Reducts of algebraic curves

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Abstract

We show that a non-locally modular reduct of the Zariski structure of an algebraic curve interprets a field. This answers a question of Zilber's.

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1 Introduction

Zilber's trichotomy principle (to be described in more detail below) has an unsual status in mathematics. Conjectured in various forms by Zilber throughout the late 1970s, essentially every aspect of the conjecture was refuted by Hrushovski [21], [20] in the late 1980s. Due to Hrushovski's zoo of counterexamples the conjecture has never been reformulated. Yet, Zilber's principle remains a central and powerful theme in model theory: it has been proved to hold in many natural examples such as differentially closed fields of characteristic 0, algebraically closed fields with a (generic) automorphism, o-minimal theories and more (see [7, 34, 31, 8, 25]). Many of these special cases of Zilber's trichotomy had striking applications in algebra and geometry ([22, 23, 37]).

A relatively recent application of one such result is Zilber's model theoretic proof [41] of a significant strengthening of a theorem of Bogomolov, Korotiaev, and Tschinkel ([3]). The model theoretic heart of Zilber's proof is Rabinovich' trichotomy theorem for *reducts* of algebraically closed fields [36]. In the concluding paragraph of the introduction to [41] Zilber writes: "It is therefore natural to aim for a new proof of Rabinovich' theorem, or even a full proof of the Restrited Trichotomy along the lines of the classification theorem of Hrushovski and Zilber [25], or by other modern methods [...]. This is a challenge for the model-theoretic community."

The conjecture referred to in Zilber's text above can be formulated as follows:

Conjecture A. *A Let (M, Ω_0) be a non-locally modular strongly minimal reduct of an algebraic curve M over an algebraically closed field K. Then, there exist definable $L, E \in \Omega_0$ such that $E \subseteq L \times L$ is an equivalence realtion with finite equivalence classes and L/E with the induced structure from (M, Ω_0) is a field K-definably isomorphic to K.*

Rabinovich [36] proved Conjecture A in the special case where $M = \mathbb{A}^1$, and her result can be extend by general principles to any rational curve. In the present paper we prove Conjecture A. Our proof (Theorem 6.13), despite Zilber's expectations as quoted above, is geometric in nature and does not use any advanced model theoretic machinery. Roughly, it proceeds in four main steps:

1. Given a reduct (M, Ω_0) of the full Zariski structure on an algebraic curve M, the non-modularity assumption a definable 2-dimensional (almost) faithful family $X \subseteq M^2 \times T$ of curves in M^2. Throughout the text we make the assumption that for almost all $t \in T$ the curve S_t is pure-dimensional in the sense of K (i.e., it does not have 0-dimensional irreducible components). This assuption considerably simplifies the discussion, and is justified in Section 6.

2. We introduce the notion of the slope of a curve $C \subseteq M^2$ at a point $P \in C$, and use it to define when two curves $C_t, C_s \in X$ incident to P are tangent at that point. The main technical observation is that this geometric notion of tangency can be detected (up to an equivalence relation with finite classes)

definably in the reduct. That is, that there exists a definable set $T_0 \subseteq T \times T$ and an equivalence relation E with finite classes, such that for any $(t, s) \in T_0$ there are $t'Et$ and $s'Es$ such that $S_{t'}$ is tangent to $S_{s'}$ at P, Proposition 5.22.

3. This allows us, using our assumption of the first clause and – by now – standard model theoretic machinery, to reconstruct a 1-dimensional algebraic group in the reduct, Subsection 5.6.

4. The above reduces us to proving Conjecture A in the context where (M, Ω_0) is a non-locally modular expansion of a 1-dimensional algebraic group, where the group operation is definable. This problem was studied by Marker and Pillay in the context of the additive group in characteristic 0, [28]. In Subection 5.8 we show how to apply the tools developed in the previous sections to generalise the result of [28] to the present, fully general, context (Theorem 5.32).

The general scheme of our proof seem to have much in common with Rabinovich' original work, though we were unable to understand significant parts of her argument which are highly technical. We cannot, therefore positively identify the source of the greater generality of our result or its considerably lower level of combinatorial complexity. We believe that our more liberal application of algebro-geometric tools such as non-reduced schemes helped in simplifying the exposition, and possibly also in extending the scope of the results.

We point out one important source of difficulty in the present paper. As in Rabinovich' work — and in most of the works which followed it — the reconstruction of a group from a 1-dimensional family $S \subseteq M^2 \times T$ of curves incident to a point $(Q, Q) \in M^2$ is obtained as follows: pick $s, t \in T$ independent generics, find $r \in T$ such that the curve S_r is tangent to (in our termonology: "has the same slope as") the curve $S_s \circ S_r$ at (Q, Q), and prove that the mapping $(s, t) \mapsto r$ is (almost) definable and is (roughly) a group operation on T. This strategy can only work if the set of slopes of the family S at the point (Q, Q) is infinite. For a field K in characteristic 0 this is fairly easy, and follows from the uniqueness of solutions of ordinary differential equations for formal power series over K. This is, of course, not the case in characteristic $p > 0$ where the kernel of derivation is non-trivial. This calls for extra care in the choice of the family S, and we were unable to avoid having to work with high-order slopes, which is the source of some additional technicalities. This allows us to reconstruct a 1-dimensional algebraic group[2], which we then – using a case by case analysis – apply to construct a 2-dimensional family of plane curves with an infinite set of 1-slopes at every generic point of the diagonal.

It seems that the tools developed in the present paper can be generalised to prove Zilber's trichotomy in other contexts as well:

1. Prove the full Restricted Trichotomy Conjecture (i.e., remove the restriction in the present work that M is a curve). This will require extending the adaptation

[2]In a similar situation Rabinovich, [36, Section 8, p.93] seems to claim that she can actually recover an additive subgroup of $(K^2, +)$, given rise to a characteristic-independent argument.

to the present context of classical intersection theory of plane curves to higher dimensions.

2. There are good indications that our techniques translate, almost verbatim, to extend the main result of [26] to any characteristic and to any 1-dimensional algebraic group. On the face of it we see no immediate obstruction to proving the full Restricted Trichotomy Conjecture for algebraically closed valued fields using the present techniques (at least in the 1-dimensional case). A more challenging problem will be to use the fine intersection theory developed in the present paper to study unstable relics of algebraically closed valued fields.

3. Extend at least some of the tools and techniques to the context of strongly minimal o-minimal relics. In particular, it seems that the results of Section 6, allowing us to assume pure dimensionality of the curves in our family is an important technical simplification, that may be adapted to other topological structures.

4. There are good indications that, at least in characteristic 0, the results of Section 6 combined with dual results (on the uniform definability of the frontier of reduct-definable curves in M^2) developed in the o-minimal context may give a new proof of the main result of the present paper, using Zariski Geometries as a black box.

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2 Model theoretic background

For readers unfamiliar with the model theoretic jargon we give a self contained explanation of Conjecture A. In order to keep this introduction as short as possible, we specialise our definitions to the setting in which they will be applied. Readers familiar with the basics of model theory are advised to skip to Subsection 2.3.

2.1 Interpretations, Zilber trichotomy

A structure $\mathcal{M}$ is a set M, called the universe of the structure, together with a collection of Boolean algebras $\mathrm{Def}(\mathcal{M})$ of subsets of M^n (for all $n \in \mathbb{N}$), called definable sets, and satisfying the following requirements:

1. $\mathrm{Def}(\mathcal{M})$ is closed under all projections and permutations of coordinates;

2. All diagonals, i.e. sets of the form $\{ (x_1, \dots, x_n) \in M^n \mid x_i = x_j \}$, are in $\mathrm{Def}(\mathcal{M})$;

3. If $D \in M^{n+m}$ is definable and $a \in M^n$ then $\{x \in M^m : (a,x) \in D\}$ is definable.

Usually a structure $\mathcal{M}$ is given by specifying the universe M and a collection of distinguished subsets of powers of M, called *atomic* definable sets. The class $\mathrm{Def}(\mathcal{M})$ is then the class of sets generated by the atomic definable sets. A function $f : M^n \to M$ is definable if its graph is a definable set. For example, given a field F with the graphs of addition and multiplication as atomic sets, the additive and multiplicative inverses are definable and so is every constructible set. If F is algebraically closed then, by a theorem of Chevalley, these are the only definable sets.

A structure $\mathcal{N}$ is a *reduct* of a structure $\mathcal{M}$ if $N = M$ and $\mathrm{Def}(\mathcal{N}) \subseteq \mathrm{Def}(\mathcal{M})$, that is, if the two structures share the same universe and any set definable in $\mathcal{N}$ is already definable in $\mathcal{M}$. Given a structure $\mathcal{M}$ and a (definable) $D \subseteq M^n$ the *induced structure* on D is the structure $\mathcal{D}$, with universe D and whose atomic[3] definable sets are all sets of the form $D^k \cap S$ where $S \subseteq M^{nk}$ is definable in $\mathcal{M}$. The structure $\mathcal{M}$ is *interpretable* in an algebraically closed field K if it is the reduct of the structure induced from K on some constructible subset of K^n (some $n \in \mathbb{N}$).

