amounts to studying definable sets in a structure, since any structure is a model of its (complete) theory.

Definition 2 (Types)**.** Let M be a model of some complete theory. Let $A \subset M$ be a subset. The space of n-types over A, denoted $S_n(A)$ is the space of ultrafilters in the Boolean algebra definable with parameters in A subsets of M^n. One can form type spaces over infinite sets of variables too.

One says that a type $p \in S_n(A)$ is *realised* in M if there exists an tuple $\bar{a} \in M^n$ such that $\bar{a} \in X$ for all definable $X \in p$.

The space of types $S_n(A)$ is endowed with the natural topology with the base of clopen sets of the form

$$\{ \ p \in S_n(A) \ \mid \ X \in p \ \}$$

where X is a definable set.

Let $\bar{a} \in M^k$ be a tuple and let A be a parameter set. The ultrafilter of definable over A sets

$$\{ \ X \in \mathrm{Def}(A) \ \mid \ \bar{a} \in X) \ \}$$

is called the complete type of $\bar{a}$ over A and is denoted $\mathrm{tp}(\bar{a}/A)$.

Theorem 1 (Compactness theorem)**.** *The type spaces $S_\alpha(A)$ are compact for any ordinal α.*

Definition 3 (Saturated models)**.** Let M be a model and let κ be a cardinal, then M is called κ-saturated if for any $A \subset M$, $|A| < \kappa$ any n-type is realised in M.

Example 1 (Algebraically closed fields)**.** Let k be an algebraically closed field of infinite transcendence degree. Then k is $\aleph_0$-saturated.

In model theory one often considers theories where a good independence notion can be defined. We will be further be interested in the simplest setting when this is possible to do, when a dimension function is defined on the definable sets.

Let us say that a model has *good dimension notion* (this terminology is *not* universal) if there exists function $\dim : \mathrm{Def}(M) \to \mathbb{Z}_{\geq 0}$ such that

1. $\dim X \cup Y = \max\{ \ \dim X, \dim Y \ \}$
2. $\dim Y > n$ if $Y \supset \sqcup X_i$ and $\dim X_i = n$ for all i;
3. if $f : X \to Y$ is definable then the set $\{ \ y \in Y \ \mid \ \dim f^{-1}(y) = n \ \}$ is definable for any n (definability of dimension);
4. if $f : X \to Y$ is definable and $\dim f^{-1}(y) = n$ then $\dim X = \dim Y + n$ (additivity of dimension);
5. quantifier "there exists infinitely many" is eliminable

Example 2. Consider a compact complex manifold M with a predicate for each analytic subset of M^n. Then M has quantifier elimination to the level of sets constructible in analytic "Zariski" topology and dimension can be extended from analytic subsets X of M^n to constructible subsets, for all n (take the maximum of dimensions of the irreducible components of the closure of X).

A *degree* of a definable set is defined as

$$\deg X = \min\{ \ n \ \mid \ \exists X_1, \ldots, X_n \in \mathrm{Def}_n(M), X = \sqcup_{i=1}^n X_i, \dim X_i = \dim X \ \}$$

The notion of degree is sensitive to the parameters used!

Given dimension function on definable sets one can assign dimension and degree to types:

$$(\dim p, \deg p) = \min(\dim X, \deg X)$$

for $X \in P$, where minimum is understood in the lexicographic ordering of pairs.

Let X be of degree 1 over A, then the *generic type of X over A* is

$$\{ \ y \in \mathrm{Def}_n(A) \ \mid \ \dim Y \triangle X < \dim X \ \}$$

Conversely, a type $p \in S_n(A)$ is determined by any definable set of dimension $\dim p$ that belongs to it, i.e. p is the generic type over A of any such set.

Lemma 2. *Let a, b realise the generic types of X, Y. Let $f : X \to Y$ be definable and let $f(a) = b$. Then there exists a meromorphic function $\hat{f} : X \to Y$ such that $\hat{f}(a) = b$.*

Definition 4. If there exists a definable function $f : X \to Y$ that maps a to b, one says that b is *in the definable closure of a*. If there exists a correspondece $f : X \vdash Y$ (i.e. a subvariety in $X \times Y$, regarded as a multivalued function) finite over Y, and $f(a) = b$, then one says that b is *in the algebraic closure of a*. Notation: $b \in \mathrm{dcl}(a), b \in \mathrm{acl}(a)$.

Example 3. Let y be a generic point of Y and let Z be a definable subset of X defined over y. How to think of Z? Z is a "generic fibre" of $\hat{Z} \subset Y \times X$ over y. Points $Z \cap \mathrm{dcl}(y)$ are meromorphic sections of the projection $\hat{Z} \to Y$.

In a compact complex manifold for any type p there exists a *unique irreducible Zariski closed (i.e. analytic)* set X such that p is the generic type of X. Call this set X the *locus* of the type (or any tuple realising it).

This is true in a the setting of Zariski geometries, which an abstract axiomatisation of a structure endowed with a topology with properties similar to Zariski topology. Natural examples of Zariski geometries are, of course, algebraic varieties over an algebraically closed field (with the all subvarieties of Cartesian powers defined over some fixed field of definition considered as atomic relations), and compact complex manifolds.

Definition 5 (Zariski geometry, [3])**.** Let X be a set and let $\tau_n(X), n \in \mathbb{N}$ be a collection of topologies on X^n.

A *Zariski geometry* is a model-theoretic structure in the language containing an n-ary predicate for each closed subset of X^n for all n, subject to the following two sets of conditions.

Topological axioms

1. singletons and diagonal sets are closed;

2. projection maps are continuous;

3. permutation of coordinates maps are homeomorphisms;

4. quantifier elimination: any definable set is constructible;

Dimension axioms There exists a function dim from the collection of constructible subsets of X^n to non-negative integers

1. $\dim\{x\} = 0$, $\dim Z \cup W = \max(\dim Z, \dim W)$;

2. given a definable continuous map of closed irreducibles $X \to Y$

$$\dim X = \dim Y + \max_y \dim X_y$$

3. given a definable continuous map of closed irreducibles $X \to Y$ the function $y \mapsto \dim X_y$ is upper semi-continuous;

Definition 6 (Forking)**.** A type $p \in S_n(A \cup \bar{b})$ forks over A if $\dim p < \dim p \restriction A$. Two tuples a, b of a model are called *independent over* A if $\mathrm{tp}(a/bA)$ doesn't fork over A, i.e. $\dim a/bA = \dim a/A$.

Proposition 3 ("Symmetry of forking")**.** *Let x, y be tuples. Then* $\dim x/yA =$ *$dimx/A$ if and only if* $\dim y/xA = \dim y/A$.

Proof. Follows from additivity and definability of dimension. ☐

If p is the generic type of X over A and y is generic in Y then there exists a type $p \restriction A$ that does not fork over A: it is the generic type of the fibre of $X \times Y$ over y.

Lemma 4. *Dimension and degree are preserved by definable bijections. Dimension is preserved by finite-to-finite correspondences.*

Lecture 1. Zilber's theorem. Definable fields

From now on we work in a structure with good dimension notion.

