Elimination of generalised imaginaries and Galois cohomology

Dmitry Sustretov[1]

Abstract

The objective of this article is to characterise elimination of finite generalised imaginaries as defined in [9] in terms of group cohomology. As an application, I consider series of Zariski geometries constructed [10, 23, 22] by Hrushovski and Zilber and indicate how their non-definability in algebraically closed fields is connected to eliminability of certain generalised imaginaries.

Contents

1 Introduction **1**

2 Generalised imaginaries **3**
2.1 Groupoids and torsors 3
2.2 Morita equivalence 5

3 Galois cohomology **8**

4 Non-standard Zariski structures **12**
4.1 Group extensions 12
4.2 Quantum Zariski geometries 13
4.3 Definability and Brauer group 16
4.4 Quantum Zariski geometries: an example 18
4.5 Eliminability of generalised imaginaries in CCM and RCF . . . 19

A Group cohomology **20**

1 Introduction

Let M be a strongly minimal non-locally modular structure. It has been a long-standing conjecture of Zilber's [26] that M interprets an algebraically closed field (this statement is one of the clauses of a statement widely known as Zilber's trichotomy principle). It has been disproved by Hrushovski [8] but then proved by

[1] The research leading to these results has received funding from the European Research Council under the European Unions Seventh Framework Programme (FP7/2007- 2013)/ERC Grant Agreement No. 291111. The author was also supported by Hill Foundation Scholarship when the work on the material present in the article has started.

Hrushovski and Zilber in the important case of Zariski geometries [10]. The notion of a Zariski geometry is a natural axiomatisation of the properties of Zariski topology on Cartesian powers of algebraic varieties over an algebraically closed field, and of compact complex manifolds ([21, 24]). A Zariski geometry is a structure M endowed with a topology on every Cartesian power of M^n such that definable sets in M are constructible sets in the topology. Moreover, the structure must have a notion of dimension (for example, Krull dimension), that satisfies certain properties.

The main result of [10] asserts that a strongly minimal non-locally modular Zariski geometry M interprets an algebraically closed field k. The next natural question to ask is that of the possibility of co-ordinatisation of M, i.e. finding a definable in M injection $M \hookrightarrow k^n$ for some n, so that M becomes a quasi-projective algebraic curve. Theorem B of [10] asserts that if one assumes just non-local modularity then one can only guarantee the existence of a definable map $M \hookrightarrow k^n$ with finite fibres, and in order to ensure that it is an injection, one has to impose an extra condition: existence of a family of one-dimensional sets in M^2 that separates points.

In Theorem C of [10] the authors give an example of a Zariski structure that projects onto an algebraic curve but the extra structure on the fibres of the projection prevents this structure from being interpretable in an algebraically closed field.

Later, many more examples of Zariski geometries that are not interpretable in an algebraically closed field have been constructed by Zilber ([23, 25, 22]), not only in Morley rank 1. The work that lead to the results presented in this article started as an attempt to find a uniform approach to proving non-interpretability of such structures in an algebraically closed field, identifying an obstruction that would appear in all known examples. It was also desirable to be able to decide if the structures in question were interpetable in a compact complex manifold considered as a first-order structure.

This article suggests the following approach to this problem, building on the notion of generalised imaginary sort introduced by Hrushovski [9]. In [17] Poizat proposed a model-theoretic generalisation of an absolute Galois group: the group $\mathrm{Gal}(\mathrm{acl}(A)/\mathrm{dcl}(A))$ of automorphisms of $\mathrm{acl}(A)$ that fix $\mathrm{dcl}(A)$ can be endowed with a topology generated by the base consisting of the stabilisers of finite subsets of $\mathrm{acl}(A)$, and in a theory that eliminates imaginaries there is a one-to-one correspondence between closed subgroups of $\mathrm{Gal}(\mathrm{acl}(A)/\mathrm{dcl}(A))$ and definably closed subsets of $\mathrm{acl}(A)$. That opens path for extending the Galois cohomological results from theory of algebraically closed fields to a general model-theoretic setting. Thus, Pillay [15] has noticed that the correspondence between isomorphism classes of torsors and cocycle classes of the first group cohomology group translates almost verbatim to model-theoretic context.

In [9] Hrushovski introduced generalised imaginaries as certain sorts related to definable groupoids. Loosely speaking, if one regards a definable groupoid as a generalisation of an equivalence relation where equivalence classes can have automorphisms, then generalised imaginary sort is something that is like an imaginary sort, but also takes into account the automorphisms.

In this article I introduce a notion of Morita equivalence of definable groupoids (quite standard in other categories) so that Morita equivalent groupoids give rise to bi-interpretable generalised imaginary sorts. Groupoids that correspond to those imaginary sorts which are interpretable in the home sort are called eliminable. The notion of Morita equivalence gives the same equivalence classes of definable groupoids as the notion of equivalence defined by Hrushovski in [9]. A generilised imaginary sort is a sort with a structure of a definable groupoid torsor, i.e. a set acted upon definably, freely and transitively by a definable groupoid.

I then prove (Theorem 3.3) that in a setting where K-definable groupoids have a split torsor over $\mathrm{acl}(K)$, Morita equivalence classes of connected K-definable groupoids that have an Abelian isomorphism group of objects are in one-to-one correspondence with classes in the second cohomology group of the absolute Galois group of K. In this correspondence, eliminable groupoids correspond to the trivial class.

In Section 4.3 I show that quantum Zariski geometries of [22] are bi-interpretable with a certain generalised imaginary sort. It is then easy to see that the definability of the whole structure depends on eliminability of the corresponding groupoid. Thus, eliminability can be decided by computing the corresponding cohomology class. It also follows easily (Section 4.5) that the interpretability of the discussed structures in a compact complex manifold is equivalent to interpretability in an algebraically closed field, since the parameters of the groupoids corresponding to the obstruction sorts lie in a projective variety, and thus their absolute Galois group is exactly the same as in the algebraically closed fields.

The same situation is observed (Section 4.1) in the case of non-standard Zariski structures defined in [23, 10], if one assumes finiteness of the group action used in their definition; there again the definability of the structure in an algebraically closed field is equivalent to eliminability of a certain generalised imaginary.

I would like to thank Boris Zilber for introducing me to the circle of problems that motivated the work presented in this article, and I am grateful to Maxim Mornev, Martin Bays and Moshe Kamensky for many helpful conversations.

2 Generalised imaginaries

2.1 Groupoids and torsors

Throughout the article we will write $X \times_{f,Z,g} Y$ for the fibre product of definable sets, i.e. the set

$$\{\ (x, y) \in X \times Y \ \mid\ f(x) = g(y)\ \}$$

sometimes dropping f and/or g when they are clear from context.

Definition 2.1 (Groupoid)**.** *A* groupoid *is a category such that all its morphisms are isomorphisms. If a groupoid is small, i.e. if its objects and its morphisms are sets, then it is defined by the following data: a tuple* $X_\bullet = (X_0, X_1)$ *of sets along with maps* s, t, m, i, e*, where* s, t *maps* X_1 *to* X_0 *(source and target objects), c maps* $X_1 \times_{s,X_0,t} X_1$ *to* X_1 *(composition of arrows), i maps* X_1 *to itself (inverse),* $e : X_0 \to X_1$*, satisfying the natural axioms.*

Let $\mathcal{C}$ *be a category that has finite products. A* groupoid $X_\bullet$ internal to a category $\mathcal{C}$ *is a pair of objects* X_0, X_1 *along with the morphisms* s, t, m, i, e *satisfying the mentioned identities.*

The set of morphisms from object x *to object* y *is denoted* $\mathrm{Mor}(x, y)$*. If* $\mathrm{Mor}(x, x)$ *is isomorphic to a group* A *for all* $x \in X_0$ *then the groupoid* $X_\bullet$ *is said to be* bounded *by* A*.*

Remark 2.2. *The notation serves to underline the fact that a groupoid is in particular a simplicial set that only has* 0*- and* 1*-simplices.*

Definition 2.3 (Definable groupoid)**.** *Let* $\mathrm{Def}(\mathcal{U})$ *be the category of sets and maps definable with parameters in a monster model* $\mathcal{U}$ *of a complete theory* T*. Then a groupoid internal to* $\mathrm{Def}(\mathcal{U})$ *is called* definable groupoid *(cf. [9]).*

A definable groupoid definable over a set of parameters K *is said to be* bounded *by a definable group* A *(definable over* K *as well) if for any* $x \in X_0$ *there exists a definable isomorphism between* $\mathrm{Mor}(x, x)$ *and* A*, definable perhaps over a bigger set of parameters* $L \subset K$*.*