Basic examples of reducts of an algebraically closed field K, are the trivial reduct, whose only atomic sets are the diagonals, and the reducts generated by the additive or multiplicative groups. More complicated examples consist of those structures generated, say, by the additive group and one non-linear polynomial. It is not too hard to show that in the first two of these examples the field K cannot be definably reconstructed. It is somewhat harder to show ([28]) that in the latter example, the field can be definably reconstructed. The problem of classifying those reducts – and, more generally, structures interpretable in algebraically closed fields – allowing a reconstruction of the field is the subject of Zilber's restricted Trichotomy conjecture.

To give a precise formulation of Zilber's trichotomy conjecture we need some more definitions. A saturated (see below) structure $\mathcal{M}$ is *strongly minimal* if all definable subsets of M are finite or co-finite. For example, it follows immediately from Chevalley's theorem that algebraically closed fields (of infinite transcendence degree) are strongly minimal. Clearly, if $\mathcal{M}$ is strongly minimal then so is every reduct of $\mathcal{M}$. However, given a reducible algebraic curve C and $\mathcal{D}$, a reduct of the induced structure on C, the resulting structure need not be strongly minimal.

From now $\mathcal{M}$ will denote a strongly minimal reduct of the full Zariski structure of an algebraic curve, M, over an algebraically closed field K. We will not assume that M is an irreducible curve.

In the present context, the *Morley rank* of a set $S \subseteq M^n$ definable in $\mathcal{M}$ can be identified with the dimension of its Zariski closure (as an affine variety) – see also Lemma 2.1 below. A definable set $S \subseteq M^n$ of Morely rank 1 is called a *curve*, and if $n = 2$ it is a *plane curve*. A *definable family of $\mathcal{M}$-definable subsets of M^n* is a collection $\{D_a : a \in S\}$ where $D \subseteq M^{m+n}$, $S \subseteq M^m$ are definable and for $a \in S$ we denote $D_a := \{x \in M^n : (a,x) \in D\}$. In the present text we will denote such a family as a function $f : D \to S$ with f onto S. It is well known (see, for

[3]In the context of the present paper, in fact, any definable set in the induced structure is of that form.

example, Lemma 6.2.20 in [27]) that if $\mathcal{M}$ is strongly minimal and $f : D \to S$ is a definable family of plane curves, then the equivalence relation on S given by $s \sim t \iff \#(D_s \triangle D_t) < \infty$ is definable. We say that the family $f : D \to S$ is *(almost) faithful* if $\sim$ is trivial on S (has only finite classes). If $f : D \to S$ is an almost faithful definable family of plane curves the dimension of the family is the dimension of S.

As a consequence of weak elimination of imaginaries in $\mathcal{M}$ (see, for example, Lemma 1.6 in [33]) for any $\mathcal{M}$-definable family $f : D \to S$ of plane curves there exists an almost faithful definable family $f' : D \to S'$ such that for every $s \in S$ there exists $s' \in S'$ such that $\#(D_s \triangle D'_{s'}) < \infty$. We say that $\mathcal{M}$ is locally modular if every almost faithful family of plane curves is at most 1-dimensional.

Clearly, algebraically closed fields are not locally modular as is witnessed by the definable family of affine lines $f : D \to K^2$ given by $f(x, y, a, b) = (a, b)$ where $D(x, y, a, b) := \{(x, y, a, b) : ax + b = y\}$. It follows that if K can be reconstructed in $\mathcal{M}$ then $\mathcal{M}$ is not locally modular. Thus, Zilber's conjecture, asserts that a field (necessarily isomorphic to K) can be reconstructed in the structure $\mathcal{M}$ if and only if $\mathcal{M}$ is not locally modular.

We remark that the model theoretic notion of a family of curves is looser than standard notions studied in algebraic geometry, e.g., no flatness conditions are imposed.

2.2 Generic parameters, imaginaries, canonical bases

We will now introduce more subtle model theoretic notions used in the text, referring to [27, 9, 6, 32] for a more thorough exposition.

The *language* (or *signature*) of a structure $\mathcal{M}$ specified by its atomic definable sets is a set of symbols (with prescribed arities), $\mathcal{L}$, and a function τ (the interpretation of $\mathcal{L}$ in M) from $\mathcal{L}$ onto the class of atomic definable sets. We say that $\mathcal{M}$ is an $\mathcal{L}$-structure.

Given an $\mathcal{L}$-structure $\mathcal{M}$, a key notion in model theory is that of a set being definable over a set of parameters $A \subseteq M$. The class of $\emptyset$-definable sets is the minimal subclass of $Def(M)$ that is closed under (fintite) boolean operations, projections and containing all diagonals. A set definable set $D \subseteq M^n$ is A-definable if there is a $\emptyset$-definable set $D' \subseteq M^{n+m}$ (some m) such that $D = \{x \in M : (x, \bar{a}) \in D'\}$ where $\bar{a}$ is a tuple from A (of length m).

An algebraically closed field K is *saturated* if it is of infinite transcendence degree over its prime field. Any structure interpretable in a saturated algebraically closed field is itself saturated. For saturated structures the notion of a definable set D being A-definable is equivalent to D being invariant (set-wise) under all automorphisms of $\mathcal{M}$ fixing A point-wise (where an automorphism of $\mathcal{M}$ is any bijection respecting all atomic sets).

A *(complete) type* over a set A is an (ultra)-filter of A-definable sets (i.e., a (maximal) collection of A-definable sets with the finite intersection property). Unless specifically stated otherwise, all types in the present paper will be complete. If $\mathcal{M}$

is saturated and $|A| < |M|$ and p is a type over A then, in fact, the intersection of all definable sets in p is non-empty. Throughout this paper all sets of parameters will be small, i.e., of cardinality smaller than that of M. An element $b \in \bigcap_{D \in p} D$ is *a realisation* of p. The *rank of a type* p is the minimum rank of a formula in p. If p is a type over A and q is a type over B we say that q extends p if $p \subseteq q$. We say that q is a non-forking extension over p if p and q have the same rank.

Given an element $b \in M$ and a set of parameters A *the type of* b *over* A is the collection of all A-definable sets D such that $b \in D$. This is denoted $\mathrm{tp}(b/A)$. In a saturated structure $\mathcal{M}$, for any set of parameters A any type p over A is the type of some element. Given an element b and parameters sets $A \subseteq B$ we say that b does not fork with B over A (or that b is independent from B over A) if the $\mathrm{tp}(b/B)$ is a non-forking extension of $\mathrm{tp}(b/A)$. In the context of algebraically closed fields this amounts to the locus of b over B being equal to the locus of b over A.

If $\mathcal{M}$ is saturated any type over A can be identified with the an orbit of the automorphism group of $\mathcal{M}$ fixing A point-wise. In this language the rank of a type is the Krull dimension of the closure (in K) of the orbit associated with that type.

Given an A-definable set D a type in D is any type p (over a parameter set $B \supseteq A$) such that $D \in p$. Since algebraic varieties have only finitely many irreducible components of maximal rank, and since $\mathcal{M}$ is a reduct of the full Zariski structure on M, it follows that for any definable set D there are finitely many types in D of maximal rank. Those are the *generic types in* D. A set D is stationary if it has a unique generic type (over any set of parameters B. E.g., if the closure of D is an absolutely irreducible variety). This implies that at the price of extending the set of parameters any definable set D can be definably split into finitely many (disjoint) stationary sets. We observe that for a stationary set D the generic type of D can be conveniently described as the type given by D and the negation of all A-definable subsets of D of rank smaller than the rank of D. We say that a type p over A is stationary if there is a definable set $D \in p$ of minimal rank which is stationary.

We will say that an $\mathcal{M}$-definable property holds for *almost all elements of* D if every generic element of D satisfies that property.

An element $b \in D$ is *generic in* D *(over* A*)* if it is a realisation of a generic type in D (equivalently, if its type over A is generic in D). Manipulations of realisations of generic types is a common technique in model theory (used non-trivially throughout Section 6, for example). This may seem unclear to non-specialists due to the fact that the term "generic point" does not refer to exactly the same notion as in the theory of schemes. For example, a generic point of an irreducible variety in the sense of scheme theory is unique, whereas two realisations x, y of the unique generic type of $\mathbb{A}^1$ may be independent so that the type of the tuple $(x, y) \in \mathbb{A}^2$ is the generic type of $\mathbb{A}^2$, but if they are dependent (x, y) will be a generic point of a subvariety of $\mathbb{A}^2$ projecting dominantly on both coordinates of the affine plane.