Theorem 5 (Baldwin-Saxl)**.** *Let G be a definable group and let H_i be a collection of subgroups. Then the intersection $\cap H_i$ is the intersection of finitely many H_i-s.*

Proof. First notice that if H is a definable subgroup of G then $(\dim H, \deg H) <$ $(\dim G, \deg G)$ in the lexicographic ordering. Therefore there cannot be infinite chains of subgroups. The conclusion of the theorem follows. ☐

Corollary 6. *Let $f : G \to G$ be a definable injective group homomorphism. Then f is surjective.*

Lemma 7 (Connected component of the identity)**.** *Let G be a definable group. Then there exists the smallest definable subgroup of G of finite index (it is denoted G^0).*

Proof. Take the intersection G^0 of definable subgroups of finite index. By Baldwin-Saxl it is a finite intersection, hence it is of finite index. The conjugates of G^0 are of the same index, hence contain G^0, hence coincide with G^0. $\square$

If G definably acts on a definable set X and p is a type such that $X \in p$ then the type $g \cdot p$ is defined to be $\{ gX \mid X \in p \}$.

Definition 7 (Stabiliser)**.** Let G be a group that acts definably on a set X. Let $p \in X$ be a type. The model-theoretic stabilizer of p is the set of elements $g \in G$ such that $g \cdot p = p$.

Proposition 8. $\mathrm{Stab}(p)$ *is definable.*

Proof. Take some $X \in p$ such that $\dim X = \dim p$. Then $\mathrm{Stab}(p) = \{ g \in G \mid \dim(X \triangle gX) < \dim X \}$, which is definable by definability of dimension. Indeed, for any other $Y \in p$, any $g \in \mathrm{Stab}(p)$

$$\dim gY \triangle X = \dim(gY \triangle gX) \triangle (X \triangle gX) \leq \dim Y \triangle X \cup X \triangle gX < \dim X$$

$\square$

Proposition 9. *Let G be a definable group and let p be a type such that $G \in p$. Then* $\dim \mathrm{Stab}(p) \leq \dim p$*. If* $\dim \mathrm{Stab}(p) = \dim p$ *then p is the generic type of a coset of* $\mathrm{Stab}(p)$*.*

Proof. Denote $\mathrm{Stab}(p)$ as H, let p be the generic type of some set $X \subset G$, and let $h \in H$, $x \in X$ be generic elements of $\mathrm{Stab}(p)$, X respectively (over G), such that g is independent from x over G ($\dim(h/x, G) = \dim h, G$). Then $\dim(h \cdot x/x, G) = \dim(h/x, G)$ since multiplication by x is a definable bijection, and since h is independent from x, $\dim(h/x) = \dim h = \dim H$. On the other hand $\dim(h \cdot x/x) \leq \dim h \cdot x$ and $\dim h \cdot x = \dim x$ since h is in the stabiliser of the type of a. If the equality is reached, then picking any $x \in X$ we get that $\dim xH \triangle X < \dim X$. $\square$

A more conventional proof. Denote $\mathrm{Stab}(p)$ as H, and suppose that $\dim H > \dim X$. Then there exists a coset aH such that $\dim aH \cap X = \dim X$. Since inside a coset H acts transitively, there exists an $h \in H$ such that $\dim haH \triangle X < \dim X$.

If $\dim H = \dim X$, then again, taking any $x \in X$ we get that $\dim xH \triangle X < \dim X$ (recall that we have taken X to be of degree 1). $\square$

Definition 8 (Indecomposable)**.** A definable subset X of a definable group G is called *indecomposable* if for any definable subgroup either X is entirely contained in a coset of H or X intersects infinitely many cosets of H (so, indecomposable="degree one with respect to definable subgroups").

A definable group is called *connected* if it is of degree one.

Lemma 10. *Let X, Y be definable subsets in a definable group G, such that* $\dim(G \setminus X) < \dim G$ *and* $\dim(G \setminus Y) < \dim G$*. Then $G = X \cdot Y$.*

Proof. For any $g \in G$ the set $\{ \ y \in G \ \mid \ \exists x \in X, x \cdot y = g \ \}$ has complement of dimension $< \dim G$, hence has non-empty intersection with Y. $\square$

Theorem 11 (Zilber's theorem)**.** *Let G be a definable group in a ranked theory. Let $\{ \ X_i \ \}$ be a collection of indecomposable definable subsets of G that contain the identity. Then the subgroup H generated by $\{ \ X_i \ \}$ is definable, connected, moreover there are finitely many X_i-s that generate H in two steps: $H = (X_{i_1} \times \ldots \times X_{i_n})^2$.*

Proof. By finiteness of rank, there will be some $X_{i_1}, \ldots, X_{i_k}$ such that $\dim \prod (X_{i_k}) \cdot X_j = \dim X_j$ for all j. Denote $K = \prod X_{i_k}$ and let p be a generic type of K. Let H be $\mathrm{Stab}(p)$.

Claim: $X_i \subset H$. If not, both X_i have non-trivial intersection (at least the identity), and since X_i is indecomposable, X_i is contained in infinitely many cosets of H, $a_i H$. Consider $a_i p$. Since H is the stabilizer of p, they are distinct. Therefore $a_i K$ are distinct, but they are all included in p.

By Proposition 9, $\dim H \leq \dim K$. Since $X_i \subset K$ for all i, $\dim K \leq \dim H$. So $\dim H \setminus K < \dim H$, and by Lemma 10, K generates H, $H = K \cdot K$. $\square$

Lemma 12. *Suppose X is preserved by conjugation by elements of H. To verify indecomposability of X it suffices to consider those subgroups H of G that are preserved by conjugation by elements of H.*

Proof. Let N be an arbitrary subgroup such that X intersects finitely many cosets of N. Then X also intersects finitely many cosets of every group of the form aNa^{-1}. Let N' be the intersection of such groups that X also intersects finitely many cosets of. The group N' is presereved by conjugation by any element of X. By Baldwin-Saxl N' is definable and since a finite intersection of groups of finite index, it is of finite index. $\square$

Proposition 13. *Let G be a definable group, H definable connected subgroup in a ranked theory and let $A \subset G$ be an arbitrary subset. The the commutator $[A, H]$, i.e. the group generated by elements of the form $[a, h], a \in A, h \in H$, is definable and connected. Moreover, $[A, H]$ is generated by finitely many such commutators.*

Proof. We aim at applying Zilber's theorem on indecomposables to $[a, H]$ where a ranges in A. In order to achieve that we show that the sets $a^H = \{ \ hah^{-1} \ \mid \ h \in H \ \}$ are indecomposable. Since g^H are preserved by conjugation by H, by Lemma 12 suffices to consider only subgroups N that are preserved by conjugation by elements of H. Suppose that for some such N there are finitely many cosets of N that cover g^H. Then H permutes these cosets transitively. But then a stabiliser of any of these cosets is a finite index subgroup in H, which cannot be as H is connected. $\square$

Theorem 14. *Let F be a definable infinite field in a structure with good dimension notion. Then F is algebraically closed.*

Proof. The group $(F,+)$ is connected, since multiplication by a non-zero element is a definable automorphism of $(F,+)$ and therefore preserves the connected component of the identity in $(F,+)$. Therefore F^0 is an ideal, and cannot be proper in F.

So F has degree 1. Therefore the subgroup $(F^\times)^n \subset F^\times$ cannot be of finite index (or otherwise F would have degree > 1). Therefore raising to the n-th power is a surjective map. In particular, F is perfect. In a similar fashion, the map $x \mapsto x^p - x$ is surjective.

So F has no Kummer or Artin-Schreier extensions. Moreover, any extension F' of F is definable in F (it is a finite-dimensional vector space over F with multiplication a linear map), hence by the same arguments has no Kummer or Artin-Schreier extensions.

Suppose F has an extension F', which we can without loss of generality suppose Galois. Then there exists $K \subset F$ such that F'/K is cyclic, contradiction. It follows that F has no finite extensions. □

Lecture 2. Field configuration

We continue investigating definable fields. These turn out to be quite rigid, they do not have definable infinite subfields, and they have really few definable automorphisms.