Definition 2.4 (Action groupoid)**.** *Let* G *be a group and let* $\cdot : G \times X \to X$ *be a group action. The* action groupoid *is defined to be the groupoid with the morphisms* $G \times X$ *and objects* X *where* $s(g, x) = x$ *and* $t(g, x) = g \cdot x$*,* $(g, x) \cdot (h, gx) = (gh, x)$*, and other structure maps defined in the obvious way.*

Definition 2.5 (Groupoid quotient)**.** *Let* $X_\bullet$ *be a groupoid. Let* E *be the equivalence relation on* X_0 *which is the image of the map* $(s, t) : X_1 \to X_0 \times X_0$*. The quotient* X_0/E *is called the* groupoid quotient*. We will denote it as* $[X_\bullet]$*.*

Definition 2.6 (Groupoid torsors)**.** *Let* $X_\bullet$ *be a groupoid. A* groupoid homogeneous space for $X_\bullet$ over Y *is a map* $p : P \to Y$ *together with the* anchor map $a : P \to X_0$ *and* action map $\cdot : X_1 \times_{s,X_0,a} P \to P$ *which commutes with the projection to* Y*. A homogeneous space is called* principal *(or a* torsor*) if for any two* $f, g \in P$ *such that* $p(f) = p(g)$ *there exists a unique* $m \in X_1$ *such that* $f \cdot m = g$*.*

A morphism of groupoid torsors P *and* Q *is a map* $\alpha : P \to Q$ *that respects the anchor map and commutes the action map:* $a(\alpha(f)) = a(f)$*,* $\alpha(m \cdot f) = m \cdot \alpha(f)$ *for any* $a \in X_\bullet)$ *and any* $m \in \mathrm{Mor}(a, s(f))$*.*

A groupoid $X_\bullet$ *is called* eliminable *if there exists a* $X_\bullet$*-groupoid torsor over* $[X_\bullet]$*.*

Informally, a groupoid torsor is a collection of arrows from Y to X_0 with a possibility to compose these arrows with morphisms of $X_\bullet$ with the suitable source object. Note that if $X_\bullet$ is a groupoid with a single object x, a groupoid torsor is the same as group $\mathrm{Mor}(x,x)$-torsor.

Let E be an equivalence relation on a definable set X_0. Transitivity, symmetry and reflexivity E imply the existence of the natural composition, inverse and identity structure maps of the groupoid with the morphisms set $X_1 = E \subset X_0 \times X_0$. Then X_0 is a groupoid torsor over X_0/E. In a theory that does not eliminate imaginaries X_0/E lives in an imaginary sort. Elimination of imaginaries is the condition that X_0/E is in definable bijection with a definable set in some of the home sorts of the theory.

In a similar vein, one might want to define generalised imaginary sorts as sorts that contain a groupoid torsor for a groupoid (which does not necessarily come from an equivalence relation).

Definition 2.7 (Generalised imaginary sort)**.** *Consider a theory with elimination of imaginaries. Let $X_\bullet$ be a definable groupoid. A* generalised imaginary sort *is an expansion of the theory with an additional sort P which is the groupoid torsor for $X_\bullet$ over $[X_\bullet]$. The expansion includes the maps $s : P \to X_0$, $t : P \to [X_\bullet]$ and the action map $a : P \times X_1 \to P$. The axioms that define P to be a torsor are clearly first-order.*

2.2 Morita equivalence

Suppose two groups G, H act freely on two spaces X, Y so that $X/G \cong Y/H$. If one declares the corresponding action groupoids equivalent then the set of action groupoids for free aciton modulo the equivalence is the same as the set of groupoid quotients up to isomorphism. One way to motivate the notion of Morita equivalence is that it generalises the equivalence just described to arbitrary groupoids, in particular to arbitrary action groupoids, taking into account the stabilisers. The following definitions and lemmas quite standard, their analogues for the differential category, for exapmple, can be found in [2].

Definition 2.8 (Morita equivalent groupoids)**.** *A* Morita morphism *$f_\bullet : X_\bullet \to Y_\bullet$ is a pair of maps $f_0 : X_0 \to Y_0, f_1 : X_1 \to Y_1$ such that the diagram*

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \times X_0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0 \times f_0} \\ Y_1 & \longrightarrow & Y_0 \times Y_0 \end{array}$$

commutes, f_0 is surjective and for any $(x_1, x_2) \in X_0 \times X_0$ the map f_1 induces a bijection between $Mor(x,y)$ and $Mor(f_0(x), f_0(y))$. If one looks at groupoids as small categories, then the above conditions say precisely that Morita morphism defines a fully faithful functor which is surjective on objects.

Two groupoids $X_\bullet$ and $Y_\bullet$ are called Morita equivalent *if there exists a third groupoid $Z_\bullet$ together with two Morita morphisms $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$.*

This is indeed an equivalence relation: given equivalences $X_\bullet \leftarrow R_\bullet \rightarrow Y_\bullet$ and $Y_\bullet \leftarrow S_\bullet \rightarrow Z_\bullet$ consider the groupoid $R_1 \times_{Y_1} S_1 \rightrightarrows R_0 \times_{Y_0} S_0$, one checks that the naturally defined Morita morphisms from this groupoid to $X_\bullet$ and $Z_\bullet$ define a Morita equivalence.

Proposition 2.9. *Let $X_\bullet$ and $Y_\bullet$ be Morita equivalent. Then $[X_\bullet] \cong [Y_\bullet]$.*

Proof. Indeed, one concludes easily from the definitions that if $f : Z_\bullet \rightarrow X_\bullet$ is a morita morphism, then $a \sim_X b$ if and only if $f_0(a) \sim_Y f(b)$ where $\sim_X, \sim_Y$ are the equivalence relations on X_0, Y_0, images of X_1, Y_1 in $X_0 \times X_0, Y_1 \times Y_1$ respectively. □

Corollary 2.10. *Connectedness is preserved under Morita equivalence.*

A class of Morita equivalent groupoids is morally a quotient space "that remembers stabilisers".

Lemma 2.11. *Let $f : X_\bullet \rightarrow Y_\bullet$ be a Morita morphism and let P be a $Y_\bullet$-torsor over S. Then there exists a $X_\bullet$-torsor $f^{-1}(P)$ over S.*

Proof. Consider the torsor $X_0 \times_{f_0, Y_0, a} P \rightarrow S$ with the natural anchor map and the action map

$$\sigma(x,p) = (x, f_1(\sigma)p)$$

The commutative diagram in the definition of Morita morphism ensures that the action map commutes with a map to X_0. □

Lemma 2.12. *Let $f : X_\bullet \rightarrow Y_\bullet$ be a Morita morphism and let P be a $X_\bullet$-torsor over S. Then there exists a $Y_\bullet$-torsor $f(P)$ over S.*

Proof. Consider the torsor $P/\sim \rightarrow S$ where $\sim$ is defined as

$$p \sim q \quad \text{if and only if} \quad \begin{array}{l} \sigma p = q \text{ for some } \sigma \in \mathrm{Mor}(a(p), a(q)), \\ \text{such that } f_0(a(p)) = f_0(a(q)) \text{ and } f_1(\sigma) = \mathrm{id} \end{array}$$

The anchor map is the composition of the anchor map with f_0. The action map is defined as $\sigma[p] = [f^{-1}(\sigma) \cdot p]$. □

Proposition 2.13. *Let $X_\bullet$ and $Y_\bullet$ be Morita equivalent definable groupoids. Then expansions of the theory with generalised imaginary sorts corresponding to $X_\bullet$ and $Y_\bullet$ are bi-interpretable.*

Proof. Let $X_\bullet$ and $Y_\bullet$ be Morita equivalent via Morita morphisms $f : Z_\bullet \rightarrow X_\bullet$ and $g : Z_\bullet \rightarrow Y_\bullet$. Let P be the generalised imaginary sort associated to $X_\bullet$. Then it follows from Lemmas 2.11 and 2.12 that the torsor $g(f^{-1}(P))$ is interpretable in the expansion of the theory with P and is the generalised imaginary sort associated to $Y_\bullet$. □

Corollary 2.14. *Generalised imaginary sort that corresponds to an eliminable groupoid is interpretable in the base structure.*

Proposition 2.15. *Let $X_\bullet, Y_\bullet$ be groupoids definable over K. Then $X_\bullet$ is Morita equivalent to $Y_\bullet$ if and only if there exists a set Q ($X_\bullet$-$Y_\bullet$-*bitorsor*) which has the structure of a $X_\bullet$-torsor over Y_0 and $Y_\bullet$-torsor over X_0 such that the actions of $X_\bullet$ and $Y_\bullet$ commute.*

Proof. Let $X_\bullet \to Y_\bullet$ be a Morita morphism. Then $Q = X_1 \times_{X_0} Y_0$ has a natural structure of a $X_\bullet$-$Y_\bullet$-bitorsor. If Q is a $X_\bullet$-$Z_\bullet$-bitorsor and P is a $Z_\bullet$-$Y_\bullet$-bitorsor then $(Q \times_{Z_0} P)/Z_1$ is a $X_\bullet$-$Y_\bullet$-bitorsor.