Further, a fibre X_a of a definable set $X \subset M^{m+n}$, where a is a realisation of the generic type of a variety $Y \subset M^m$, corresponds to the base change of X to the generic point of Y in the scheme-theoretic sense. The points of X_a in a saturated model are then generic points of subvarieties of X that project dominantly on Y.

Note that if D is $\mathcal{M}$-definable it is certainly K-definable, so it makes sense to talk about a K-generic element of D. Such an element is also generic in the sense of $\mathcal{M}$. Throughout this text, unless specifically stated otherwise, all generics will be taken with respect to the full Zariski structure.

A type p over a parameter set A is algebraic if it has rank 0. In the context of saturated structures this is equivalent to the orbit associated with p being finite (which is the same as the type having only finitely many realisations in $\mathcal{M}$). An element b is algebraic over A if $\mathrm{tp}(b/A)$ is algebraic. In the context of algebraically closed fields an element is algebraic over A in the model theoretic sense precisely if it is algebraic over the field generated by A in the usual algebraic sense. However, if $\mathcal{M}$ is a reduct of the full Zariski structure on M the algebraic closure in the sense of $\mathcal{M}$ will, in general, be smaller than in the algebraic sense. A set A is *algebraically closed* if any element algebraic over A is in A. In the context of algebraically closed fields a set A is algebraically closed precisely if it is an algebraically closed subfield.

The above description of the algebraic closure in a structure $\mathcal{M}$ is adequate in the context of algebraically closed fields, but is not quite strong enough for our purposes. This can be better explained by considering the action of the automorphism group of $\mathcal{M}$ fixing a parameter set A not only on elements of the universe, but also as definable sets. In the context of algebraically closed fields, if V is an affine variety (or, more generally, a constructible set) and A is an algebraically closed field containing the field of definition of V then either $\mathrm{Aut}(K/A)$ fixes V set-wise or it has an infinite orbit. If $\mathcal{M}$ is a reduct of the full Zariski structure on M this need not be the case.

To addres this problem recall that if D is an A-definable set then $D = D'(\bar{a})$ for some $\emptyset$-definable set D' and $\mathrm{dom}(\bar{a}) \subseteq A$. Let T be the projection of D' onto the coordinates corresponding to $\bar{a}$. Consider the (definable) equivalence relation on T given by $t \sim s$ if $D(s) = D(t)$ (as sets). It is clear that the orbit of D under $\mathrm{aut}(M/A)$ is in bijection with the quotient space $T/\sim$. In the context of algebraically closed fields $T/\sim$ can be naturally identified with a constructible set, and it follows that if $T/\sim$ is finite then the elements of this algebraic set are algebraic over A. We say that a structure has *elimination of imaginaries* if for any $\emptyset$-definable set T and any $\emptyset$-definable equivalence relation E on T there is a $\emptyset$-definable set S and a $\emptyset$-definable function $f : T \to S$ such that $f(x) = f(y)$ if and only if $E(x, y)$. The above discussion can be readily adapted to show that algebraically closed fields admit elimination of imaginaries.

In the more general context we are working in there is no reason to assume that $\mathcal{M}$ has elimination of imaginaries. There is a standard model theoretic technique for adding, to any structure $\mathcal{M}$, any $\emptyset$-definable set T and any $\emptyset$-definable equivalence relation E on T a new *sort* (which we identify with T/E) and a new function symbol $\pi_E : T \to T/E$ (which we identify with the natural projection map). The resulting structure is denoted $\mathcal{M}^{eq}$ and it is an easy exercise to verify that $\mathcal{M}^{eq}$ admits elimination of imaginaries. The elements of the new sorts are called *imaginary elements.* The new elements we added to $\mathcal{M}$ are referred to as *imaginary elements* and they should be thought of as canonical names for equivalence classes of $\emptyset$-definable equivalence relations. Throughout this text, when referring to the algebraic closure of a parameter set A we will be implicitly referring to the algebraic closure including imaginary elements (i.e., including names for definable equivalence relations with a finite number of classes). In the next subsection we will give a more concrete description of the treatment of imaginaries in the present paper.

One of the most striking and powerful applications of imaginary elements is that they allow us to treat definable sets as elements of our structure. That is, if $D = D'(\bar{a})$ is definable (with D' $\emptyset$-definable), T and $\sim$ are as in the previous paragraph, then D is identified with $[\bar{a}]/\sim$, and the latter is called *a code for* D, or *a canonical parameter for* D. Note that the canonical parameter of a definable set is not uniquely determined, but the rank of its type is, which will suffice for all our applications.

2.3 Remarks on the model-theoretic set up

Throughout the paper K is an algebraically closed field of infinite transcendence degree. We fix M, an affine algebraic curve over K and $\mathcal{M}$ a non-locally modular reduct of the full Zariski structure on M. The following lemma justifies the correspondence between Morley rank and Krull dimension introduced above:

Lemma 2.1. *For any $\mathcal{M}$-definable set $Z \subset M^n$ the Morley rank of Z in $\mathcal{M}$ coincides with the Krull dimension of the Zariski closure of Z.*

Proof. The claim is clear if $Z = M^k$. In general Z has Morley rank k if and only if k is maximal such that some projection $\pi : Z \to M^k$ contains an $\mathcal{M}$-generic point of M^k. So such a projection of Z contains a K-generic point, so the Morley rank is bounded from above by the Krull dimension of K. As the other inequality is obvious (K-generic points being obviously $\mathcal{M}$-generic), the lemma is proved. $\square$

We proceed with a few basic reductions and conventions that will simplify the exposition and the notation. First, there is no harm assuming that $\mathcal{M} = (M, X)$, where X is a predicate naming the total space of a 2-dimensional (almost) faithful family $X \to T$ of plane curves (in the sense of the structure $\mathcal{M}$). Indeed, since $\mathcal{M}$ is non-locally modular, there exists a 2-dimensional family of plane curves, X, and (M, X) is a reduct of $\mathcal{M}$. If a field is interpretable in (M, X) it is necessarily interpretable already in $\mathcal{M}$. Next we need the following easy observation:

Remark. Let D be a strongly minimal set definable in $\mathcal{M}$. Then D with the full induced structure is non-locally modular.

It is an easy exercise to verify that the notion of interpretablity is transitive, namely, that if a structure $\mathcal{N}$ is interpretable in $\mathcal{M}$ and $\mathcal{D}$ is interpretable in $\mathcal{N}$ then $\mathcal{D}$ is interpretable in $\mathcal{M}$. Thus, if D is a strongly minimal set definable in $\mathcal{M}$ and $\mathcal{D}$ is the $\mathcal{M}$-induced structure on D, in order to show that $\mathcal{M}$ interprets a field it will suffice to show that $\mathcal{D}$ interprets a field. Therefore, we may assume, e.g., that the curve M is regular (by removing its non-regular locus, which is finite, and in particular definable in any structure on M). Similarly, after showing that a 1-dimensonal group is interpretable in $\mathcal{M}$ we may assume – by replacing $\mathcal{M}$ with the universe of the group with its full induced structure – that $\mathcal{M}$ itself expands an algebraic group.

While algebraically closed fields have elimination of imaginaries there is no reason for the same to be true in $\mathcal{M}$. However, by the previous paragraph, any strongly minimal structure $\mathcal{D}$ interpretable (allowing imaginary elements) in $\mathcal{M}$ is already interpretable in K. Since K does have elimination of imaginaries, the structure $\mathcal{D}$ can be identified with the reduct of the full Zariski structure on some algebraic curve D over K. In particular, if $\mathcal{D}$ is non-locally modular then it falls in the scope of Conjecture A, justifying the reduction of the previous paragraph.

The above allows us to tacitly assume that the structure $\mathcal{M}$ has elimination of imaginaries. This is merely a matter of convenience. As explained in the previous paragraphs, using elimination of imaginaries for algebraically closed fields, and changing the ground structure as we go, we can avoid almost any usage of imaginaries. There is, however, one exception. We cannot assure the existence of a faithful 2-dimensional family of plane curves in $\mathcal{M}$ without allowing the parameter space T of the family to range over imaginary sorts. Though all proofs in the present work could go through essentially unaltered if X were an almost faithful family of plane curves, this could somewhat hamper the clarity of the exposition by adding simple, unnecessary, technicalities which we prefer to avoid.

3 Correspondences and slopes

The main object of study in the present section is that of a correspondence and its slopes (of various orders) at a regular point. After introducing these notions we study the behaviour of slopes under composition (subsection 3.17) and under taking sums of compositions in the presence of an underlying group structure (Subsection 3.2). Most of the algebro-geometric definitions in this (and the next) section are well known. We give them here specialised to the context in which they will be used.