Lemma 15. *A commutative integral domain is a field.*

Proof. Definable ideals of R are in particular subgroups of the additive group, therefore there is a minimal ideal $I \subset R$. Then $Rx = I$ for any $x \in I$ (since I is minimal). But then (x) cannot be a proper ideal as $(x) \cap (y) = 0$ since R is integral. □

Lemma 16. *A definable field K has no proper definable subring.*

Proof. A subring would have to be integral domain, hence a subfield, hence algebraically closed k. The extension K/k is of infinite degree, so the dimension of K cannot be finite. □

Lemma 17. *Let K be a definable field. If* char $= 0$ *then K does not have non-trivial definable automorphisms. If* char $= p > 0$ *then all definable automorphisms of K are powers of Frobenius.*

There is no non-trivial definable groups that act non-trivially by field automorphisms on K.

Proof. Any automorphism of K is determined by its restriction to the algebraic closure of the prime field: since any definable automorphism has a definable fixed field, and we have just proved that there are no infinite such, for automorphisms σ, τ that coincide on $\overline{\mathbb{F}_p}$, $\sigma\tau^{-1}$ can only fix K, i.e. $\sigma = \tau$.

Then in characteristic 0, there are simply no non-trivial definable automorphisms (because they fix an infinite field).

In characteristic $p > 0$, a definable group of automorphisms is a subgroup of $\hat{\mathbb{Z}}$.

Let G be a group acting on K by definable automorphisms. Any definable Abelian torsion-free group G is divisible: multiplication by n is an automorphism (it is injective therefore the images has the same dimension and degree). Therefore G is trivial. □

We now proceed to a group of results that determine properties of a definable group or a definable group action solely from assumpitons about dimensions.

Theorem 18 (Reineke)**.** *Let G be a definable group such that all definable subgroups of G are finite. Then G is Abelian.*

Proof. Suppose G non-commutative. Recall that centraliser $C(g)$ of $g \in G$ is the set of all elements commuting with a. By descending chain condition on definable subgroups the centraliser of an element (the set of elements commuting with it) is definable. It follows from the assumptions of the theorem that all centralisers of non-central elements are finite.

For any $g, h \in G$ where g is non-central consider the set $\{ x \in G \mid g^x = g^h \}$ which is equal to $\{ x \in G \mid xh^{-1} \in C(g) \}$, and therefore finite. Therefore, the set g^G has the same dimension as G, and since G is of degree 1, its complement is of strictly smaller dimension.

Let $g, h \in G \setminus Z(G)$. Then $g^G \cap h^G \neq \emptyset$. Let x, y be such that $g^x = h^y$, then $g^{xy^{-1}} = h$. Therefore all elements not in $Z(G)$ are in the same conjugacy class.

Taking the quotient $G/Z(G)$ we obtain an infinite group where all non-identity elements are in the same conjugacy class. Therefore all (non-identity) elements have the same order.

The order cannot be 2 for all elements. Indeed, take $g, h \in G/Z(G)$, then since gh is of order 2,

$$gh = (gh)^{-1} = h^{-1}g^{-1} = hg$$

so any two elements commute, which contradicts our assumption of non-commutativity of G.

Take an element $a \in G \setminus \{ 1 \}$. Since a and a^{-1} are conjugate, there exists b such that $a^b = a^{-1}$ (and $(a^{-1})^b = a$), and therefore $a^{b^2} = a$, so $a \notin C(b)$ but $a \in C(b^2)$. Since b and b^2 are, like all distinct non-identity elements of G, conjugate, there is c such that $b^c = b^2$.

Then the sequence of groups

$$C(b) \subset C(b^c) \subset C(b^{c^2}) \subset \dots$$

is strictly increasing. Bet a centraliser of b^c is conjugate to the centraliser of b, which leads to contradiction. □

Corollary 19. *A group of dimension 1 is Abelian-by-finite.*

Lemma 20. *Let G be a group such that all its definable proper subgroups are finite. Then the ring of automorphisms of G is an integral domain.*

Proof. Suffices to notice that images must be the whole of A or 0, and kernels are 0 or finite. So when $\sigma\tau = 0$ either σ or τ must be 0. $\square$

Definition 9. Recall that a group G is *solvable* if there exists a chain of normal subgroups $G \triangleright G_1 \triangleright \ldots$ such that G/G_i is Abelian.

Definition 10 (Morley rank)**.** Let M be an $\aleph_0$-saturated structure. The *Morley rank* is defined to be the minimal function $\mathrm{MR} : \mathrm{Def} \to \mathrm{Ord}$ that satisfies the following conditions:

1. $\mathrm{MR}(X) = 0$ for any finite set X;
2. if there exist disjoint definable sets $\{ X_i \}_{i=1}^{\infty}$ such that $\mathrm{MR}(X) = \alpha$ and $Y \supset X_i$ for all i then $\mathrm{MR}(Y) \geq \alpha + 1$;
3. if there exist disjoint definable sets $\{ X_i \}$ such that $\mathrm{MR}(X) < \beta$ and $Y \supset X_i$ for all i then $\mathrm{MR}(Y) \geq \beta$.

Proposition 21. *Let M be a structure of Morley rank and degree 1. Then Morley rank is a good notion of dimension in M.*

Theorem 22. *(Cherlin) A definable connected group of Morley rank 2 is solvable.*

Theorem 23 (Nesin, Zilber)**.** *Let $(H, \cdot)$ and $(A, +)$ be infinite Abelian groups, and suppose that there is a faithful action of H on A such that no inifinite definable subgroup $B \subset A$ is H-invariant. Then a field K is interpretable.*

Proof. Since A^0 is H-invariant, A must be equal to A^0, so A is connected.

Consider the stabilisers of the points of A, by descending chain condition, their (trivial) intersection is an intersection of finitely many of them, say, of stabilisers of points $a_1, \ldots, a_n$. Action of an element $h \in H$ on A is therefore determined by its action on $a_1, \ldots, a_n$. Since H is infinite, for one of them, call this element a, Ha is infinite.

Let us show that $Ha \cup \{ 0 \}$ is indecomposable. Suffices to check that this set is not partitioned into finitely many cosets by H-invariant subgroups of A, but these are finite, so this is true.

By Zilber's theorem, $Ha \cup \{ 0 \}$ generates a definable group, which is H-invariant, therefore it must coincide with A.

Let R be the subring of the ring of endomorphisms of A generated by H. Each element of A, as follows from Zilber's theorem, is of the form $g \cdot a + h \cdot a$ where $g, h \in H \cup \{ 0 \}$. If $r \in R$ then

$$r(g \cdot a + h \cdot a) = g \cdot (r \cdot a) + h \cdot (r \cdot a)$$

(note that R is commutative). The action of r on A is therefore determined by its action on a. It follows that any r is of the form $g + h$ where $g, h \in H$ (this presentation is not unique!).

Define equivalence relation on $(H \cup \{ 0 \})^2$:

$$(g, h) \sim (g', h') \text{ iff } (g + h) \cdot a = (g' + h') \cdot a$$

Then $R \cong H^2/\sim$ with addition and multiplication definable.

By an argument similar to that of Theorem 14, multiplication by any element of r is surjective, so R is an integral domain, and by Lemma 15, is a field.