Conversely, let Q be a $X_\bullet$-$Y_\bullet$-bitorsor. Then $Y_1 \times_{Y_0} Q \times_{X_0} X_1 \rightrightarrows Q$ has a natural structure of a groupoid, with natural Morita morphisms to $X_\bullet$ and $Y_\bullet$. □

Corollary 2.16. *Let $X_\bullet$ be a groupoid definable over K and suppose $X_\bullet$ is bounded by A. Then $X_\bullet$ is eliminable if and only if it is Morita equivalent to the trivial action groupoid over $[X_\bullet]$.*

Hrushovski defines equivalence of definable groupoids in [9] as follows: $X_\bullet$ is equivalent to $Y_\bullet$ if there exist full definable functors $X_\bullet \hookrightarrow Z_\bullet, Y_\bullet \hookrightarrow Z_\bullet$ that are injective on objects and such that images of $X_\bullet$, $Y_\bullet$ in $Z_\bullet$ meet every isomorphism class in $Z_\bullet$. Such a pair of functors gives rise naturally to a $X_\bullet$-$Y_\bullet$-bitorsor of arrows from objects of $X_\bullet$ to objects of $Y_\bullet$ therefore, by Proposition 2.15, $X_\bullet$ and $Y_\bullet$ are Morita equivalent. The inverse implication is slightly less straightforward.

Proposition 2.17. *Let $X_\bullet \leftarrow Z_\bullet \to Y_\bullet$ be a Morita equivalence. Then there exist definable full faithful functors from $X_\bullet$ and $Y_\bullet$ into a groupoid $W_\bullet$ which are injective on objects and such that the images meet every isomorphism class of objects of $W_\bullet$.*

Proof. Let Q be the $X_\bullet$-$Y_\bullet$-bitorsor witnessing the Morita equivalence of $X_\bullet$ and $Y_\bullet$ (Lemma 2.15).

Define $W_0 = X_0 \sqcup Y_0, W_1 = X_1 \sqcup Q \sqcup Q^{-1} \sqcup Y_1$ where Q^{-1} is a copy of Q. There are natural source and target maps defined on Q, Q^{-1} coming from bitorsor structure. Thinking of elements of Q as arrows from X to Y and of elements of Q^{-1} as arrows from Y to X define composition of arrows: the composition of arrows in X_1 and Y_1 are given by composition in respective groupoids, the composition of arrows in Y_1 and X_1 and arrows in Q, Q^{-1} is given by bitorsor structure.

The fact that inclusions $X_\bullet \hookrightarrow W_\bullet, Y_\bullet \hookrightarrow W_\bullet$ are fully faithful follows from the fact that action of X_1 and Y_1 on Q, Q^{-1} is free. □

The following statement generalises the lemma of Lascar and Pillay about elimination of imaginaries up to finite ones in strongly minimal theories.

Proposition 2.18. *Let M be strongly minimal and let $\mathrm{acl}(\emptyset) \cap M \neq \emptyset$. Let $X_\bullet$ be a groupoid with $X_0 \subset M^n$. Then there exists a Morita equivalent groupoid $Y_\bullet$ such that $Y_0 \to [Y_\bullet]$ is finite.*

Proof. Let R be the image of X_1 in $X_0 \times X_0$. By Lemma 1.6 [16], there exists a definable $Y_0 \subset X_0$ such that the R-equivalence classes of Y_0 are finite. The groupoid which is the restriction of $X_\bullet$ to $Y_\bullet$ clearly satisfies the requirements of Propostion 2.17 and hence is Morita equivalent to $X_\bullet$. □

Finally, let us remark that the the notion of *retractability* defined in [6] is equivalent to having a Morita morphism to a groupoid such that $X_0 = [X_\bullet]$; one uses the straightforward generalisation of the argument in Proposition 1.11, *loc.cit.*, applying it to groupoids that are not necessarily connected.

3 Galois cohomology

I will work in a theory that eliminates imaginaries, so the Galois correspondence for strong types applies:

Theorem 3.1 (Poizat [17], Theorem 14)**.** *Let* $K \subset L \subset \mathrm{acl}(K)$*, then* $\mathrm{Aut}_L(\mathrm{acl}(K))$ *is the closed subgroup of* $\mathrm{Aut}_L(\mathrm{acl}(K))$ *that fixes* $\mathrm{dcl}(L)$*.*

For any set K denote $G_K := \mathrm{Aut}(\mathrm{acl}(K)/\,\mathrm{dcl}(K))$. If $G_L \subset G_K$ is normal I will denote the quotient $\mathrm{Gal}(L/K)$. If A is a definable group then I denote $A(K)$ the group of tuples definable over K that belong to A; I will also denote the algebraic closure of a set K as $\overline{K}$. If $f : X \to Y$ is a map between sets defined over K and σ is an element of G_K, then $\sigma(f) : \sigma(X) \to \sigma(Y)$ will be the map obtained from f by conjugation by σ: $\sigma(f) = \sigma^{-1} f \sigma$.

For a connected groupoid $X_\bullet$ let us say that a $X_\bullet$-torsor P is *split over* L if it has a point $x \in P$ definable over L.

Lemma 3.2. *Let* P *be a torsor of a connected groupoid* $X_\bullet$ *bounded by a definable group* A *and split over* L*. The group of definable (over some set* $M \supset L$*) automorphisms of* P *is in bijective correspondence with elements of the group* $A(M)$*.*

Proof. Straightforward. □

Theorem 3.3. *Let* K *be a set of parameters. Let* A *be a definable over* K *Abelian group. There exists a bijective correspondence between Morita equivalence classes of connected groupoids definable over* K*, eliminable over* $\mathrm{acl}(K)$*, with torsors split over* $\mathrm{acl}(K)$ *and bounded by* A *and cohomology classes in* $H^2(G_K, A(\overline{K}))$*. Eliminable groupoids correspond to the trivial cohomology class.*

Proof.
Step 1: describe a map that sends a cocycle that reperesents a class in $H^2(G_K, A(K))$ to a groupoid with a torsor over $\overline{K}$.
Let $X_\bullet$ be a groupoid and let P be an $X_\bullet$-torsor definable over a set of parameters $L \supset K$. If L is minimal such that G_L is normal in G_K then the orbit of P under the action of G_K consists of several copies of P that are in one-to-one correspondence

with elements of $\mathrm{Gal}(L/K)$. I will denote the Galois conjugates of P as P_σ for all $\sigma \in G_K$ even though by doing so I denote a particular Galois conjugate by several different names. Until the end of this proof the composition will be written without the symbol $\circ$ and from left to right.

Pick a point $x \in P$ definable over $\overline{K}$. Choose a continuous section $j : \mathrm{Gal}(L/K) \to G_K$. Let $v_\sigma : P \to P_\sigma$ be the isomorphism of $X_\bullet$-torsors that sends x to $\sigma(x)$. Let $Q = \bigcup\limits_{\sigma \in G_K} P_\sigma$ and define $u_\sigma : Q \to Q$ as follows:

$$u_\sigma(y) = v_{j(\alpha)^{-1}} u_\sigma v_{j(\alpha)}$$

for $y \in P_\alpha$. The maps u_σ can be thought of as the *scindage* in the terminology of [5], IV.3.5. Note that $u_\sigma f = \sigma(f) u_\sigma$ for any $f : Q \to Q$ any $\sigma \in G_K$.

Define

$$h(\sigma, \tau) = u_\sigma u_\tau u_{\sigma\tau}^{-1}$$

The expression on the right is an automorphism of P and therefore can be identified with an element of $A(\overline{K})$ by Lemma 3.2.