Definition 3.1 (Correspondence)**.**

1. *If X and Y are schemes, a* correspondence, α, from X to Y *is a closed subscheme of $X \times Y$ whose projection on X is dominant. We refer to the closed subscheme itself as the* graph *of the correspondence, denoted $\Gamma(\alpha)$. A correspondence α is* finite-to-finite *if the projections p_X, p_Y restricted to $\Gamma(\alpha)$ are quasi-finite morphisms.*

2. *If X and Y are definable sets a* definable correspondence *from X to Y is any definable set $Z \subseteq X \times Y$ projecting generically onto X.*

3. *When we wish to emphasize that we regard the correspondence α as a multi-valued map from X to Y we write $\alpha : X \vdash Y$.*

In order to define the slope of a correspondence X at a regular point $P \in X$ we need the notion of a local coordinate system at P.

Definition 3.2 (Local coordinate system)**.** *Let P be a regular point of a variety X of dimension n. A* local coordinate system *at P is an isomorphism $\widehat{\mathcal{O}_{X,P}} \xrightarrow{\sim} k[[x_1, \ldots, x_n]]$.*

By Corollary to Theorem 30.5 [29] any variety over a field is generically regular, and by Theorem 29.7 *loc.cit.* a completion of a regular local ring over a field is a formal power series ring.

If X is one-dimensional and a local coordinate system is chosen at P then for all $n > 0$ the inclusion $\mathcal{O}_{X,x} \to \widehat{\mathcal{O}_{X,x}}$ followed by the reduction maps $k[[x]] \to k[[x]]/(x^{n+1}) \cong k[x]/(x^{n+1})$ gives rise to closed embeddings $\operatorname{Spec} k[x]/(x^{n+1}) \hookrightarrow M$ with the closed point of $\operatorname{Spec} k[x]/(x^{n+1})$ mapped to P.

If $i_1 : \widehat{\mathcal{O}_{X,P_1}} \xrightarrow{\sim} k[[x]], i_2 : \widehat{\mathcal{O}_{Y,P_2}} \xrightarrow{\sim} k[[y]]$ are local coordinate systems at regular points $P_1 \in X, P_2 \in Y$ then there is a natural local coordinate system at the point $(P_1, P_2) \in X \times Y$,

$$i_1 \otimes i_2 : \mathcal{O}_{X,\widehat{P_1 \otimes \mathcal{O}}_{Y,P_2}} \to \varprojlim (k[x]/(x^n) \otimes k[y]/(y^n)) \cong k[[x, y]].$$

Remark. From now on, whenever we mention that we choose a local coordinate system at $(P_1, P_2) \in M^2$ for M a curve, we do so by choosing local coordinate systems i_1, i_2 at $P_1, P_2 \in M^2$ and then passing to $i_1 \otimes i_2$.

Finally, let $f : X \to Y$ be a morphism of schemes. Recall ([14] EGA IV.17.1) that a morphism f is *formally étale* if for any scheme T, closed subscheme T' defined by a nilpotent ideal and any two compatible morphisms $\lambda : T' \to X$ and $\iota : T \to Y$ there is a unique morphism $\bar{\iota} : T \to X$ such that the following diagram commutes

$$\begin{tikzcd} T' \arrow[r, "\lambda"] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r, "\iota"] \arrow[ur, dashed, "\bar{\iota}"] & Y \end{tikzcd}$$

Observe that a closed embedding ι lifts to a closed embedding $\bar{\iota}$ and it follows automatically that f induces an isomorphism between two copies of T embedded into X and Y.

A morphism $f : X \to Y$ is *étale* if it is flat and unramified.

An étale morphism of schemes $f : X \to Y$ is formally étale (SGA1 [16], Corollaire I.5.6), and a formally étale morphism which is locally of finite presentation is étale ([14] EGA IV, Corollaire 17.6.2). Since in what follows we deal mainly with schemes of finite type over fields, we will not distinguish between these two notions.

Lemma 3.3. *Let $f : X \to Y$ be an étale morphism. Assume Y is the spectrum of a local Artinian algebra over a field. Then sections of f are in one-to-one correspondences with closed points in the fibre over the closed point of Y. For any such section s the composition $f \circ s$ is an isomorphism.*

Proof. By definition of formally étale morphism, given any morphism $\lambda : Y' \to X$ where Y' is a reduced subscheme supported at a closed point of X yields a unique section s that makes the following diagram commute:

$$\begin{tikzcd}
Y' \arrow[r, "\lambda"] \arrow[d] & X \arrow[d, "f"] \\
Y \arrow[r, "id"] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

That $s \circ f$ is an isomorphism follows from Corollary I.5.3, SGA1 [16]. $\square$

Definition 3.4 (Scheme-theoretic image)**.** *Let $f : X \to Y$ be a morphism of schemes. The* scheme theoretic image of X in Y *is the smallest closed subscheme of Y through which f factors.*

Lemma 3.5. *Let $\mathcal{O}_X$ be a local ring with residue field k and let $f : \mathcal{O}_{X,x} \to k[\varepsilon]/(\varepsilon^{n+1})$ be a morphism. Then f factors through $\mathcal{O}_{X,x}/\mathfrak{m}^{n+1}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,x}$.*

Proof. One observes easily that $f^{-1}(\mathfrak{p}) \subset \mathfrak{m}^{n+1}$ where $\mathfrak{p}$ is the radical ideal of $k[\varepsilon]/(\varepsilon^{n+1})$, since $\mathfrak{p}^{n+1} = 0$. This implies the statement of the lemma. $\square$

Let M be a curve, and assume a local coordinate system is chosen at a regular point $P = (P_1, P_2) \in M^2$. Let $Z \subseteq M \times M$ be a closed one-dimensional reduced subscheme such that the projection p_1 restricted to Z is étale in an open neighbourhood of P. Consider the diagram

$$\begin{tikzcd}
\mathcal{O}_{M,P_2} \arrow[r, "f_2"] \arrow[d] \arrow[ddr, "g"] & \mathcal{O}_{M,P_2}/\mathfrak{m}_2^{n+1} \arrow[r, equal, "\sim"] \arrow[dd, "\tau_n"] & k[\eta]/(\eta^{n+1}) \\
\mathcal{O}_{Z,P} \arrow[dr, "\gamma"] & & \\
\mathcal{O}_{M,P_1} \arrow[r, "f_1"] \arrow[u] & \mathcal{O}_{M,P_1}/\mathfrak{m}_1^{n+1} \arrow[r, equal, "\sim"] & k[\varepsilon]/(\varepsilon^{n+1})
\end{tikzcd}$$

where $\mathfrak{m}_1, \mathfrak{m}_2$ are maxmial ideals of $\mathcal{O}_{M,P_1}, \mathcal{O}_{M,P_2}$, respectively. The isomorphisms on the right are provided by the local coordinate systems at P_1 and P_2. The morphism γ is a lifting of f_1 that follows from etaleness of $\mathcal{O}_{Z,P}$ over $\mathcal{O}_{M,P_1}$. The morphism g is the composition of the structure morphism of the $\mathcal{O}_{M,P_2}$-algebra $\mathcal{O}_{Z,(P_1,P_2)}$ and γ. By Lemma 3.5, g factors through $\mathcal{O}_{M,P_2}/\mathfrak{m}^{n+1}$. The morphism τ_n that comes out of this factorization, can be regarded as an endomorphism of $k[\varepsilon]/(\varepsilon^{n+1})$ after one identifies $k[\eta]/(\eta^{n+1})$ with $k[\varepsilon]/(\varepsilon^{n+1})$.

Definition 3.6 (Slope)**.** *Let M be a curve, and let $Z \subset M \times M$ be a curve as above. The* n-th order slope of Z at (P_1, P_2) *is the endomorphism $\tau_n : k[\varepsilon]/(\varepsilon^{n+1}) \to k[\varepsilon]/(\varepsilon^{n+1})$ arising from the construction above. In general, we will denote the slope of Z at P, which is an element of* $\mathrm{End}(k[\varepsilon]/(\varepsilon^{n+1}))$*, as $\tau_n(Z,P)$.*

Similarly, in the above setting if N is an algebraic variety over k and $Z \subset M^2 \times N$ a curve, $P = (P_1, P_2, P_3) \in Z$ a regular point and such that the projection p_1 on the first factor M is étale in an open neighbourhood of (P_1, P_2, P_3), one can consider the diagram as above, and consider the morphism γ obtained as a lifting of f_1 that follows from etaleness of $\mathcal{O}_{Z,P}$ over $\mathcal{O}_{M,P_1}$. By Lemma 3.5, g factors through $\mathcal{O}_{M,P_2}/\mathfrak{m}^{n+1}$. The morphism τ_n that comes out of this factorization, can be regarded as an endomorphism of $k[\varepsilon]/(\varepsilon^{n+1})$ after one identifies $k[\eta]/(\eta^{n+1})$ with $k[\varepsilon]/(\varepsilon^{n+1})$.