$\square$

Theorem 24 ("Field configuration". Cherlin, Hrushovski)**.** *Let G be a connected group acting faithfully transitively on X (i.e. $\cap Stab(x) = \{ 1 \}$), and let $\mathrm{MR}\, X = 1, \deg X = 1$. Then there are three possibilities:*

1. $\mathrm{MR}\, G = 1$*, G is Abelian and G^0 acts freely on X;*
2. $\mathrm{MR}\, G = 2$*, then one can define the structure of a field K on X, and $G = \mathbb{G}_a(K) \rtimes \mathbb{G}_m(K)$;*
3. $\mathrm{MR}\, G = 3$*, then one can define the structure of a field K on X, and $G = \mathrm{PSL}_2(K)$;*

Proof. Suppose $\mathrm{MR}\, G = 1$, then by Reineke's theorem it is Abelian. The set X is partitioned into finitely many orbites under G^0. Since G^0 is normal in G, the orbits are permuted by the action of G, but since they are infinite and $\deg X = 1$, there is only one orbit. G^0 acts freely on this orbit, since the stabiliser of any point is the stabiliser of any other point (by Abelian-ness), and is trivial.

Suppose $\mathrm{MR}\, G = 2$. Then by Theorem 22 G is solvable. Let H be the commutant $[G^0, G^0]$, it is a definable connected group of dimension 1. Applying the reasoning of the previous paragraph, we deduce that H acts freely transitively on X.

Pick an element $x \in$ then $G/\operatorname{Stab}(x)$ (set-theoretic quotient)is in bijection with the orbit $Gx = X$. There fore $\mathrm{MR}\, G/\operatorname{Stab}(x) = \mathrm{MR}\operatorname{Stab}(x) = 1$, so $\operatorname{Stab}(x)^0$ is Abelian. For each $g \in G$ there is a unique $h \in H$ and a unique $s \in \operatorname{Stab}(x)$ such that $g = hs$. We get the split exact sequence

$$1 \to H \to G \to \operatorname{Stab}(x) \to 1$$

$\operatorname{Stab}(x)$ acts on H by inner automorphisms (recall that H is normal). The action is faithful (or otherwise G would be a direct product). Since H is of dimension and degree 1, it satisfies the conditions of Theorem 23, which can be applied to the action of $\operatorname{Stab}(x)^0$ on H, $\operatorname{Stab}(x)^0$ we construct a definable field K with $\mathbb{G}_a(K) = H, \mathbb{G}_m = \operatorname{Stab}(a)^0$. In fact, $\mathbb{G}_m = \operatorname{Stab}(a)$, since we know that $\mathbb{G}_a$ is connected and it is of the same degree as $\mathbb{G}_m$. Then $\operatorname{Stab}(x)$ acts by inner automorphisms on $\operatorname{Stab}(x)^0 = \mathbb{G}_m(K)$, but by Lemma 17 we know that these must be trivial, so $\operatorname{Stab}(x)$ is commutative and by the $\mathrm{MR} = 1$ argument acts freely and faithufully on $\mathbb{G}_a$ by linear maps, it is therefore $\mathbb{G}_m(K)$.

Now suppose $\mathrm{MR}\, G = 3$.

Pick an $x \in X$. Since G acts transitively, all stabilisers are conjugate and of the same dimension, therefore, by additivity of dimension $\mathrm{MR}\operatorname{Stab}(x) = 2$. Then again by dimension considerations re are finitely many points $x_1, \ldots, x_n$ with finite orbits under the action of $\operatorname{Stab}(x)$, and their complement X_0, an infinite orbit.

Points $x_1, \ldots, x_n$ are fixed by $\mathrm{Stab}(x)^0$. Note that $\mathrm{Stab}(x)^0$ acts strictly 2-transitively on X_0 (acts freely transitively on pairs of distinct points). Notice that the equivalence relation

$$y \sim z \text{ iff } \mathrm{Stab}(x)^0 = \mathrm{Stab}(y)^0$$

is preserved by the action of G. In view of strict 2-transitive action of $\mathrm{Stab}^0(x)$ on X_0, all equivalence classes on X_0 consist of singletons. But then all equivalence classes on the entire X consist of singletons too. Therefore $X_0 = X \setminus \{ x \}$. This is true for all points $x \in X$.

Then G acts strictly 3-transitively on X (freely transitively on triples of points). Indeed, for any triple (a_1, a_2, a_3) there exists $f \in \mathrm{Stab}(a_3)$ that maps it to (b_1, b_2, a_3), and $g \in \mathrm{Stab}(b_1)$ that maps the latter triple to (b_1, b_2, b_3). By strict transitivity of point stabiliser for any $h \in G$ that maps (a_1, a_2, a_3) to (b_1, b_2, b_3) must be such that $g^{-1} \circ h$ and $h \circ f^{-1}$ coincide with f and g respectively, therefore $h = g \circ f$.

Apply the case $\mathrm{MR} = 2$ to $\mathrm{Stab}(x)$ acting on X_0. We have a structure of a field K on X_0 and $\mathrm{Stab}(x)^0 = \mathbb{G}_a(K) \rtimes \mathbb{G}_m(K)$. Denote x as ∞ and pick two other points $0, 1 \in X \setminus \infty$.

The group $\mathrm{Stab}(0, \infty)$ is the multiplicative group of the field K, hence is of dimension and degree 1.

Since G is 3-transitive then there exits an element σ that sends $(\infty, 0, 1)$ to $(0, \infty, 1)$, σ^2 fixes three points, and is the identity, and $\mathrm{Stab}(0, \infty)^\sigma = \mathrm{Stab}(0, \infty)$.

By Lemma 20, the ring of automorphisms is an integral domain. In an integral domain the only involutive automorphisms are identity and inverse (solve the equation $\sigma^2 - 1 = (\sigma - 1)(\sigma + 1) = 0$). So, conjugation by σ is the multiplicative inverse $x \mapsto 1/x$.

Now we aim to see that σ together with $\mathrm{Stab}(0, \infty)$ generates $\mathrm{PSL}_2(K)$.

If $(-)^\sigma = \mathrm{id}$ on $\mathrm{Stab}(0, \infty)$ then $\sigma(ga) = ga = 1$, so σ transposes 0 and ∞ and fixes all other points. Since G is 3-transitive, using σ we can find an element in G acting by *any* permutation of X with finite support. This contradicts descending chain condition on centralisers. Therefore σ acts like $a \mapsto 1/a$ on X.

From this one concludes that G is indeed $PSL_2(K)$.

Let us now show that it is impossible that $\mathrm{MR}\, G > 3$.

By considering stabilisers of finite sets of points as above, we find a subgroup of G of dimension 4 acting transitively on X without finitely many points, and we show as above that it acts strictly 4-transitively on a definable subset of X' of X, of dimension 1. Consider points $0, 1, \infty_1, \infty_2 \in X'$. Let σ be an involution that exchanges ∞_1 and 0 and fixes $\infty_2, 1$. Let τ be an involution that exchanges ∞_2 and 0, and fixes ∞_2.

Then $\sigma\tau$ is a cycle of order three, and G contains all even permutations of X with finite support. This contradicts descending chain condition on centralisers. $\square$

Lecture 3. Borel-Tits theorem

Definition 11 (Imaginaries)**.** Let $R \subset X \times X$ be a definable equivalence relation on X. Then Y is called a *definable quotient* of X by R if there exists a map $X \to Y$ such that $(x, y) \in R$ if and only if $f(x) = f(y)$. The points of Y are called *imaginary elements*.

If in a structure (or a theory) every definable equivalence relation has a definable quotient, one says that the structure or theory has *elimination of imaginaries*.

It is possible to formally adjoin definable quotients to a structure (these quotients are then called *imaginary sorts*), and typically the notion of dimension extends to them. Notation: the structure M with definable quotients formally adjoint is M^{eq}.