Let us check that this indeed defines a cocycle, i.e. that the equality

$$\alpha(h(\sigma, \tau))h(\alpha, \sigma\tau) = h(\alpha\sigma, \tau)h(\alpha, \sigma)$$

holds. Indeed,

$$u_\alpha u_\sigma u_\tau = h(\alpha, \sigma) u_{\alpha\sigma} u_\tau = h(\alpha, \sigma) h(\alpha\sigma, \tau) u_{\alpha\sigma\tau}$$

but on the other hand

$$u_\alpha u_\sigma u_\tau = u_\alpha h(\sigma, \tau) u_{\sigma\tau} = h(\sigma, tau) u_\alpha u_{\sigma\tau} = \alpha(h(\sigma, \tau)) h(\alpha, \sigma\tau) u_{\alpha\sigma\tau}$$

Step 2: describe a map that puts a groupoid and a torsor into correspondence to a cocycle in $H^2(G_K, A(\overline{K}))$.
By Lemma A.3 for any $\alpha \in H^2(G_K, A(\overline{K}))$ there exists $\beta \in H^2(\mathrm{Gal}(L/K), A(L))$ for some $L \subset \overline{K}$ such that $\alpha = \inf(\beta)$. Let G be the definable group, an extension of $\mathrm{Gal}(L/K)$ by A, that corresponds to (Theorem A.5) the cocycle β. Consider the action of G on a $\mathrm{Gal}(L/K)$-torsor P where an element $g \in G$ acts as its projection $p(g) \in \mathrm{Gal}(L/K)$, and let $X_\bullet$ be the associated action groupoid. Then $\cup \mathrm{Mor}(y, x)$ for $x \in X_0$ is definable over L (it is a union of $|\mathrm{Gal}(L/K)|$ copies of A) and is naturally an $X_\bullet$-torsor over a singleton.

Step 3: show that groupoids corresponding to cohomologous cocycles are Morita-equivalent.
This amounts to showing (by Step 2 and Theorem A.5) that if $f : G \to G'$ is an isomorphism of group extensions, i.e. if

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 1 \\ & & \| & & \downarrow f & & \| & & \\ 1 & \longrightarrow & A & \longrightarrow & G' & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 1 \end{array}$$

is a commutative diagram then the action groupoids for the actions of G and G' on a $\mathrm{Gal}(L/K)$-torsor are Morita equivalent. In fact, it is clear by construction of the action groupoid that they are definably isomorphic.

Step 4: show that cocycles corresponding to a groupoid $X_\bullet$ and two different $X_\bullet$-torsors P, P' are cohomologous.

Pick some tuples $x \in P, y \in P'$ defined over $\overline{K}$ such that $a(x) = a(y)$ (where a is the anchor map). We will identify P and P' via an isomorphism that sends x to y. Let $\eta : P' \to P$ the isomorphism of torsors that sends x to y. Let u_σ, u'_σ be the maps as in Step 1 used to obtain the cocycles h, h' for P and P' respectively. Define a cochain $g : G_K \to A(\overline{K})$:

$$g(\sigma) = u'_\sigma u_\sigma^{-1}$$

(here $g(\sigma)$ is identified with an element of $A(\overline{K})$ by Lemma 3.2). Let $h, h' \in H^2(G_K, A(\overline{K})$ be cocycles that are obtained using procedure from Step 1 from torsors P and Q. Then

$$\begin{aligned} h'(\sigma,\tau) &= g(\sigma)u_\sigma g(\tau)u_\tau u_{\sigma\tau}^{-1} g(\sigma\tau)^{-1} = \\ &= g(\sigma)u_\sigma g(\tau)u_\sigma^{-1}u_\sigma u_\tau u_{\sigma\tau}^{-1} g(\sigma\tau)^{-1} = \\ &= g(\sigma)\sigma(g(\tau))h(\sigma,\tau)g(\sigma\tau)^{-1} = \\ &= g(\sigma)\sigma(g(\tau))g(\sigma\tau)^{-1}h(\sigma,\tau) \end{aligned}$$

and therefore h and h' are cohomologous.

Consequently, if a groupoid has a torsor definable over K then the associated cocycle is cohomologous to zero. Indeed, in the latter case, as all the maps u_σ of Step 1 are automorphisms of the K-definable torsor P, and $u_\sigma = \sigma$, therefore the corresponding cocycle is the zero cocycle.

Step 5: show that cocycles corresponding to Morita equivalent groupoids are the same.

Let $Y_\bullet$ have a torsor Q and let $f : X_\bullet \to Y_\bullet$ be a Morita morphism. Then by Lemma 2.11 $P = X_0 \times_{f_0, Y_0, a} Q$ is a $X_\bullet$-torsor. Let $u_\sigma : Q \to Q_\sigma$ be a collection of isomorphisms of torsors, then

$$v_\sigma(x, q) = (x, u_\sigma(q))$$

is a collection of isomorphisms of Galois conjugates of Q. It is easily checked that v_σ gives rise to exactly the same cocycle.

Similarly, if $X_\bullet$ has a torsor P and $f : X_\bullet \to Y_\bullet$ is a morphism of torsors then $Q = P/\sim$ is a $Y_\bullet$-torsor where $\sim$ is the equivalence relation defined in Lemma 2.12. The morphisms u_σ are determined by the choice of a point $x \in P$ and they descend to morphisms v_σ between Galois conjugates of Q determined by the equivalence class $x/\sim \in Q$. One checks that again v_σ give rise to the same cocycle as u_σ.

Step 6: show that the correspondence is a bijection.

Let P be a torsor of a groupoid $X_\bullet$. If P is definable over a set of parameters L such that G_L is normal in G_K then the cocycle h representing an element in

$H^2(G_K, A(\overline{K}))$ constructed in Step 1 is the inflation of a cocycle η representing an element in $H^2(\mathrm{Gal}(L/K), A(L))$. Let G be the definable group extension of $\mathrm{Gal}(L/K)$ by A corresponding to η. Let us show that the group action groupoid for G acting on a $\mathrm{Gal}(L/K)$-torsor via projection $G \to \mathrm{Gal}(L/K)$, call it $Y_\bullet$, is Morita equivalent to $X_\bullet$.

Let $Q = \bigsqcup_{\sigma \in \mathrm{Gal}(L/K)} P_\sigma$ and let B be the $\mathrm{Gal}(L/K)$-orbit of the tuple of elements (defined over L) used to define P. Then, by Galois invariance, there is a projection $p : Q \to B$ defined over K. Let $Z_\bullet$ be the groupoids such that $Z_0 = X_0 \sqcup Y_0$ and $Z_1 = X_1 \sqcup Y_1 \sqcup Q \times Q'$ where Q' is a copy of Q. The composition of arrows in X_1 and in Y_1 is defined as in groupoids $X_\bullet$ and $Y_\bullet$, the elements of X_1 compose with elements of Q as follows:

$$f \cdot \sigma(x) = \sigma(f \cdot x)$$

the elements of Q compose with elements of Y_1 as follows

$$x \cdot g = u_{p(g)}(x)$$

and similarly for inverse arrows.

Now by Lemma 2.17 $X_\bullet$ and $Y_\bullet$ are Morita equivalent.

Let h be a cocycle representing an element of $H^2(G_K, A(\overline{K}))$ that is an image of an element of $\eta \in H^2(\mathrm{Gal}(L/K), A(L))$, let $X_\bullet$ be the action groupoid corresponding to the extension described by the element η. It is straightforward to check, by following the construction of the Step 1, that the cocycle that corresponds to $X_\bullet$ is η. □

Remark 3.4. *If* $\mathrm{acl}(K)$ *is a model then the theorem allows to classify all connected groupoids bounded by an Abelian group definable over* K.

Remark 3.5. *Note that as a part of the proof (Step 6) we have seen that a connected groupoid that has a torsor, defined over* $\mathrm{acl}(K)$*, is Morita equivalent to an action groupoid for an extension* G *of the group* $\mathrm{Gal}(L/K)$ *for some* $L \supset K$ *by an Abelian definable group* A*, acting on a* $\mathrm{Gal}(L/K)$*-torsor via the projection on* $\mathrm{Gal}(L/K)$.

Proposition 3.6. *Let* $f : A \to B$ *be a definable map of definable groups and let* $\alpha \in H^2(G_K, A(\overline{K}))$*. Then the generalised imaginary sort corresponding to* $f^*(\alpha) \in H^2(G_K, B(\overline{K}))$ *is interpretale in the generalised imaginary sort corresponding to* α.

Proof. Let $X_\bullet$ be the groupoid definable over K corresponding to the cocycle α with a torsor P and let $Y_\bullet$ be the groupoid corresponding to the cocycle $f^*(\alpha)$. We may assume that $X_\bullet$ and $Y_\bullet$ are group action torsors, so in particular the action of A on P is defined. Then $Y_\bullet$ has the torsor $Q = (P \times B)/A$ where the action of A is defined as

$$a \cdot (p, b) = (a^{-1} \cdot p, f(a)b)$$

One checks that Q is a $Y_\bullet$-torsor. □

4 Non-standard Zariski structures

In this section I will look at two series of examples of Zariski structures constructed by Hrushovski and Zilber, and examine their interpretability in various theries. It will turn out that intrepretability is closely related to eliminability of certain generalised imaginaries.

4.1 Group extensions

Let X be an algebraic variety defined over an algebraically closed field k. Let H be an abstract group that acts on $X(k)$ such that the stabilizer of any point is either G or the trivial subgroup.