Definition 3.7 (Relative slope)**.** *The* n-th order slope of Z at (P_1, P_2, P_3) relative to *N is the endomorphism $\tau_n : k[\varepsilon]/(\varepsilon^{n+1}) \to k[\varepsilon]/(\varepsilon^{n+1})$ arising from the construction above. We will denote the relative slope by $\tau_n(Z/N, P) \in \mathrm{End}(k[\varepsilon]/(\varepsilon^{n+1})$.*

The need for this seemingly artificial definition will become apparent in the next section.

In the above definition, and later in this article, we assume that a choice of local coordinate systems for M at the relevant points has been made.

A first-order slope is a map $k[\varepsilon]/(\varepsilon^2) \to k[\varepsilon]/(\varepsilon^2)$, which is determined by its action on ϵ, $\epsilon \mapsto a \cdot \epsilon$. We observe that the scalar a is just a component of the normalised Plücker coordinates of the tangent subspace in the given local coordinate system. Clearly, two curves having the same first order slope at a point are tangent at this point.

We will also need to consider projective limits of slopes of all orders, which are formal power series.

Definition 3.8 (Formal power series expansion)**.** *In the setting of Definition 3.6 consider the endomorphism*

$$A = \varprojlim_n \tau_n(Z, (P_1, P_2)) \in \varprojlim(\mathrm{End}(k[\varepsilon]/(\varepsilon^{n+1}) \cong \mathrm{End}(k[[x]])$$

We call $A(x)$ the formal power series expansion *of Z at (P_1, P_2).*

Proposition 3.9. *Assume local coordinate systems are chosen at (P_1, P_2), then there is a canonical isomorphism $\widehat{\mathcal{O}_{(P_1,P_2),M^2}} \to k[[x, y]]$. The formal power series expansion of X at (P_1, P_2) is $f \in k[[\varepsilon]]$ if and only if the morphism $\mathcal{O}_{(P_1,P_2),M} \to k[[\varepsilon]]$ given by $x \mapsto \varepsilon, y \mapsto f$ factors through $\mathcal{O}_{(P_1,P_2),X}$.*

Proof. Straightforward from the definitions. □

Later on we will needing the following charactersiation of invertible endomorphisms from $\mathrm{End}(k[\varepsilon]/(\varepsilon\ /n + 1)$:

Proposition 3.10. *Let* $\mathrm{Aut}(k[\varepsilon]/(\varepsilon^{n+1}))$ *denote the set of automorphisms of* $k[\varepsilon]/(\varepsilon^{n+1})$. *Consider the restriction map* $\mathrm{End}(k[\varepsilon]/(\varepsilon^{n+1})) \to \mathrm{End}(k[\varepsilon]/(\varepsilon^2))$ *defined by* $\varphi \mapsto (f \mapsto \varphi(f)/(x^2))$. *Then* $\mathrm{Aut}(k[\varepsilon]/(\varepsilon^{n+1}))$ *is the pre-image of* $\mathrm{Aut}(k[\varepsilon]/(\varepsilon^2))$.

Proof. An endomorphism $\varphi \in \mathrm{End}(k[\varepsilon]/(\varepsilon^{n+1}))$ is invertible if and only if there exists f such that $\varphi(f) = x$. By Corollary 7.17, [11], an endomorphism of $k[[x]]$ defined by sending x to f is an automorphism if and only if $f \in (x)$ but not $f \in (x^2)$. As any $\varphi \in \mathrm{End}(k[\varepsilon]/(\varepsilon^{n+1}))$ extends uniquely to $\mathrm{End}(k[[x]])$, the conclusion follows. $\square$

Definition 3.11 (Graph of a morphism)**.** *Let $f : X \to Y$ be a morphism of schemes over some base scheme S. The* graph *of the morphism f is the unique closed subscheme $Z \subset X \times_S Y$ such that the projection on X restricted to Z is an isomorphism and $f = p_Y \circ p_X^{-1}$.*

Lemma 3.12. *Let $Z \subset M \times M$ be a locally closed set of dimension 1 such that the slope is well-defined at $P \in Z$. Let $\mathcal{O}_{M^2,P} \to k[\varepsilon,\eta]/(\varepsilon^{n+1},\eta^{n+1})$ be the natural morphism induced by the local coordinate system.*

Define

$$R := \mathcal{O}_{Z,P} \otimes_{\mathcal{O}_{M^2,P}} k[\varepsilon,\eta]/(\varepsilon^{n+1},\eta^{n+1}) \text{ and } \bar{Z} := \operatorname{Spec} R$$

Then $\bar{Z}$, identified with a closed subscheme of $\operatorname{Spec} k[\varepsilon,\eta]/(\varepsilon^{n+1},\eta^{n+1})$*, is the graph of $\tau_n(Z,P)$. If $p_2 : Z \to M$ is étale in an open neighbourhood of P then it is is an automorphism of $k[\varepsilon]/(\varepsilon^{n+1})$.*

Proof. Consider R in relation to the objects in the diagram used in the definition of slope. The ring R has a natural $\mathcal{O}_{M^2,P}/\mathfrak{m}^{n+1}$-algebra structure, and hence a natural $\mathcal{O}_{M,P_1}/\mathfrak{m}_1^{n+1}$-algebra and $\mathcal{O}_{M,P_2}/\mathfrak{m}_2^{n+1}$-algebra structure:

$$\begin{tikzcd}
 & \mathcal{O}_{M,P_2} \arrow[r, "f_2"] \arrow[d] \arrow[ddr, "g"] & \mathcal{O}_{M,P_2}/\mathfrak{m}_2^{n+1} \arrow[r, equal, "\sim"] \arrow[dd, "\tau_n"] \arrow[dll, bend right, "p_2"'] & k[\eta]/(\eta^{n+1}) \\
R & \mathcal{O}_{Z,P} \arrow[l] \arrow[dr, "\gamma"] & & \\
 & \mathcal{O}_{M,P_1} \arrow[u] \arrow[r, "f_1"] & \mathcal{O}_{M,P_1}/\mathfrak{m}_1^{n+1} \arrow[r, equal, "\sim"] \arrow[ull, bend left, "p_1"] & k[\varepsilon]/(\varepsilon^{n+1})
\end{tikzcd}$$

The map $p_1 \otimes p_2$ is factors through quotient of $k[\varepsilon,\eta]/(\varepsilon^{n+1},\eta^{n+1})$. It follows from the fact that $\mathcal{O}_{Z,P}$ is étale over $\mathcal{O}_{M,P_1}$ that the morphism p_1 is an isomorphism, and if $\mathcal{O}_{Z,P}$ is supposed étale over $\mathcal{O}$, p_2 is an isomorphism too. The statement of the lemma follows from the commutativity of the diagram. $\square$

Lemma 3.13. *Let $Z \subset M^2 \times N$ be a locally closed set of dimension 1 such that relative slope is well-defined at $P \in Z$, and let $p : Z \to M^2$ be the projection on the factor M^2, with p_1, p_2 projections on the first and second copies of M. Let $\mathcal{O}_{M^2,P} \to k[\varepsilon,\eta]/(\varepsilon^{n+1},\eta^{n+1})$ be the natural morphism induced by the local coordinate system.*

Define

$$R := \mathcal{O}_{Z,P} \otimes_{\mathcal{O}_{M^2,P}} k[\varepsilon, \eta]/(\varepsilon^{n+1}, \eta^{n+1}) \text{ and } \bar{Z} := \operatorname{Spec} R$$

Then the scheme-theoretic image of $\bar{Z}$ under π is the graph of $\tau_n(Z/N, P)$. If $p_2 : Z \to M$ is étale in an open neighbourhood of P then it is an automorphism of $k[\varepsilon]/(\varepsilon^{n+1})$.

Proof. Similar to the proof of the previous lemma. In the situation of the present lemma, as R is isomorphic a quotient $k[\varepsilon, \eta]/(\varepsilon^{n+1}, \eta^{n+1})$ of, $\operatorname{Spec} R$ is by definition the scheme-theoretic image of $\bar{Z}$ under p. □

3.1 Composition of correspondences

We have identified the slope of a curve at a regular point with an automorphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^{n+1}$. We note that the group of automorphisms of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ is $\mathbb{G}_m(k)$, and that in general an automorphism group of a fat point $\operatorname{Spec} k[\varepsilon]/\varepsilon^{n+1}$ is unipotent of rank n. Our goal in Section 5 will be to recover $\mathbb{G}_m(k)$ by identifying its points with the (first order) slopes of a family of curves in M^2. In positive characteristic we will have to resort to higher order slopes and we will not be able assure that the group recovered will indeed be $\mathbb{G}_m(k)$, but the general outline of the construction remains the same. The group operation on slopes arises from the operation of composition of correspondences. In this subsection we develop the necessary machinery.