By analogy with algebraicity over regular sets of parameters, we will say that y is algebraic a set of parameters a and an equivalence class x/R if there exists a definable correspondence $f : x/R \times Y \vdash M$ finite over M, where $a \in Y$, such that $f(x/R, a) = y$.

One can check that this extended algebraicity relation behaves similarly to the usual one, in particular: if $y \in \mathrm{acl}(x/R, a, b)$, and $b \in \mathrm{acl}(x/R, a)$ then $y \in \mathrm{acl}(x/R, b)$.

Theorem 25. *The theory of algebraically closed fields has elimination of imaginaries.*

Proof sketch. Let k be an algebraically closed field, $X \subset k^n$ be definable, and $R \subset X \times X$ be an equivalence relation.

Then there exists a definable correspondence with finite fibres $f : X \vdash Y$ such that $f(x) = f(y)$ implies $(x, y) \in R$.

For any point $x \in X$ we will find a definable set $U \ni x$ and a correspondence $f_x : U_x \vdash X$ with finite fibres over X, such that $f_x(u) = v$ implies $(u, v) \in R$ and $(u, v) \in R$ implies $f(u) = f(v)$.

The equivalence class of x is definable over x. Therefore, by Nullstellensatz, or just by the fact that $\mathrm{acl}(x)$ is a model, x/R has a point algebraic over x. This defines a correspondence $U \vdash x/R$ in some neighbourhood of x. The set on which the conditions on f hold is definable.

Then by compactness finitely many U_{x_i} cover X. Therefore we may "sew together" f_{x_i} into a single definable function $f : X \to X$. It follows from the definition of f that $f(X)$ intersects every equivalence class of X in finitely many points.

Existence of finite definable quotients follows from existence of finite quotients of affine varieties and quantifier elimination. □

Definition 12 (Algebraic varieties as definable sets)**.** An *abstract algebraic variety* over a field k is a set X together with a cover $\cup U_i = X$ and bijective maps $f_i : U_i \to V_i$ (V_i-s are called *charts* and the data (U_i, f_i) is called an *atlas*) where V_i are affine algebraic varieties and $f_j \circ f_i^{-1} : f_i(U_i \cap U_j) \to f_j(U_i \cap U_j)$ are isomorphisms of affine varieties. The notion of Zariski topology is extended to abstract algebraic varieties: sets Z such that $f_i(Z \cup U_i)$ are closed in all V_i are declared closed in this topology.

Let X and Y be abstract algebraic varieties with atlases (U_i, g_i), (O_j, h_j). A continuous map $f : X \to Y$ is called a *morphism* if the following additional data is specified. Let $W_{ij} = f^{-1}(O_j) \cap U_i$, then a collection of morphisms of quasi-affine varieties $f_{ij} : g_i(W_{ij}) \to h_j(O_j)$ specifies the morphism f if $h_j \circ f \circ g_i^{-1} = f_{ij}$ on W_{ij}.

This definition in the spirit of Weil allows to adopt a point of view that (sets of points of) varieties over algebraically closed fields can be seen as definable sets. Given an abstract algebraic variety, one can consider the disjoint union of V_i-s factored by the equivalence relation that identifies $f_i(U_i \cap U_j)$ with $f_j(U_i \cap U_j)$ using isomorphisms $f_j \circ f_i^{-1}$. By elimination of imaginaries the quotient can be identified with a contstructible set.

Note that an arbitrary definable (contsructible) set in an algebraically closed field k does not posses a natural structure of an algebraic variety (there are, a priory, many different ways to cover it with open affines).

Theorem 26 (Lie-Kolchin)**.** *A solvable algebraic group can be embedded into the group of upper triangular matrices.*

Proof. A solvable group acting on a proper variety has a fixed point. Indeed, proceed by induction on derived series length. For Abelian groups, that's clear: take a minimal closed orbit, the stabilizer of any point x is the stabiliser of any other point, so factoring it out we get a free action. But an affine group cannot act freely on a proper closed orbit, hence, x is a fixed point.

Now notice that a solvable G acts on the proper full flag variety. □

The following theorem is a generalisation of Theorem 23.

Theorem 27. *Let G be a group that faithfully acts on an abelian group V. Suppose also that G has a definable Abelian normal subgroup H, and V has a definable connected subgroup A that is G-invariant and such that A and $H/\operatorname{Stab}(A)$ satisfy the conditions of Theorem 23. Suppose further that sets gA generate V. Then there exists a definable field K, $A = (K^n, +)$ is a vector group and $G \subset \mathrm{GL}_n(K)$.*

Theorem 28 (model-theoretic "Lie-Kolchin")**.** *Let G be a solvable connected group that acts faithfully on an Abelian group V. Then every definable subgroup of V which either*

1. *has noinfinite $[G, G]$-invariant subgroups, or*

2. *has noinfinite G-invariant subgroups*

is pointwise fixed by $[G, G]$.

Proof. Induction on the length of derived series.

<u>Case 1.</u> Let W be a G'-invariant subgroup of V that has no infinite G'-invariant subgroups. Assume for contradiction that G' acts non-triviall on A'.

By induction hypothesis, G'' acts trivially on W. Let W' be the subgroup generated by indecomposable gW-s. Apply Theorem 27 to G' and G acting on W' (factoring out G'' to get faithful action). We get vector space structure on W' with G' acting by scalar matrices (as Therem 27 tells), of determinant 1 (since $[H, H] \subset \mathrm{SL}_n$ for any linear group H). Therefore $G' = \mu_n$, the group of roots of unity, but commutant is connected, so it acts trivially.

Case 2. Suppose that W has no infinite G-invariant subgroups. Let W' be an infinite subgroup of W that has no infinite G'-invariant subgroups. By induction hypothesis, G'' acts trivially on W'. Then $G \cdot W'$ is also fixed by G''. □

Theorem 26 follows from Theorem 28: let G be a solvable linear subgroup of $\mathrm{GL}_n(k)$, so it acts faithfully on $V = (k^n, +)$. It suffices to find G-invariant 1-dimensional subspace W and then proceed by induction, considering V/W.

Take $W \subset V$, a minimal G-invariant vector subspace. Let H be a definable subgroup of W that has no infinite definable G-invariant subgroups. Then $[G, G]$ acts trivially on H, and on W, which it generates (since $[G, G]$ acts by linear transformations). If G also acts trivially then all vector subspaces of V are G-invariant, and since W is minimal, it is of dimension one. Otherwise, by Nesin-Zilber Theorem (Theorem 23), there exists a structure of a field on W, such that multiplication by elements of the field are linear maps $W \to W$. So W must be a finite extension of k, but k is algebraically closed.

Definition 13 (Nilpotent)**.** A group G is called *nilpotent* if there exists a chain of normal subgroups $G \triangleright G_1 \triangleright \dots$ such that G_i/G_{i+1} is central in G/G_{i+1}.

Proposition 29. *Define increasing central series:* $Z_1 = Z(G)$

$$Z_{i+1} = \{\ g \in G \ \mid\ [g, Z_i] \in Z_{i+1}\ \}$$

A group is nilpotent of class n *if* $Z_n = G$.

Theorem 30. *If G is a solvable, non-nilpotent definable group then it interprets a field, i.e. there exists a definable set $K \subset G^n$ and definable maps $+ : K \times K \to K$ and $\cdot : K \times K \to K$ that make K into a field.*

Proof. First let us reduce to the case when G is centerless. Consider the increasing series of subgroups $Z_1 = Z(G)$, $Z_{i+1} = Z(G/Z_i)$. Since the rank of G is finite for some n we will have Z_{n+1}/Z_n finite. Take $a \in Z_{n+2}$, i.e. $[a, g] \in Z_{n+1}$. Then a^G consists of finitely many cosets of Z_n. A quotient of a connected definable group by a definable subgroup is connected, then so is G/Z_n. The centraliser of a is of finite index in G/Z_n hence is just G, and $a \in Z_{n+1}$. Therefore Z_{n+1} has stabilised.