Consider some extension $1 \to A \to G \to H \xrightarrow{\pi} 1$ with A finite. Let $\mathrm{Fix}(H)$ denote the set of fixed points of H. Each orbit of H on $X \setminus \mathrm{Fix}(H)$ is a principal homogeneous space. Pick a representative x_α in each orbit and let D be the set

$$\bigsqcup (G \cdot x_\alpha) \sqcup \mathrm{Fix}(H)$$

Let $p : D \to X$ be a map that maps elements of the form $g \cdot x_\alpha$ to $\pi(g) \cdot x_\alpha$ and is identity on $\mathrm{Fix}(H)$. The group G acts naturally on D. If one declares the pre-images of closed subsets of D^n under p and graphs of the action by elements of G closed, this defines a topology on D which satisfies the axioms of a Zariski geometry, as proved in [10, Proposition 10.1]. The Proposition also asserts that the isomorphism type of the structure D does not depend on the choices of representatives in the orbits of H. Let us denote this structure $D(X, G)$. Hrushovski and Zilber further prove (Theorem C) that there exists a variety X and a group extension such that $D(X, G)$ is not interpretable in an algebraically closed field. More examples of structures of the form $D(X, G)$ were considered in [23].

If one assumes that H, G are finite (and hence definable) it is easy to see that construction is equivalent to adding a generalised imaginary sort for an action groupoid. Indeed, define $X_\bullet$ to be the action groupoid for the lifting of the action of H on X to G. Then D is a $X_\bullet$-torsor, with anchor map p:

$$\begin{array}{ccccc} X_1 = & X \times G & & D \\ & s \Big\downarrow \Big\downarrow t & \overset{p}{\swarrow} & \Big\downarrow q \circ p \\ X_0 = & X & & X/H \end{array}$$

where $q : X \to X/H$ is the natural projection map. The action of $X_\bullet$ is given by the action of G on D.

Proposition 4.1. *Let G be finite. The structure $D(X, G)$ is interpretable in an algebraically closed field if and only if $X_\bullet$ is eliminable.*

Proof. The left to right direction tautologically follows from the definition of the structure.

Let us prove the right to left direction. As follows from trichotomy for Zariski geometries, $D(X, G)$ interprets an algebraically closed field $k \subset D^n$, and moreover, there exists a definable embedding $X \to k^m$ (for 1-dimensional X, see [10, Sections 6 and 9], for X arbitrary, see [24, Section 4.4]). By a theorem of Poizat ([18, Theorem 4.15]), if $D(X, G)$ is interpretable in an algebraically closed field F then k is definably isomorphic to F. Therefore, one may assume that D is a definable set in k and the projection p is definable. But that means precisely that $X_\bullet$ is eliminable. □

Corollary 4.2. *If X is a curve then any structure $D(X, G)$ with G finite is interpretable in an algebraically closed field.*

Proof. As follows from Tsen's theorem, $k(X)$ is quasi-algebraic and Galois cohomology of quasi-algebraic fields with coeffecients in torsion modules vanishes beyond degree 1 ([19],II.§3.2), so $H^2(k(X), \mu_n) = 0$. It follows that a restriction of $X_\bullet$ to some subset $Y \subset X$ is eliminable. The restriction of $X_\bullet$ to $X \setminus Y$ is a groupoid with finitely many objects and hence is eliminable, perhaps after adding some parameters from $\mathrm{acl}(\emptyset)$. □

4.2 Quantum Zariski geometries

The paper [22] considers a large class of Zariski geometries that are constructed from a certain piece of data that (as will be shown below) defines an Azumaya algebra over a variety. We now recall the definition of a quantum Zariski geometry to fix notation (Section 2, [22]).

Definition 4.3 (Data for quantum Zariski geometry)**.** *Let k be an algebraically closed field, and A be an associative unital finitely. The input data for a quantum Zariski geometry is*

1. *an associative algebra A finitely generated over its center $\mathrm{Z}(A)$, and the center is a commutative algebra of finite type over k which is the coordinate ring of a variety X;*

2. *a collection of irreducible modules M_x, all of fixed dimension n over k, where x ranges in the maximal spectrum of $\mathrm{Z}(A)$, such that M_x is annihilated by x;*

3. *a choice of generators of A, $\mathbf{u}_1, \ldots, \mathbf{u}_d$, a choice of bases $e_i^\alpha(x) \in M_x, 1 \le d \le n$ (called* canonical bases*), where α ranges in some finite set B, and a system of polynomials $\{\ f_l(t, x)\ \}, t = \{t_{ij}^k\}, 1 \le i, j \le n, 1 \le k \le d, x \in X$ such that for any $x \in X$ and any t_{ij}^k that satisfy the equations $f_l(t, x) = 0$ there is $\alpha \in B$*

such that $\mathbf{u}_i$ *have the form*

$$\mathbf{u}_k\, e^\alpha_j(x) = \sum_{i=1}^n \, t^k_{ij}\, e^\alpha_i(x)$$

4. *a finite group* Γ *and a partial map* $g : X\times\Gamma \to \mathrm{GL}_n(k)$ *such that for all* $\gamma\in\Gamma$*, the map* $g(-,\gamma)\to \mathrm{GL}_n(k)$ *is regular on some open subset of* X *for every* γ*, and for any* $x\in X$*,* $g(x,-)$ *is defined on a subgroup of* Γ *and is injective. For any* $\alpha,\beta\in B$*, there exists* $\lambda\in k^\times$ *and* $\gamma\in\Gamma$

$$e^\beta(x) = \lambda\sum_{j=1}^n g_{ij}(x,\gamma)e^\alpha(x)$$

The equations, the map g *and the defining relations of the algebra* A *can be defined over a subfield* $k_0\subset k$*.*

Remark 4.4. *Note that the clause 4 effectively says that* g *is a 1-cocycle of group cohomology of* Γ *with values in the projective group* $\mathrm{PGL}_n(k[U])$ *for some open* Γ*-invariant* $U\subset X$ *with trivial action of* Γ*. In fact, nothing prevents one from considering a cocycle with values in* $\mathrm{PGL}_n(k[Y])$ *where* Y *is the Galois cover of* U *with Galois group* Γ *defined by equations* $f(x,t)=0$ *from clause 3, so this is further allowed.*

Definition 4.5 (Quantum Zariski geometry)**.** *A* quantum Zariski geometry *associated to the data described in the previous definition is a structure with two sorts* V,k *and a projection map* $p : V\to X(k)$ *where*

1. k *has the structure of an algebraically closed field, and* X *is an affine variety with coordinate ring* $Z(A)$*, viewed as a definable subset in* k^m*;*

2. V *has a fibrewise structure of a* k*-vector space, i.e. the language on* V *has graphs of operations* $V\times V\to V$ *and* $k\times V\to V$ *that restrict to graphs of addition and multiplication by a scalar on every fibre* $V_x = p^{-1}(x), x\in X$*;*

3. *the structure contains graphs of maps* $\mathbf{u}_i : V\to V$ *that restrict to linear self-maps on fibres of* p*, and are defined as follows. For every* $x_0\in X$*, for every solution* t^k_{ij} *of the system of equations* $f_l(t,x_0)=0$ *there exists a basis in* V_{x_0} *such that* $\mathbf{u}_k$ *acts on* V_{x_0} *by the matrix* (t^k_{ij}) *in this basis.*

Every fibre V_x *is therefore isomorphic to* M_x *as an* A*-module.*

Zilber has shown in [22], Lemma 2.4, that the structure is unique up to isomorphism, once the base algebraically closed field is fixed.

I am going to show that the described structure is bi-interpretable with an algebraically closed field with an added generalised imaginary sort. In order to do that I will argue that a quantum Zariski geometry encodes an Azumaya algebra over the variety X.

Definition 4.6 (Azumaya algebra)**.** *Let A be an algebra over the ring of regular functions $k[U]$ of an affine variety U. The algebra A is called* twisted form of a matrix algebra *if for some morphism of varieties $Y \to X$ the base change $A \otimes_{k[U]} k[Y]$ is isomorphic to the matrix algebra $M_n(k[Y])$. An algebra A over $k[X]$ is called an* Azumaya algebra *if for any point $x \in X$ there exist an open affine neighbourhood U such that $A \otimes_{k[X]} k[U]$ is a twisted form of a matrix algebra. The rank of an Azumaya algebra is its rank as a module.*

From now on I adopt a simplifying assumption that the morphism $Y \to X$ is Galois and the function $g : \Gamma \times X \to GL_n$ is regular everywhere on $\Gamma \times X$. I will show that then the input data of a quantum Zariski geometry defines two objects: a twisted matrix algebra over X and an injective morphism of A into this algebra.