Definition 3.14 (Scheme-theoretic composition of correspondences)**.** *Let X, Y, Z be schemes over a field, and let $\alpha : X \vdash Y, \beta : Y \vdash Z$ be closed subschemes of $X \times Y$ and $Y \times Z$ respectively, regarded as correspondences. Denote, p_{12}, p_{23}, p_{13} the projections from $X \times Y \times Z$ onto the respective products of schemes. let $\mathcal{I}_\alpha, \mathcal{I}_\beta$ be the ideal subsheaves of $\mathcal{O}_{X\times Y}$ and $\mathcal{O}_{Y\times Z}$ cutting out the graphs of α and β respectively. Define an ideal sheaf, a subsheaf of $\mathcal{O}_{X\times Z}$*

$$\mathcal{I}_{\beta\circ\alpha} = (p_{13})_*(p_{12}^*\mathcal{I}_\alpha \otimes p_{23}^*\mathcal{I}_\beta)$$

and define $\beta \circ \alpha$ to be the correspondence with the graph cut out by $\mathcal{I}_{\beta\circ\alpha}$.

Definition 3.15 (Composition of definable correspondences)**.** *Let X, Y, Z be sets definable in M, and let $\alpha \subset X \times Y, \beta \subset Y \times Z$ be definable correspondences. We will denote by $\beta \circ \alpha$ the composition of α and β*

$$\beta \circ \alpha = \{ \ (x, z) \in X \times Z \ \mid \ \exists y \in Y \ (x, y) \in \alpha \ \text{ and } (y, z) \in \beta \ \}$$

and α^{-1} the inverse correspondence,

$$\alpha^{-1} = \{ \ (y, x) \ \mid \ (x, y) \in \alpha \ \}$$

These two definitions are closely related: the graph of the scheme-theoretic correspondence of Definition 3.14, $\Gamma(\beta \circ \alpha)$, is the Zariski clousre of the graph of the composition $\Gamma(\beta \circ \alpha)$ in the sense of Definition 3.15. By the Proper Mapping Theorem the two definitions agree if the projections $\Gamma(\alpha) \to X$ and $\Gamma(\beta) \to Z$ are proper.

We point out that regularity is not always preserved under composition. That is, even if Z_1 is regular at (P_1, P_2) and Z_2 is regular at (P_2, P_3) the composition $Z_2 \circ Z_1$ is not necessarily regular at (P_1, P_3). This is where the notion of relative slope introduced in the previous section becomes useful.

Consider two curves V, W regular at $(P_1, P_2) \in V$ and (P_2, P_3), and let $S \subset M^3$ be the curve cut out by the ideal sheaf $p_{13}^* I_V \otimes p_{23}^* I_W$; then $W \circ V$ is the image of S under the projection p_{13} on the first and third factor M in M^3. Proposition 3.17 below gives the relative slope at a point on S in terms of slopes of W and V. The proofs uses the projection formula for coherent sheaves and finite morphisms, as stated in the next lemma:

Lemma 3.16 (Projection formula)**.** *Let $p : X \to Y$ be an affine morphism of schemes, and let $\mathcal{E}, \mathcal{F}$ be coherent sheaves on X, Y respectively. Then the natural morphism*

$$p_* \mathcal{E} \otimes \mathcal{F} \to p_*(\mathcal{E} \otimes p^* \mathcal{F})$$

is an isomorphism.

Proof. Follows from Corollaire I.9.3.9, [15]. □

Proposition 3.17. *Let M be an algebraic curve over a field. Let V, W be curves incident to $(P_1, P_2) \in M^2$ and $(P_2, P_3) \in M^2$, respectively, and regular at these points. Fix local coordinate systems at P_1, P_2, P_3 and assume that the slopes of V and W are well-defined at (P_1, P_2) and (P_2, P_3), respectively. Let S be the variety cut out by the ideal sheaf $p_{12}^* I_V \otimes p_{23}^* I_W$. For any integer $n > 0$*

$$\tau_n(S, (P_1, P_2, P_3)) = \tau_n(W, (P_1, P_2)) \circ \tau_n(V, (P_2, P_3))$$

In particular, by Lemma 3.13, if $W \circ V = p_{13}(S)$ is regular at (P_1, P_3),

$$\tau_n(W \circ V, (P_1, P_3)) = \tau_n(W, (P_1, P_2)) \cdot \tau_n(V, (P_2, P_3))$$

Proof. Let X, Y, Z be copies of $\operatorname{Spec} k[\varepsilon]/(\varepsilon^{n+1})$ and let i_X, i_Y, i_Z be the closed embeddings of X, Y, Z into M that map the unique closed points of these schemes to P_1, P_2, P_3 respectively. Let $\overline{W}, \bar{V}$ be restrictions of W, V to infinitesimal thickenings $X \times Y, Y \times Z$ of the points (P_1, P_2) and (P_2, P_3). More precisely,

$$\overline{W} = W \times_{i_X \times i_Y} (X \times_k Y) \qquad \bar{V} = V \times_{i_Y \times i_Z} (Y \times_k Z)$$

By Lemma 3.12

$$\overline{W} = \Gamma(\tau_n(W, (P_2, P_3))) \qquad \bar{V} = \Gamma(\tau_n(V, (P_1, P_2)))$$

Denote p_{XY}, p_{YZ}, p_{XZ} the projections of $X \times Y \times Z$ onto the respective products of factors. Denote $f : X \to Y$ and $g : Y \to Z$ the morphisms such that $\overline{W} = \Gamma(f), \bar{V} = \Gamma(g)$. Let $\mathcal{I}_f, \mathcal{I}_g, \mathcal{I}_{g \circ f}$ be the ideal subsheaves of $\mathcal{O}_{X \times Y}$, $\mathcal{O}_{Y \times Z}$ and $\mathcal{O}_{X \times Z}$ that cut out the graphs of f,g and $g \circ f$ respectively.

Now notice that $\overline{W} \circ \bar{V}$ is the scheme-theoretic image of $(W \times_{p_2} V) \times_{i_X \times i_Z} (X \times Z)$ under p_{XZ}, and that $W \times_{p_2} V$ is regular at (P_1, P_2, P_3) since its projection to V is étale in the neighbourhood of this point and V is regular. Notice also that the natural projection of $W \times_{p_2} V$ to W is étale, so its composition with projection p_1 to M is étale too, so the relative slope is well-defined.

Therefore, by Lemma 3.12, in order to prove the first statement of the proposition it suffices to show that

$$(p_{XZ})_*(p_{XY}^* \mathcal{I}_f \otimes p_{YZ}^* \mathcal{I}_g) = \mathcal{I}_{g \circ f}$$

Denote $\gamma_f : X \to X \times Y$, $\gamma_g : Y \to Y \times Z$ and $\gamma_{g \circ f} : X \to X \times Z$ the morphisms sending the domains of f, g, $g \circ f$ onto their respective graphs.

Then

$$(p_{XZ})_*(p_{XY}^* \mathcal{I}_f \otimes p_{YZ}^* \mathcal{I}_g) = p_{XZ*}(p_{XY}^* \mathcal{I}_f \otimes (\mathrm{id}_X \times \gamma_g)_* \mathcal{O}_{X \times Y})$$

and by Lemma 3.16 (which we can apply because the map $\mathrm{id}_X \times \gamma_g$ is affine and all the sheaves involved are coherent),

$$\begin{aligned} & (p_{XZ})_*(p_{XY}^* \mathcal{I}_f \otimes p_{YZ}^* \mathcal{I}_g) = \\ = \; & (p_{XZ})_*((\mathrm{id}_X \times \gamma_g)_*((\mathrm{id}_X \times \gamma_g)^* p_{XY}^* \mathcal{I}_f \otimes \mathcal{O}_{X \times Y})) = \\ = \; & (p_{XZ})_*(\mathrm{id}_X \times \gamma_g)_*(\mathrm{id}_X \times \gamma_g)^* p_{XY}^* \mathcal{I}_f \end{aligned}$$

Taking into account the commutativity of the following diagrams

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{id}_X \times \gamma_g"] \arrow[dr, "\mathrm{id}_X \times g"'] & X \times Y \times Z \arrow[d, "p_{XZ}"] \\ & X \times Z \end{tikzcd} \qquad \begin{tikzcd} X \times Y \arrow[r, "\mathrm{id}_X \times \gamma_g"] \arrow[dr, "\mathrm{id}_{X \times Y}"'] & X \times Y \times Z \arrow[d, "p_{XY}"] \\ & X \times Y \end{tikzcd}$$

we conclude that

$$(p_{XZ})_*(p_{XY}^* \mathcal{I}_f \otimes p_{YZ}^* \mathcal{I}_g) = (\mathrm{id}_X \times g)_* \mathcal{I}_f.$$