We may now take a quotient by Z_{n+1} which will live in an imaginary sort. A field interperted in it will be interpreetable in G itself.

In a centerless solvable non-nilpotent group take A, an Abelian subgroup such that it does not have infinite normal subgroups. Apply Lie-Kolchin theorem (considering conjugation action of G on A). Then A is fixed pointwise by $[G, G]$. So action of $G/[G, G]$ is faithful, and we can apply Theorem 23. □

Lecture 4. Borel-Tits theorem, continued

Lemma 31. *Let G be simple linear group. Let B be a maximal solvable group. Then B is not nilpotent.*

Proof. Any automorphism of G constant on B can be factorised through G/B which is proper, while G is affine. Therefore, it is constant on G.

Suppose B is nilpotent. Proceed by induction on the length of lower central series. Since G/B is proper, if $B = \{ 1 \}$ then $G = G/B$ is proper and affine, and since G is connected, $G = B = \{ 1 \}$.

Induction step: there is an element of lower central series $N \subset B$ such that $[N, B] = \{ 1 \}$ therefore N is central in G. Then $B/N = G/N$ by induction hypothesis. $\square$

Proposition 32. *A definable morphism of $f : G_1 \to G_2$ of algebraic groups is an algebraic morphism composed with an inverse power of Frobenius morphism. A definable subgroup of an algebraic group is therefore a closed subgroup.*

Proof. By quantifier elimination, there exists an affine open subset $U \subset G_1$ such that f is a morphism composed with Frobenius. Since G_1 is quasi-compact, it can be covered by finitely many translates of U, $g_1U, \dots, g_nU$. Since f is a homomorphism, $f(g_iu) = f(g_i)f(u)$ so it is regular composed with a power of Frobenius on each g_iU, therefore it is such on the whole of G_1.

For the second statement, recall that the image of a morphism of algebraic groups is a closed subgroup. $\square$

Proposition 33 (Rosenlicht)**.** *Let G be a connected algebraic group with trivial center. Then G is a linear group.*

Proof. Consider the adjoint representation of G on T_eG: if $\mathcal{O}_{G,e}$ is the local ring at identity with maximal ideal $\mathfrak{m}$, then $TG = \mathfrak{m}/\mathfrak{m}^2$, with G acting on it as $g \cdot f(x) = f(x^g)$. More generally, G acts on all spaces $\mathfrak{m}/\mathfrak{m}^n$ for all $n > 1$, which are all finite-dimensional. Since G is centerless and connected, for any g there exists x such that x^g and so for n big enough g acts non-trivially. Therefore G is embedded into $\operatorname{proj\,lim} \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^n)$. On the other hand, the kernels $G \to \mathrm{GL}(\mathfrak{m}/\mathfrak{m}^n)$ must satisfy the descending chain condition, so in fact G embeds int GL_n for some n. $\square$

Theorem 34 (Poizat)**.** *Let K be an infinite field definable in an algebraically closed field k. Then K is definably isomorphic to k.*

Proof. Consider $\mathbb{G}_a(K) \rtimes \mathbb{G}_m(K)$. By Theorem 39 we may assume that it is an algebaric group. This is a centerless connected group, therefore, by Proposition 33, is a subgroup of $\mathrm{GL}_n(K)$.

Then $\mathbb{G}_a(K)$ is a commutative linear group by Proposition 32. It is connected by Lemma 16. By Jordan decomposition $\mathbb{G}_a(K) = G_u \cdot T$ where $\mathbb{G}_u$ consists of unipotent elements and T is an algebraic torus. As an additive group of K, $\mathbb{G}_a(K)$ does not have elements of finite order in characteristic 0, and of arbitrary finite order in characteristic $p > 0$ as torus does, so $\mathbb{G}_a(K)$ is unipotent.

A commutative unipotent group is isomorphic to a vector group. Indeed, the logarithm map

$$\log X = (X-1) - (X-1)^2/2 + \ldots + (-1)^n (X-1)^n/n!$$

is a bijection with the Lie algebra, and BCF formula shows that if the group is commutative then its inverse, the exponent, is an isomorphism of groups. So we have seen that $\mathbb{G}_a(K)$ is isomorphic to $(k^n, +)$.

We can observe already that $\operatorname{char} K = \operatorname{char} k$, and K is a finite extension of k. By Macintyre's theorem, k is algebraically closed, so must coincide with K. $\square$

Definition 14 (Orthogonality, internality)**.** Let X, Y be two definable sets. X and Y are called orthogonal if any definable set $Z \subset X \times Y$ there exist sets $Z_1 \subset X$, $Z_2 \subset Y$ such that $\dim(Z \triangle Z_1 \times Z_2) < \dim Z$.

A set U is *internal* to a set X if there exists an equivalence relation R on X^n and a definable injection $U \hookrightarrow X^n/R$.

Proposition 35. *In a definable simple group G of finite dimension no two definable sets are orthogonal. Moreover, if a set U is non-orthogonal to G, then G is internal to U.*

Corollary 36. *Let G be a simple definable group. Then it interprets a field K, G is definable in K and there exists a definable in G isomorphism $G \to G(K)$.*

Proof. By Lemma 31 and Theorem 30 a field is interpetable in G. By Proposition 35, there exists a non-trivial correspondence $f : G \vdash K$. By elimination of (finite) imaginaries there exists an incusion $G \to K^n$. $\square$

Theorem 37 (weak Borel-Tits)**.** *Let G_1, G_2 be two simple algebraic groups over algebraically closed field K and L, and let $f : G_1(K) \to G_2(L)$ be an abstract group isomorphism. Then there exists a field isomorphism $\sigma : K \to L$ and a map $f : G_1 \to G_2 \times_{\sigma^{-1}} \operatorname{Spec} L$, which is an algebraic group isomorphism followed by an inverse power of Frobenius.*

Proof. By Theorem 30 G_1 interprets a field k and by Proposition 35 and elimination of imaginaries in K, there exists an embedding $G_1 \hookrightarrow k^n$ definable in G_1. By Theorem 34, k is definably isomorphic to K, so we just identify it with K. Similarly, G_2 definably embeds into L^n, where L is definable in G_2.

Since G_2 is isomorphic to G_1 it is interpretable in K, and so L interpretable in K. We thus have an isomorphism $\tilde{\sigma} : K \to L$ of fields definable in K, and the isomorphism between $G_1(K)$ and $G_2(L) \times_{\tilde{\sigma}} \operatorname{Spec} K$ is definable in K, so by Proposition 32, a group morphism followed by inverse power of Frobenius. $\square$

We will now prove the model-theoretic version of Weil's theorem on group chunks.

Definition 15 (Group chunk)**.** A group chunk is a non-empty topological space G equipped with partal multiplication and inversion maps (m and i respectively) defined on subsets $W \subset G \times G$, $V \subset G$, satisfying the following conditions:

1. multiplication is a morphism $W \to V$;

2. for any $x \in V$ there exists an open subset $O_x \subset G$ such that $\forall y \in O_x$, $(x, y) \in W$ and $(y^{-1}, y \cdot x) \in X$;

3. the map i is a homeomorphism of V and a dense subset of G;

4. for x, y, z the identities $(xy)z = x(yz), (xz)z^{-1} = x, z^{-1}(zx) = x$ hold whenever both sides are defined;

Lemma 38. *Let G be a definable group. Let (W, V, m, i) be a group chunk of an algebraic group, with multiplication and inversion regular maps on U and V. Then for any $a, b \in G$ the set*

$$U_{a,b} = \{\ (x, y) \in V \times V \ \mid\ axby \in V\ \}$$

is open.