Indeed, the input data specifies a Galois cover $Y \to X$, where Y is the subvariety of $X \times \mathbb{A}^{n^2 \cdot d}$ defined by the equations $f_l(x,t) = 0$ (clause 3 of the Definition 4.3), and a 1-cocycle $g : \Gamma \to \mathrm{PGL}_n(k[Y])$ with values in the group of regular functions from Y to PGL_n (clause 4). Consider the matrix algebra $M_n(k[Y])$. The group Γ acts on $M_n(k[Y])$ by conjugation by $g(\gamma)$ for $\gamma \in \Gamma$. Let B be the $k[X]$-algebra $M_n(k[Y])^\Gamma$ of elements fixed by this action.

By Galois descent ([11], II.§5) there exists an isomorphism $\iota : B \otimes_{k[X]} k[Y] \to M_n(k[Y])$ such that for any $\gamma \in \Gamma$, $\iota^{-1} \circ \gamma \circ \iota = g(\gamma)$. As the equations $f_l(x,t) = 0$ define Y, the variables t^k_{ij} are regular functions on Y, and the matrices $(t^k_{ij}) \in M_n(k[Y])$ are part of the input data. Since these matrices are Γ-invariant, they descend to elements of B. This defines a map $\eta : A \to B$.

Note that for a closed point $x \in X$, $B \otimes k(x) \cong M_n(k(x))$ since $B \otimes k(x)$ is a central simple algebra over the residue field at a point $k(x)$ (Proposition IV.2.1, [13]) and $k(x)$ is algebraically closed, though the isomorphism is not canonical.

Proposition 4.7. *The map η is injective.*

Proof. It follows in a straightforward way from the definitions that the A-module given by compostion of η and the reduction map $B \to B \otimes k(x) \cong M_n(k)$ is isomorphic to M_x.

Since the annihilator of M_x is the maximal ideal $\mathfrak{m}_x \subset k[X]$, the kernel of η therefore is contained in the intersection of all maximal ideals of $k[X]$ which is zero since $k[X]$ is reduced. □

A classical fact from linear algebra (for proof see, for example, [12]) helps establish that η is surjective if the fibre modules are irreducible.

Proposition 4.8 (Burnside's theorem on matrix algebras)**.** *Let V be a vector space over a field k, and let $A \subset \mathrm{End}(V)$. Suppose that A does not have invariant subspaces. Then $A \cong M_n(V)$.*

Proposition 4.9. *If all modules M_x are irreducible then the image of the morphism ι from the statement of Proposition 4.7 coincides with B.*

Proof. By Burnside's theorem the maps $\eta \otimes k(x) : A \otimes k(x) \to B \otimes k(x)$ are isomorphisms for all $x \in X$. Therefore A is an Azumaya algebra of the same rank as B (see, for example, Proposition III.2.1 in [13]). Since f is an inclusion, it is an isomorphism. □

4.3 Definability and Brauer group

There is a natural groupoid that one can assosciate to a quantum Zariski geometry.

Definition 4.10 (Splitting groupoid)**.** *Let us keep notation for the data that defines a quantum Zariski geometry from Definition 4.5.*

The splitting groupoid $S_\bullet$ has the objects set $S_0 = Y$ and the morphisms set S_1 is $(Y \times_X Y) \times k^\times$ with the obvious source and target maps. The composition is defined as follows:

$$(y_0, \gamma \cdot y_0, a) \circ (\gamma \cdot y_0, \delta\gamma \cdot y_0, b) = (y_0, \delta\gamma y_0, g(\gamma)\gamma(g(\delta))(g(\gamma\delta))^{-1}ab)$$

Theorem 4.11. *The generalised imaginary sort corresponding to the splitting groupoid of a quantum Zariski geometry is interpretable in the quantum Zariski geometry. A quantum Zariski geometry is definable in an algebraically closed field expanded with the generalised imaginary sort. In particular, if the splitting groupoid is eliminable, then the quantum Zariski geometry is interpretable in the pure algebraically closed field.*

Proof. Suppose $V \to X$ is a quantum Zariski geometry. Let P be the set of all canonical bases in all fibres V_x up to multiplication, it is a definable set and an $S_\bullet$-torsor. Indeed, let Σ be the set of definable maps $h : V \times_X Y \to Y \times \mathbb{A}^n(k)$ such that $\mathbf{u}_1, \ldots, \mathbf{u}_d$ under these maps correspond to operators that act by matrices (t^k_{ij}) that satisfy the equations $f_l(x,t) = 0$ from the clause 3 of the Definition 4.3. The set P is defined as the set

$$\{\ (v_1, \ldots, v_n) \in V^n \ \mid\ \exists h \in \Sigma\, \exists y \in Y : h(e_i) = (y, e_i)\ \forall i\ 1 \leq i \leq n\ \}$$

where $\{e_i\}_{i=1}^n$ is the standard basis in $Y \times \mathbb{A}^n(k)$ regarded as a family of vector spaces. The projection map is the restriction of the map $p : V \to X$ to P, and the anchor map $a : P \to Y$ is defined as

$$a(\overline{v}) = \{\ y \in Y \ \mid\ \exists h \in \Sigma : h(e_i) = (y, e_i)\ \forall i\ 1 \leq i \leq n\ \}$$

The action of $S_\bullet$ is definable in V^{eq}:

$$(y_0, \gamma \cdot y_0, \alpha) \cdot (v_1, \ldots, v_n) = (\alpha \cdot g(\gamma) \cdot v_1, \ldots, \alpha \cdot g(\gamma) \cdot v_n)$$

Conversely, let $P \to [S_\bullet] = X$ be a $S_\bullet$-torsor with the anchor map $a : P \to S_0$. Let $W = P \times k^n/\mathbb{G}_m$ where $\mathbb{G}_m$ acts by the formula $a \cdot (p, x) = (a \cdot p, a^{-1} \cdot x)$. Define the action of the groupoid $S_\bullet$ on W as follows:

$$(y_0, \gamma y_0, a) \cdot (p, x) = ((y_0, \gamma y_0, a) \cdot p, g(\gamma)^{-1} x)$$

In particular, Γ acts on W if one lets an element $\gamma \in \Gamma$ acts on an element (p, x) like the arrow $(a(p), \gamma \cdot a(p), 1)$. Let V be the quotient W/Γ. The matrices of $\mathbf{u}_k$ define endomorphisms of W which are Γ-invariant, and hence descend to V. □

Definition 4.12 (Brauer group)**.** *Two Azumaya algebras A, A' over X are called* Morita equivalent *if there exists a vector bundle $V \to X$ such that $A \otimes \mathrm{End}(V) \cong A' \otimes \mathrm{End}(V)$ where $\mathrm{End}(V)$ is the $k[X]$-algebra of endomorphisms of the vector bundle V. The Morita equivalence classes of Azumaya algebras over X form a group under the tensor operation called the* Brauer group of X*.*

Remark 4.13. *It is not essential in the definition of a quantum Zariski geometry that X be affine. One could propose a straightforward generalisation of the definition of a quantum Zariski geometry for an arbitrary variety using sheaves of $\mathcal{O}_X$-algebras. Using such definition it would be possible to realise an arbitrary Azumaya algebra in a quantum Zariski geometry (in the sense of Propostion 4.9).*

Proposition 4.14 (Theorem 2.5, [13])**.** *Let X be a variety. Then $\mathrm{Br}(X)$ injects canonically into $H^2(X_{\text{ét}}, \mathbb{G}_m)$.*

Proposition 4.15 (Proposition 2.7, [13])**.** *An equivalence class of an Azumaya algebra of rank n^2 specified by a 1-cocycle $\alpha \in \check{H}^1(X_{\text{ét}}, \mathbb{G}_m))$ is $\delta(\alpha)$ where δ is the connecting morphism in the following long exact sequence:*

$$\ldots \to \check{H}^1(X_{\text{ét}}, \mathrm{GL}_n) \to \check{H}^1(X_{\text{ét}}, \mathrm{PGL}_n) \xrightarrow{\delta} \check{H}^2(X_{\text{ét}}, \mathbb{G}_m) \to \ldots$$

Proposition 4.16. *Every n-torsion element of $H^2(X_{\text{ét}}, \mathbb{G}_m)$ is the image of an element of $H^2(X_{\text{ét}}, \mu_n)$.*

The fact that the image of the injection is the torsion part of $H^2(X_{\text{ét}}, \mathbb{G}_m)$ when X is the spectrum of a field is classical (in this case étale cohomology coincides with Galois cohomology). This fact is established for some other classes of varieties, fore example, when X is smooth variety over a field (Grothendieck, [7]), when X is affine (due to Gabber, unpublished, but see [3]).