Notice further that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\gamma_f"] \arrow[drr, "g \circ f"'] & X \times Y \arrow[r, "\mathrm{id}_X \times g"] & X \times Z \arrow[d, "p_Z"] \\ & & Z \end{tikzcd}$$

and conclude

$$(p_{XZ})_*(p_{XY}^* \mathcal{I}_f \otimes p_{YZ}^* \mathcal{I}_g) = (\mathrm{id}_X \times \gamma_g)_* \gamma_{f*} \mathcal{O}_X = \gamma_{g \circ f *} \mathcal{O}_X = \mathcal{I}_{g \circ f}.$$

The second part of the lemma follows from the first part and Lemma 3.13. $\square$

The previous lemma, since it applies to n-slopes for all n, extends to the composition of the formal power series expansion of curves. More precisely, recall that formal power series can be composed in the following sense: given an element $y \in k[[x]]$ from the maximal ideal, there exists a unique homomorphism of topological rings $f_y : k[[x]] \to k[[x]]$ mapping x to y (see [4], IV,§3 for example). The image of a power series $z \in k[[x]]$ under f_y is the composition of the power series y with the power series z. Since the expansion of a curve Z at a regular point is defined through the inverse system of n-slopes, it is then easy to verify that, in the notation of Proposition 3.17, the power series expansion or $W \circ V$ at P is the composition of the power series expansion of V at P with the power series expansion of W at P.

3.2 Sum of correspondences

We are now going to define another binary operation on correspondences, more precisely on correspondences between a fixed variety and an algebraic group. This operation amounts to application of group law to the second coordinates of points in M^2 and we call it, abusing terminology and notation, "the sum of correspondences". For correspondences with étale projection on the source it can be defined in a compatible way scheme theoretically. When applied to maps from the double point to an algebraic group, the operation amounts to addition of tangent vectors which can be taken as a justification for the terminology. Ultimately, this operation will allow us in Section 5.8, after recovering $\mathbb{G}_m(k)$ using compositions, to recover the additive group of the field and the action of the former on the latter.

Definition 3.18 (Sum of definable correspondences)**.** *Let T be a definable set, $(G, +)$ be a definable group and let $\alpha, \beta : T \vdash G$ be definable correspondences. Define:*

$$\alpha + \beta := \{(x, u) \in T \times G \mid \exists y, z \in G \quad (x, y) \in \Gamma(\alpha) \textit{ and } (x, z) \in \Gamma(\beta) \textit{ and } u = y + z)\}$$

Remark. The notation above suggests that G is commutative. The definition applies even if it is not the case, although in this paper we will only deal with G which are Abelian or Abelian-by-finite.

Note that the sum of correspondences does not in general define a group law on the class of all definable correspondences between a fixed set T and a group G, in fact, the obvious candidate for the "opposite" correspondence

$$-\alpha := \{\ (x, -y) \ \mid \ (x, y) \in \Gamma(\alpha)\ \}$$

is an opposite with repsect to "+" only if α is a graph of a function.

As in the case of composition of correspondences, where we had both scheme theoretic and model theoretic definitions of composition we introduce a scheme theoretic version of the sum of correspondences.

Definition 3.19 (Scheme-theoretic sum of correspondences)**.** *Let G be an algebraic group, $\alpha, \beta : T \vdash G$ finite-to-finite correspondences with graphs étale over T. We define $\alpha + \beta$ to be the correspondence $T \vdash G$ with the graph cut out by the ideal*

$$\mathcal{I}_{\alpha+\beta} = (\mathrm{id} \times a)_*(p_{12}^* \mathcal{I}_\alpha \otimes p_{13}^* \mathcal{I}_\beta)$$

where $a : G \times G \to G$ is the group law morphism and $p_{i,j}$ are the natural projections from $T \times G \times G$.

It follows from the definition that the scheme-theoretic sum of correspondences is the scheme-theoretic image under the morphism $\mathrm{id} \times a$ of the scheme $\Gamma(\alpha) \times_{p_1, M, p_1} \Gamma(\beta)$. It therefore coincides with Zariski closure of its definable counterpart.

Proposition 3.20. *Let (G, a) be an algebraic group over a field k and let $\rho : G(\operatorname{Spec} k[\varepsilon]/\varepsilon^2) \to G(k)$ be the natural restriction map. Then $\operatorname{Ker} \rho$ is isomorphic to $\mathbb{G}_a^n(k)$ where $n = \dim G$.*

Proof. The set $\operatorname{Ker} \rho$ is naturally identified with the tangent space $T_0 G$, and the differential of the group law is some bi-linear map $\bar{a} : T_0 G \otimes T_0 G \to T_0 G$. It follows from bi-linearity that

$$\bar{a}(x \otimes y) = Ax + By$$

for some $A, B \in GL_n(k)$.

It follows from associativity of the group law that:

$$A(Ax + By) + Bz = Ax + B(Ay + Bz)$$

for all $x, y, z \in T_0 G$. From that we easily deduce that $A^2 = A, B^2 = B$. Therefore necessarily, A and B are the identity maps, and $a(x, y) = x + y$. □

Remark. Proposition 3.21 and Proposition 3.20 are manifestations of the fact that the first-order truncation of one-dimensional formal group law is always of the form $(x, y) \to x + y$. In general, higher-order truncations depend on the local coordinate system chosen, and are not always isomorphic to $\mathbb{G}_a$. For example, completion in the standard coordinate system at the identity of $\mathbb{G}_m$ is of the form $(x, y) \mapsto x + y + xy$, and in positive characteristic it is impossible to choose a coordinate system such that it becomes of the form $x + y$, as follows from Cartier's theory of formal group laws (see for example Section 10 of [10]).

An element of $\operatorname{End}(k[\varepsilon]/(\varepsilon^{n+1}))$ is completely determined by the truncated polynomial withe zero constant term which is the value of the endomorphism on the generator ε; addition of such truncated polynomials defines addition on $\operatorname{End}(k[\varepsilon]/(\varepsilon^{n+1}))$ such that composition of endomorphisms is distributive over it, and makes $\operatorname{End}(k[\varepsilon]/(\varepsilon^{n+1}))$ into a ring.

Proposition 3.21. *With the ring structure described above, $\operatorname{End}(k[\varepsilon]/(\varepsilon^2))$ is isomorphic to k.*

Proof. This is a consequence of Proposition 3.20 and Proposition 3.17. $\square$

Proposition 3.22. *Let G be an algebraic group over an algebraically closed field k with the group law morphism $a : G\times G \to G$, and assume a local coordinate system is chosen at the identity, which induces an isomorphism of* $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ *with* $\mathcal{O}_{G,e}/\mathfrak{m}^2$*. Let f, g be endomorphisms of $k[\varepsilon]/(\varepsilon^2)$ and let $\Gamma(f)$ and $\Gamma(g)$ be graphs of induced morphisms from the scheme* $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ *to itself. Then the scheme cut out by the ideal*

$$(\mathrm{id} \times a \circ p_{23})_*(p_{12}^* I_f \otimes p_{13}^* I_g)$$

is the graph of the morphism from $\operatorname{Spec} k[\varepsilon]/(\varepsilon^{@})$ *to itself induced by the endomorphism $f + g$, where addition in* $\operatorname{End}(k[\varepsilon]/(\varepsilon^2))$ *is in the sense defined above.*

Proof. Denote $T = \operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$. Since f, g factor through T embedded into G, we will identify them with corresponding endomorphisms of T, we will also denote the restriction of a to $T \times T$ also by a. Let $\gamma_f, \gamma_g : T \to T$ be the maps defined as the analogous maps in the proof of Lemma 3.17. Then

$$\begin{aligned}
& (\mathrm{id}_1 \times a \circ p_{23})_*(p_{12} I_f \otimes p_{13}^* I_g) = \\
= \; & (\mathrm{id}_1 \times a)_*(p_{12} I_f \otimes (\mathrm{id}_2 \times \gamma_g)_* \mathcal{O}_{T\times T}) = \\
= \; & (\mathrm{id}_1 \times a \circ p_{23})_*(\mathrm{id}_2 \times \gamma_g)_*((\mathrm{id}_2 \times \gamma_g)^* p_{12}^* I_f \otimes \mathcal{O}_{T\times T}) = \\
= \; & (\mathrm{id}_1 \times a \circ p_{23})_*(\mathrm{id}_2 \times \gamma_g)_*(\mathrm{id}_2 \times \gamma_g)^* p_{12}^* I_f
\end{aligned}$$