Proof. Indeed, pick a point $(x_0, y_0) \in U_{a,b}$. We can fiend $b_1, b_2 \in V$ such that $b_1 b_2 = b$. Consider the set $O \subset G$ such that $\forall z \in O$ (za, x_0), (zax, b_1), (zax_0b_1, b_2), $(zax_0b_1b_2, y_0) \in W$ that exists by successively applying point 2 of the definition of group chunk. Then the set

$$X = \{\ (x, y) \in V \times V \ \mid\ \forall z \in O(za, x), (zax, b_1), (zaxb_1, b_2), (zaxb_1b_2, y) \in W\ \}$$

is an open as a pre-image of an open set under a composition of multiplication maps, and contains (x_0, y_0). Taking the union of such for all x_0, y_0 such that $ax_0by_0 \in V$ we find that $U_{a,b}$ is open. □

Theorem 39 (van den Dries)**.** *Let G be a definable group in an algebraically closed field. Then G is definably isomorphic to an algebraic group.*

Proof. We will only prove the characteristic 0 version.

Step 1 — construction of a group chunk. By quantifier elimination there exists a locally closed subset $V_1 \subset G$ such that $\dim V_1 = \dim G$. Moreover, there exists $V_2 \subset V_1$ such that inversion is regular on V_2. Further, again by quantifier elimination there exists $W_1 \subset V_2 \times V_2$ such that multiplication is regular on W_1. By definability of dimension there exists

$$\begin{aligned} V_3 = \{\ x \in V_2 \ \mid\ & \dim V \times \{x\} \cap W_1 = \dim V_2 \text{ and} \\ & \dim\{\ (y^{-1}, yx) \in W_1 \ \mid\ (x, y) \in V_2 \times G\ \} = \dim V_2\ \} \end{aligned}$$

which can be shrunk to an open. Let $V = V_3 \cap V_3^{-1}$, and $W = \{\ (x, y) \in W_1 \ \mid\ x, y, xy \in V\ \}$.

By construction, W has the following properties:

1. for any $x \in V$ the set of y such that $(y, x) \in W$ and $(y^{-1}, y \cdot x) \in W$ is dense;

2. the map $(x, y) \mapsto x \cdot y$ from W to V is a morphism.

Step 2 — endowing G with a variety structure. Cover G by shifts of V, by compactness $G = \cup_{i=1}^{n} a_i V$.

By Lemma 38 for any $g, h, k \in G$, the set $U_{k^{-1}g,h} = \{ (g, h) \in V \times V \mid gxhy \in kV \}$ is open and the multiplication is a morphism on it. By letting g, h, k range in $a_1, \ldots, a_n$ we establish that multiplication is a morphism. $\square$

Lecture 5. Hrushovski's group configuration

Definition 16 (Pregeometry)**.** Let X be a set and $\mathrm{cl} : P(X) \to P(X)$ an operation that satisfies the following properties:

1. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for any $A \subset X$;
2. if $A \subset B$, then $\mathrm{cl}(A) \subset \mathrm{cl}(B)$;
3. if $x \in \mathrm{cl}(A \cup \{y\})$ then $y \in \mathrm{cl}(A \cup \{x\})$ or $x \in \mathrm{cl}(A)$;
4. if $x \in \mathrm{cl}(A)$ then there exists finite $A_0 \subset A$ such that $x \in \mathrm{acl}(A_0)$.

Definition 17. A pregeometry M is called *trivial* if $\mathrm{cl}(X) = \cup_{x \in X} \mathrm{cl}\{x\}$. A pregeometry is called *modular* if for all $X, Y \subset M$ such that $\mathrm{cl}(X) = X, \mathrm{cl}(Y) = Y$

$$\dim X + \dim Y - \dim X \cap Y = \dim X \cup Y$$

It is called locally modular if it is modular after naming some parameters, i.e. for the pregemetry $\mathrm{cl}_A(X) = \mathrm{cl}(X \cup A)$ for some set $A \subset M$.

Definition 18 (Strongly minimal)**.** A definable set X is called strongly minimal if it contains no infinite definable subset Y such that $X \setminus Y$ is infinite.

Proposition 40. *Let M be a strongly minimal definable set. The model-theoretic algebraic closure* acl *defines a pregeometry on M.*

Definition 19. A family $X \to T$ of subsets of a definable set Y, $X \subset T \times Y$ is called normal if for any $s, t, s \neq t$, $\dim X_t \triangle X_n \geq \dim X_s$. It is called almost normal if for any t there exist only finitely many s with such property.

Definition 20 (Algebraic dimension)**.** A tuple a of elements of a model M is called *algebraically independent over B* if for any i, $a_i \notin \mathrm{acl}(a_{\neq i} B)$. A tuple a is said to have *algebraic dimension n* over B if any indpendent subtuple $b \subset a$ is of size $\leq n$.

The *algebraic dimension of a definable set* $[\varphi]$, is the supremum of algebraic dimensions of tuples $a \in M$ such that $M \models \varphi(a)$.

Lemma 41. *In a strongly minimal theory for any definable set D,* $\dim(D) = \mathrm{MR}(D)$.

So algebraic dimension is a good dimension notion. It will be denoted dim in this lecture.

Starting from any family in a strongly minimal structure one can construct a normal family (in M^{eq}) by identifying those t and s for which $\mathrm{MR}\, X_t \triangle X_n < \mathrm{MR}\, X_s$ (it is possible by definability of Morley rank in a strongly minimal structure).

Proposition 42. *Let M be a strongly minimal trivial set. Then there exist no normal definable families $X \to T$ such that* $\mathrm{MR}\, T > 0$.

Proof. Let $X \to T$ be a normal family $X \subset T \times M^2$. Let (x_1, x_2, t) be a generic point of X. Then by triviality

$$x_2 \in \mathrm{cl}(x_1, t) = \mathrm{cl}(x_1) \cup \mathrm{cl}(t)$$

If $x_2 \in cl(x_1)$ then there exists finitely many curves with generic x_1, x_2 and since in a normal family there is exactly one curve for each generic tuple, T is finite.

If $x_2 \in \mathrm{cl}(t)$ by the same considerations then there exists finitely many generics that occur in a family, and by normality of the family T is finite. $\square$

Definition 21 (Linear and one-based structures)**.** A structure M is called *linear* if for any normal family $X \subset T \times Y$ of definable subsets of Y of dimension n, there exists a definable subset $Y' \subset Y$ such that for any $P \in Y'$ the set

$$\{\ t \in T \ \mid\ X_t \ni P\ \}$$

is of dimension 0.

A structure M is called *one-based* if for any sets $A, B \subset M$, A is independent from B over $\mathrm{acl}(A) \cap \mathrm{acl}(B)$.

Lemma 43. *A strongly minimal structure is locally modular if and only if it is one-based.*

Proof. In view of Lemma 41, a tuple a is independent from b over $\mathrm{acl}(a) \cap \mathrm{acl}(b)$ if and only if

$$\dim\{\ a, b\ \} - \dim b = \dim a - \dim(\mathrm{acl}(a) \cap \mathrm{acl}(b)) = \dim a + \dim b +$$

which is precisely modular equality.