Proposition 4.17. *The class of the splitting groupoid in $H^2(k(X), \mathbb{G}_m)$ coincides with the restriction of the class of the Azumaya algebra associated to a quantum Zariski geometry to the generic point.*

Proof. The class of the restriction of the splitting groupoid to the generic point of X lies in the image of the map $H^2(\mathrm{Gal}(k(Y)/k(X)), \mathbb{G}_m) \to H^2(k(X), \mathbb{G}_m)$. The map $g : \mathrm{Gal}(Y/X) \to \mathrm{PGL}_n(Y)$ of Definition 4.3 is a 1-cocycle that defines the gluing data for the Azumaya algebra. The 2-cocycle of the element of an Azumaya algebra associated to the quantum Zariski geometry is the coboundary of a the lifting of a 1-cocycle in $H^1(X, \mathrm{PGL}_n)$ to GL_n (see [13], Theorem 2.5). From the definition of splitting groupoid and the proof of Theorem 3.3 it follows that the 2-cocycle corresponding to it is $(\sigma, \tau) \mapsto g(\sigma)\sigma(g(\tau))(g(\sigma\tau)^{-1})$. One checks that this coincides with the definition of the coboundary. □

For a smooth variety X, the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(k(X))$ is an injection (Auslander and Goldman, [1]), so one can check the eliminability of the splitting groupoid at the generic point.

Remark 4.18. *In view of Propositions 4.16 and 3.6, in order to eliminate the splitting groupoid of a quantum Zariski geometry it suffices to add a generalised imaginary sort for a subgroupoid of the splitting groupoid bounded by μ_n.*

4.4 Quantum Zariski geometries: an example

As an application of the main result of last section I am going to show that the quantum Zariski geometry corresponding to the quantum torus algebra (Example 2.1, [22]) is not interpretable in an algebraically closed field.

The input data is as follows. The quantum torus algebra is the algebra

$$A = k\langle \mathbf{u}, \mathbf{v}, \mathbf{u}^{-1}, \mathbf{v}^{-1} \mid \mathbf{uv} = q\mathbf{vu}, \mathbf{uu}^{-1} = \mathbf{u}^{-1}\mathbf{u} = 1, \mathbf{vv}^{-1} = \mathbf{v}^{-1}\mathbf{v} = 1\rangle$$

where $q^n = 1$ for some integer $n > 1$. The cover $Y \to X$ is defined by the equations $\mu^n = x, \nu^n = y$, where x, y are the coorinate function on $X = \mathbb{G}_m \times \mathbb{G}_m$. The matrices of $\mathbf{u}, \mathbf{v}$ are

$$\mathbf{u} = \begin{pmatrix} \mu & 0 & \dots & 0 \\ 0 & q\mu & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & \dots & q^{n-1}\mu \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0 & 0 & \dots & 0 & \nu \\ \nu & 0 & \dots & 0 & 0 \\ 0 & \nu & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & & \dots & \nu & 0 \end{pmatrix}$$

and the matrices of $\mathbf{u}^{-1}, \mathbf{v}^{-1}$ are inverses of these. The Galois group of the cover is the product of two cyclic groups of order n, and the cocycle g is specified on generators α, β of cyclic factors of Γ by matrices

$$g(\alpha) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \qquad g(\beta) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \zeta & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \zeta^{n-1} \end{pmatrix}$$

Definition 4.19. *Let k be a field that contains n-th root of unity and such that n is invertible in $k^\times$. A* cyclic algebra of rank n *corresponding to elements $a, b \in k^\times$ is the algebra*

$$k\langle x, y \mid x^n = a, y^n = b, xy = \zeta yx\rangle$$

where ζ is a primitive n-the root of unity.

Proposition 4.20 (Gille and Szamuely [4], Corollary 4.7.4)**.** *The class of a cyclic algebra that corresponds to an element $b \in k^\times$ is trivial in $Br(k)$ if and only if $b \in N_{K/k}(K^\times)$ where $K = k(\sqrt[n]{a})$ and $N_{K/k}(x) = \prod_{\sigma \in \mathrm{Gal}(K/k)} \sigma(x)$.*

The restriction of the Azumaya algebra of the quantum torus Zariski structure to the generic point is clearly a cyclic algebra corresponding to coordinate functions x, y of X. In order to check if this algebra is split one has to verify if $y \in N_{K(\mu)/K}(K(\mu)$ where $K = k(X)$ which is clearly false as for $f \in K(\mu)$ $N_{K(\mu)/K}(f(\mu))$ only has terms with coefficients that contain y to the power that divides $|\,\mathrm{Gal}(Y/X)|$.

Let us now illustrate the point made in Remark 4.18. The cohomology class $h \in H^2(X, \mathbb{G}_m)$ that corresponds to the quantum torus structure is the image of a cocycle in $H^2(X, \mathbb{G}_m \times \mathbb{G}_m)$ which corresponds to the central extension of the Galois group of the cover $Y \to X, (\mu, \nu) \mapsto (\mu^n, \nu^n)$ by μ_n:

$$1 \to \mu_n \to G \to \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \to 1$$

where $G = \langle u, v \mid u^n = v^n = 1, [u, [u, v]] = [v, [u, v]] = 1, [u, v]^n = 1 \rangle$. Let us interpret the quantum Zariski geometry in the group extension Zariski geometry M associated to this short exact sequence.

Following the construction of Theorem 4.11, let W be the quotient $M \times k^n/\mu_n$ and let V be the quotient of W by an action of G (which is the same as the action of an action groupoid) defined as follows. The action of G on M is already defined, so we have to define the action on k^n depending the point of y . Let the action of an element $\gamma \in G$ be given by $g(\gamma/\mu_n)$. Define the action of $\mathbf{u}, \mathbf{v}$ to be the action of the generators u, v of the group G on V. This defines an interpretation of the quantum torus geometry in M.

4.5 Eliminability of generalised imaginaries in CCM and RCF

We have seen that generalised imaginaries serve as an obstruction to intrerpetability of quantum Zariski geometries in the theory of algebraically closed fields. One easily observes that groupoids definable in the projective sort of the compact complex manifolds structure are not any more eliminable in this structure than they are in algebraically closed fields.

Proposition 4.21. *Let $X_\bullet$ be a groupoid definable in the field $\mathbb{C}$. Consider the same groupoid as a definable groupoid in the projective sort of the compact complex spaces structure, call it $Y_\bullet$. Then $X_\bullet$ is eliminable if and only if $Y_\bullet$ is eliminable.*

Proof. By Proposition 2.18 any groupoid is Morita equivalent to a groupoid $X_\bullet$ such that $X_0 \to [X_\bullet]$ has finite fibres. Their eliminability depends on the Galois groups of generic points of definable subsets of X_0 (Theorem 3.3).

The statement of the proposition follows from purity of the field $\mathbb{C}$ intepreted in the projective sort in the CCM. As a consequence all Galois groups of sets of parameters in $\mathbb{C} \subset \mathbb{P}^1$ coincide with that of algebraically closed field, and eliminability does not change. □

On the other hand, ACF-definable groupoids are always eliminable in RCF.

Proposition 4.22. *Let $X_\bullet$ be a groupoid bounded by an Abelian group definable in an algebraically closed field k over a set of parameters K and let $X_\bullet$ be definable in R in such a way that the structure maps are definable in the field k identified with $R+\sqrt{-1}R$. Then $X_\bullet$ is eliminable in R (over the same set of parameters K).*

Proof. Let $p : X_0 \to [X_\bullet]$ be the projection on the definable set of connected components. Since p has a section j definable in R there exists a definable $X_\bullet$-torsor over $[X_\bullet]$: $Q = \cup_{x \in X_0, y \in \mathrm{Im}(j)} \mathrm{Mor}(x,y)$ with the action of X_1 given by composition of arrows. □

In particular, quantum Zariski geometries are interpetable in $\mathbb{R}$, and they are not interetabile in the structure of compact complex spaces any more than they are interpretable in an algebraically closed field.

A Group cohomology

In this appendix the necessary facts about group cohomology are recalled, for detailed exposition see [19, 14].

Definition A.1 (Group cohomology)**.** *Let G be a profinite group. A G-*group *A is a disceret group that is endowed with a continuous action of G, i.e. a continuous homomorphism $G \to \mathrm{Aut}(A)$. If A is Abelian then A is called a G-*module.

Let A be a G-module. The group cohomology functors $H^i(G,A)$ are the derived functors of the functor $(-)^G$ that takes a G-module A to its submodule of G-invariant elements: $A^G = \{a \in A \mid \forall g \in G\ ga = a\}$.

Any G-module has a standard acyclic resolution that gives rise to the *homogeneous* cochain complex that computes the cohomology. This complex is quasi-isomorphic to the *inhomogeneous* cochain complex. Computing its cohomology of the latter complex the following elementary definition of the functors $H^i(G,-)$.