We leave it as an exercise to see that one-basedness is preserved under adding parameters. $\square$

Proposition 44. *A one-based structure is linear.*

Proof. Let $X \to T$ be a normal family $X \subset T \times Y$. Let (x, t) be a generic tuple of X. Then x is independent from t aver $\mathrm{acl}(x) \cap \mathrm{acl}(t)$, so

$$\dim x, t - \dim \mathrm{acl}(x) \cap \mathrm{acl}(t) = \dim x, t - \dim t$$

Hence $t \in \dim \mathrm{acl}(x) \cap \mathrm{acl}(t) = \dim t$, and in particular that $t \in \mathrm{acl}\, x$, which gives the desired statement. $\square$

It follows that in a non-locally modular strongly minimal set there exists a normal family $X \to T$ such that $\mathrm{MR}\, T \geq 2$. Such a family of "curves" is also called a *pseudoplane.*

A substructure of a one-based structure is obviously one-based.

Proposition 45. *Let G be a one-based group. Then all definable subsets of G^n for all n are Boolean combinations of definable subgroups of G.*

Proof. Let us prove this when $\dim G = 1$.

Suppose there exists a definable subset $X \subset G^2$, $\deg X = 1$, such that X is not a Boolean combination of definable subgroups of G. Then by Stabiliser theorem, the stabilizer of a generic type of X is finite. Form the family $Z \to X \times G$ where

$$Z_{x,g} = X - x + g$$

By finiteness of stabiliser this family is almost normal. Its dimension is $\dim X \times G = 2$, which contradicts local modularity. $\square$

One says that a set is infinitely definable if it is an intersection of infinitely many definable sets.

Theorem 46 (Hrushovski-Weil)**.** *Let G be infinitely definable and let m be a definable map defined on $X \supset G$ such that $m(x, m(y, z)) = m(m(x,), z)$, and $m(x, y) = m(x, z)$ implies $y = z$, for $x, y, z \in G$. Then there exists a definable group $G' \supset G$.*

The following theorem is a far reaching generalisation of the above result.

Theorem 47 (Group configuration, Hrushovski)**.** *Suppose that there exist tuples x, y, z, a, b, c all of Morley rank 1, and such that*

1. *all elements are pairwise independent;*
2. *for triples $(x, y, z), (x, a, b), (z, a, c), (y, c, b)$ every element of a triple is algebraic over the other two elements;*
3. *for any proper subset $X \subset \mathrm{acl}(x)$ the type $\mathrm{tp}(ab/\mathrm{acl}(x))$ doesn't fork over X, for any proper subset $Y \subset \mathrm{acl}(y)$ the type $\mathrm{tp}(bc/\mathrm{acl}(Y))$ doesn't fork over Y, for any proper subset $Z \subset \mathrm{acl}(z)$ the type $\mathrm{tp}(ac/\mathrm{acl}(z))$ doesn't fork over Z;*
4. *any other triple consists of independent elements;*

Then there exists a group G, a definable set X, a faithful action of G on X, and generic elements x', y' of G and b' of X such that

1. *$x' \in \mathrm{acl}(x)$, $y' \in \mathrm{acl}(y)$, $x' \cdot y' \in \mathrm{acl}(z)$;*
2. *$b' \in \mathrm{acl}(b)$, $x' \cdot b' \in \mathrm{acl}(a)$, $x'y' \cdot b' \,\mathrm{acl}(c)$;*

Instead of proving the above theorem we will give a sketch of the proof of a statement equivalent to a higher-dimensional version of it that is used in the Marker-Pillay theorem which concludes the lecture.

Proposition 48. *Let M eliminate imaginaries. Let $f_t : X \vdash Y$ be a normal family of correspondences parametrised by a definable set T. Let $g_{s,t} : X \vdash X$ be the family, parametrised by $T \times T$, of correspondences of the form $g_{s,t} = f_t \circ f_s$. Let $\sim$ be the equivalence relation*

$$(s,t) \sim (s',t') \text{ iff } \dim \Gamma(g_{s,t}) \triangle \Gamma(g_{s',t'}) < \dim \Gamma(g_{s,t})$$

And suppose $\dim(T \times T)/\sim = \dim T$*. Then there exists a group G that shares generic element with $T \times T/\sim$, a morphism with generically finite fibres $\pi : Z \to X$ and an action $G \times Z \to Z$ such that*

$$g_{s,t}(x) = y \text{ iff } \pi((s,t) \cdot \pi^{-1}(x)) = y$$

Proof sketch. Whenever there exists a family of correspondence $X \vdash Y$ with generically finite image of size n, then there exists a function $X \to S^n Y$ where $S^n Y$ is a set definable in a (finite) imaginary sort. Applying this argument twice, first time to $f_s : X \vdash Y$ and getting $f'_s : X \to S^n Y$, second time to the invers $f_s'^{-1} : Y' \subset S^n Y \vdash X$ we may assume that $f_s : X \to X$ is a bijection.

Now we introduce a group operation (generically) on $T \times T/\sim$ by noticing that for any two classes $(s_1,t_1)/\sim$ and $(s_2,t_2)/\sim$ there exists, by dimension considerations $u \in T$ and $s_3, t_3 \in T$ such that $(s_1,t_1) \sim (s_3,u)$ and $(s_2,t_2) \sim (u,t_3)$. Define the product of $(s_1,t_1)/\sim$ and $(s_2,t_2)/\sim$ to be (s_3,t_3).

Applying Theorem 46 we get a definable group that shares a generic with $T \times T/\sim$. □

Theorem 49 (Marker-Pillay)**.** *Let G be a one-dimensional algebraic group defined over an algebraically closed field K, and let $X \subset G$ be a constructible set that is not a Boolean combination of cosets of subgroups of $G \times G$. Then G interprets a field.*

Proof. Suppose X smooth or else pass to an open subset throwing away singular points.

Let $Y \to X$ be the family of shifts of X, i.e. $Y_x = X - x$ (by the argument like in Proposition 45, it will be normal). All curves Y_x pass through the point $(1,1)$ where 1 is the identity of the group.

We want to interpret the affine group of the field — $(\mathbb{G}_a \rtimes \mathbb{G}_m)(K)$ by looking at the tangent vectors of curves in Y_0. Suppose that no curve in the family Y has a tangent with coordinates $(0,1)$ (or else throw all such points away). We identify the projectivisation of the (2-dimensional) tangent space $T_{(1,1)}$ with the elements of the field K. Note that the tangent vectors of curves $Y_x, Y_{x'}$ are multiplied when the curves are composed and add up when the curves are added up. Let $\sim$ be the (a priory not definable) relation

$$x \sim x' \text{ iff } Y_x \text{ touches } Y_{x'} \text{ at } (1,1)$$

The set $H = X \times X/\sim$ parametrises a family of correspondences $M_{(a,b)}$: $X \vdash X$:

$$d \in M_{(a,b)}(c) \text{ if } Y_d \text{ touches } Y_a \circ Y_c + Y_b$$

The correspondences $M_{(a,b)}$ are not definable themselves (a priori), but one can define correspondences $\tilde{M}_{(a,b)}(x)$ differs from $M_{(a,b)}(x)$ by finitely many points for almost all $x \in X$, as follows. For a generic parameter $(a, b) \in X \times X$ let n be the generic number of intersections of Y_x and $Y_a \circ Y_c + Y_b$. Define $d \in \tilde{M}_{(a,b)}$ if $\#|(Y_d \cap Y_a \circ Y_c + Y_b)| < n$.

One checks that $\tilde{M}_{(a,b)}$ satisfies the conditions of Proposition 48, and thus we recover some two-dimensional group G acting on a one-dimensional set that X covers with finite fibres.

By Theorem 24, G is the affine group of a field. $\square$

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