Definition A.2. *Let A be a G-module. The n-th term of the inhomogeneous cochain complex $C^n(G,A)$ is defined to be the set of all continuous maps $G^n \to A$ with $C^n(G,A)$ supposed formally to be 0 for $n < 0$ and $C^0 = A$. The differential $d_n : C^n(G,A) \to C^{n+1}(G,A)$ is defined as follows*

$$\begin{aligned}(dh)(\sigma_0,\dots,\sigma_n) &= \sigma_1 \cdot h(\sigma_1,\dots,\sigma_n) + \\ &\quad + \sum_{i=1}^{n} h(\sigma_0,\dots,\sigma_{i-1}\sigma_i,\dots,\sigma_n) + (-1)^n h(\sigma_0,\dots,\sigma_{n-1})\end{aligned}$$

The n-cohomology group of the complex, $\mathrm{Ker}\, d_n / \mathrm{Im}\, d_{n-1}$ *is called the n-th cohomology group of G with coefficients in the module A, and is denoted $H^n(G,A)$. In particular, the 1-cocycles are maps $h : G \to A$ such that*

$$h(\sigma\tau) = h(\sigma) + \sigma \cdot h(\tau)$$

modulo the equivalence relation: $h \sim h'$ if and only if there exists $g \in A$ such that $h = \sigma(g) + h' - g$. This definition also makes sense when A is non-Abelian, though H^1 has no group structure and is just a set with a distinguished element of cocycles cohomolougous to the zero cochain.

The second cohomology group is the set of maps $G^2 \to A$ such that

$$h(\alpha\sigma, \tau) = h(\alpha, \sigma\tau) - h(\alpha, \sigma) + \alpha \cdot h(\sigma, \tau)$$

and two 2-cocycles h, h' are cohomologous if there exists a map $g : G \to A$ such that

$$h(\sigma, \tau) = h'(\sigma, \tau) + g(\sigma) - g(\sigma\tau) + \sigma \cdot g(\tau)$$

The cohomology of profinite group is related to the cohomology of element of the inverse systems in the expected way.

Proposition A.3 ([19], Chapter I,Corollary 2.2)**.** *Let G be a profinite group and let A be a G-module. Then*

$$H^i(G, A) = \varprojlim_{U \subset G} H^i(G/U, A^U)$$

where the limit is taken over all closed subgroups $U \subset G$.

The low degree cohomology groups have natural geometric and group-theoretic interpretations.

Let A be an Abelian algebraic group. Then the set of A-torsors over K which have a point in an extension L forms an Abelian group called *Weil-Châtelet* group, which is isomorphic to $H^1(\mathrm{Gal}(L/K), A)$. The group operation is defined as follows. Let P, Q be two A-torsors. Then the result of the group operation is the quotient of $P \times Q$ by the action of A: $a \cdot (p, q) = (a \cdot p, a^{-1} \cdot q)$ (note the similarity with the Baer sum of group extensions or groupoids). If A is 0-dimensional then the existence of the quotient is straightforward, while in general it takes some work to construct it (which involves Weil's theorem on birational group laws, see [20] for details).

Theorem A.4 (Kummer theory)**.** *Let K be a field that contains n-th roots of unity with $n \not| \operatorname{char} K$. Then $H^1(K, \mu_n) = K^\times/(K^\times)^n$.*

Let A be an Abelian group and let $1 \to A \to G \xrightarrow{p} H \to 1$ be a group extension. Regard A as an H-module: an element $b \in H$ acts on a by conjugation, $b \cdot a = j(b) \cdot a \cdot j(b)^{-1}$, for some section j of p, the action is independent of the choice of section since A is normal. Now fix a section and associate to it a cocycle:

$$f(\sigma, \tau) = j(\sigma) \cdot j(\tau) \cdot j(\sigma\tau)^{-1}$$

Conversely, given a cocycle f one defines a group law on $A \times H$

$$(a, \sigma) \cdot (b, \tau) = (a + \sigma \cdot b + f(\sigma, \tau), \sigma\tau)$$

Theorem A.5 (Neukirch et al. [14], Theorem 1.2.5)**.** *There is a bijective correspondence between elements of $H^2(H, A)$ and extensions of H by A such that A has the prescribed H-module structure.*

Note that if A is a definable group and H is finite, the group extension is also a definable group.

Proposition A.6. *Let A be an Abelian group and let $1 \to A \to G \xrightarrow{p} H$ be a split group extension. Then the set of all sections of $p : G \to H$ modulo conjugation by elements of H is a torsor under $H^1(H, A)$: given a section j put $(h \cdot j)(a) = j(a)(h(ab))^{-1}$.*

Indeed, the cocycle condition tells us how $h(a)$ commutes with $j(a)$ and

$$j(ab)h(ab) = j(a)j(b)(j(b))^{-1}h(a)j(b)h(b) = j(a)h(a)j(b)h(b)$$

so $(h \cdot j)$ is a homomorphism; similarly, one checks that acting by coboundaries conjugates a section by an element of H.

References

[1] M. Auslander and O. Goldman. The Brauer group of a commutative ring. *Transactions of the American Mathematical Society*, 97(3):367–409, 1960. URL `http://www.jstor.org/stable/1993378`.

[2] K. Behrend and P. Xu. Differentiable Stacks and Gerbes. *arXiv preprint*, 2006. arXiv:math/0605694.

[3] A. de Jongh. A result of Gabber. `http://www.math.columbia.edu/~dejong/papers/2-gabber.pdf`, 2004.

[4] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2006.

[5] J. Giraud. Cohomologie non abélienne. *C. R. Acad. Sci. Paris*, 260, 1965.

[6] J. Goodrick and A. Kolesnikov. Groupoids, covers, and 3-uniqueness in stable theories. *Journal of Symbolic Logic*, 75(3):905–929, 2010.

[7] A. Grothendieck. Le groupe de Brauer I-III. In *Dix exposés sur la cohomolgie des schémas*. North-Holland, Amsterdam, 1968.

[8] E. Hrushovski. A new strongly minimal set. *Annals of Pure and Applied Logic*, 62(2):147 – 166, 1993.

[9] E. Hrushovski. Groupoids, imaginaries and internal covers. *arXiv preprint*, 2006. arXiv:math/0603413.

[10] E. Hrushovski and B. Zilber. Zariski geometries. *J. Amer. Math. Soc.*, 9(1): 1–56, 1996.

[11] M.-A. Knus and M. Ojanguren. *Thérie de la descente et algèbres d'Azumaya.* Springer-Velag, 1974.

[12] V. Lomonosov and P. Rosenthal. The simplest proof of Burnside's theorem on matrix algebras. *Linear Algebra and its Applications*, 383:45–48, 2004.

[13] J. Milne. *Étale cohomology.* Princeton University Press, 1980.

[14] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of Number Fields.* Grundlehren der mathematischen Wissenschaften. Springer, 2008.

[15] A. Pillay. Remarks on Galois cohomology and definability. *Journal of Symbolic Logic*, 62(2):487–492, 1997.

[16] A. Pillay. Algebraically closed fields and model theory. In E. Bouscaren, editor, *Model theory and algebraic geometry.* Springer-Verlag, 2002.

[17] B. Poizat. Une théorie de Galois imaginaire. *Journal of Symbolic Logic*, 48(4), 1983.

[18] B. Poizat. *Stable groups.* Number 87 in Mathematical Surveys and Monographs. American Mathematical Society, 2001.

[19] J.-P. Serre. *Galois cohomology.* Springer-Verlag, 1964.

[20] A. Weil. On algebraic groups and homogeneous spaces. *American Journal of Mathematics*, pages 493–512, 1955.

[21] B. Zilber. Model theory and algebraic geometry. In M. Weese and H. Walter, editors, *Proceedings of the 10th Easter conference on model theory.* Humboldt universität, Berlin, 1993.

[22] B. Zilber. A class of quantum Zariski geometries. In Z. Chatzidakis, H. Macpherson, A. Pillay, and A. Wilkie, editors, *Model Theory with applications to algebra and analysis, I and II.* Cambridge University Press, 2008.

[23] B. Zilber. Non-commutative Zariski geometries and their classical limit. *Confluentes Mathematici*, 2(2):265–291, 2010.

[24] B. Zilber. *Zariski Geometries: Geometry from the Logician's Point of View.* London Mathematical Society Lecture Note Series. Cambridge University Press, 2010.

[25] B. Zilber, V. Solanki, and D. Sustretov. Quantum harmonic oscillator as a Zariski geometry. *arXiv preprint*, 2011. arXiv:0909.4415.

[26] B. I. Zil'ber. The structure of models of uncountably categorical theories. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 359–368, Warsaw, 1984. PWN.

Einstein Institute of Mathematics
Edmond J. Safra Campus
The Hebrew University of Jerusalem
Givat Ram. Jerusalem, 91904, Israel
sustretov@ma.huji.ac.il