Here we denote by expressions like $\mathrm{id}_2 \times \gamma_g$ the maps that act as identity on the second factor in the product T^3 and as γ_g on the rest. Since $(\mathrm{id}_2 \times \gamma_g) \circ p_{12}$ is the identity morphism on T^2, and since $I_f = (\gamma_f)_* \mathcal{O}_T$, the last expression is equal to

$$(\mathrm{id}_1 \times a \circ p_{23})_*(\mathrm{id}_2 \times \gamma_g)_* \gamma_f \mathcal{O}_G == \gamma_{a\circ(f,g)} \mathcal{O}_G$$

And so, the subvariety cut out by this ideal sheaf in G^2 is the graph of $a \circ (f \times g) : T \to G$, which is, by Proposition 3.20, the morphism from T to G associated to $f + g$. $\square$

Propositions 3.20 and 3.22 in particular imply that if $X + Y$ is regular at $(P_1, P_2, P_3, P_2 \cdot P_3)$ then

$$\tau_1(X + Y, (P_1, P_2 \cdot P_3)) = \tau_n(X, (P_1, P_2)) + \tau_n(Y, (P_2, P_3))$$

Unfortunately, this is not enough for our purposes, since $W + V$ might be not regular at $P_1, P_2 + P_3$ even if X is regular at (P_1, P_2) and Y is regular (P_1, P_3). Moreover, we will need to combine the sum of correspondences with composition of correspondences, considering correspondences of the form $X \circ Y + Z$.

We will therefore follow a strategy similar to that of Proposition 3.17: given correspondences X, Y and Z from G to G, construct an auxiliary curve W in G^5 which we will call *affine combination* $\operatorname{Aff}(X, Y, Z)$ *of X, Y and Z* that will have the necessary relative first-order slope at a certain point that projects to the possibly non-regular point in G^2. On the level of points this curve is given by

$$\mathrm{Aff}(X,Y,Z) := \{(a,b,c,d,c+d) \in G^4 \mid (a,b) \in X, (b,c) \in Y, (a,d) \in Z\}$$

and scheme-theoretically it is the subscheme cut out by the ideal

$$(\mathrm{id}_{1234} \times a \circ p_{24})_*(p_{12}^* I_X \otimes p_{23}^* I_Y \otimes p_{14}^* I_Z)$$

Proposition 3.23. *Let G be a one-dimensional algebraic group over an algebraically closed field k, and let $a : G \times G \to G$ be the group law morphism. Let $X \subset G^2 \times N, Y \subset G^2$ be curves such that the slope of X relative to N is well-defined at $(P_1, P_2, P_3) \in G$ and the slope of Y is well-defined at (P_1, P_4). Consider the curve $Z \subset G^2 \times N \times G$ cut out by the ideal*

$$(\mathrm{id}_{1234} \times a \circ p_{24})_*(p_{123}^* I_X \otimes p_{14}^* I_Y)$$

where id_{123} is identity map on the first three factors in $G \times G \times N \times G \times N$.

Proof. As in Proposition 3.17 and 3.22, we reduce the statement to a statement about graphs of maps from $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$, and apply Lemmas 3.12 and 3.13.

Let $\gamma_f, \gamma_g : T \to T$ be the maps defined as the analogous maps in the proof of Lemma 3.17. Then

$$\begin{aligned}
& (\mathrm{id}_1 \times a \circ p_{24})_*(p_{123}^* I_f \otimes p_{14}^* I_g) = \\
= \; & (\mathrm{id}_1 \times a)_*(p_{12} I_f \otimes (\mathrm{id}_{23} \times \gamma_g)_* \mathcal{O}_{T \times T}) = \\
= \; & (\mathrm{id}_1 \times a \circ p_{24})_*(\mathrm{id}_{23} \times \gamma_g)_*((\mathrm{id}_{23} \times \gamma_g)^* p_{123}^* I_f \otimes \mathcal{O}_{T \times T}) = \\
= \; & (\mathrm{id}_1 \times a \circ p_{24})_*(\mathrm{id}_{23} \times \gamma_g)_*(\mathrm{id}_{23} \times \gamma_g)^* p_{123}^* I_f
\end{aligned}$$

Since $(\mathrm{id}_{23} \times \gamma_g) \circ p_{123}$ is the identity morphism on T^3, and since $I_f = (\gamma_f)_* \mathcal{O}_T$, the last expression is equal to

$$(\mathrm{id}_1 \times a \circ p_{23})_*(\mathrm{id}_2 \times \gamma_g)_*(\gamma_f)_* \mathcal{O}_G == \gamma_{a \circ (f \times g)} \mathcal{O}_G$$

And so, the subvariety cut out by this ideal sheaf in G^2 is the graph of $a \circ (f \times g) : T \to G$, which is, by Proposition 3.20, the morphism from T to G associated to $f + g$.

□

Proposition 3.24. *Let G be a one-dimensional algebraic group as in the previous lemma. Let X, Y, Z be three curves in G^2. Then*

$$\tau_1(\mathrm{Aff}(X,Y,Z)/G^3, ()) = \tau_1(X,)\tau_1(Y,) + \tau_1(Z,)$$

Proof. The statement to be proved is a combination of Proposition 3.17 and 3.23. □

4 Differential equations in formal power series

4.1 Uniqueness of solutions

We will be using the uniqueness of solutions of differential equations in formal power series over a field of characteristic 0 on several occasions. Though it is well-known we include the short proof for the sake of completeness. It is stated precisely in the form needed for us.

Lemma 4.1 (Picard-Lindelöf for formal power series)**.** *Let k be a field of characteristic 0. Let $f(x,y) \in k[[x,y]]$ be a formal series. Then there exists a unique $y \in xk[[x]]$ such that*

$$y' = f(x,y)$$

where y' is a formal derivative.

Proof. Denote $f(x,y) := \sum_{i,j=0}^{\infty} f_{i,j}x^i y^j$ and define a map $\varphi : xk[[x]] \to xk[[x]]$ by setting

$$y \mapsto \int_0^x f(\alpha, y)d\alpha := \sum_{i,j=0}^{\infty} f_{i,j} \int_0^x \alpha^i y^j d\alpha$$

where $\int_0^x$ denotes formal antiderivative with zero constant term. Then φ is well defined, and we claim that it is a contracting map. Indeed, let $v : k[[x]] \to \mathbb{Z}$ be the valuation on the ring of formal series, $\|\cdot\|_v$ the associated norm $\|x\|_v = e^{-v(x)}$ (which is trivially 1 on constants).

$$\|\varphi(y_1) - \varphi(y_0)\|_v = \|\sum_{i,j=0}^{\infty} f_{i,j} \int_0^x \alpha^i (y_1^j - y_0^j)\|_v d\alpha \leq \frac{1}{e}\|y_1 - y_0\|_v$$

As $(k[[x]], ||\cdot||_v)$ is a complete metric space, the Banach fixed point theorem asserts that φ has a unique fixed point. By definition of ϕ this fixed point is a solution of the given differential equation. $\square$

4.2 Differential equations for slopes

On $\mathbb{A}^1$ there are natural local coordinate systems at any point x_0 given by the projective system of morphisms $k[x] \to k[\varepsilon]/(\varepsilon^n)$, $x \mapsto \varepsilon + x_0$. We would like to have in a similar vein a coherent choice of local coordinate systems at every point of a dense open subset of an arbitrary curve. Let U is a curve and suppose an étale morphism $u : U \to \mathbb{A}^1$ is chosen, then for any closed point $y \in U$ the pullback of $x - u(y) \in k[x]$ gives canonical choice of uniformizer in $\mathcal{O}_{U,y}$, and hence local coordinate systems are chosen at each such y; these are the local coordinate systems that come from isomorphisms $\widehat{\mathcal{O}_{U,y}} \cong \widehat{\mathcal{O}_{\mathbb{A}^1,u(y)}}$. We will refer to such local coordinate systems as *liftings* of local coordinate systems on $\mathbb{A}^1$.

Lemma 4.2. *Let u, v be two étale maps from U, V to $\mathbb{A}^1$, and let p be the product map $p := u \times v : U \times V \to \mathbb{A}^2$. Then for any $n > 0$, for any closed subscheme Z of $U \times V$ and any point $Q \in Z$*

$$\tau_n(Z, Q) = \tau_n(p(Z), p(Q))$$

where the local coordinate systems on $U \times V$ and on $\mathbb{A}^2$ are chosen as above.

Proof. By Lemma 3.12, $S := \operatorname{Spec} \mathcal{O}_{Z,Q} \otimes_{\mathcal{O}_{U\times V,Q}} k[\varepsilon, \eta]/(\varepsilon^n, \eta^n)$ is the graph $\tau_n(Z, Q)$, and $\operatorname{Spec} \mathcal{O}_{p(Z),p(Q)} \otimes_{\mathcal{O}_{\mathbb{A}^2,p(Q)}} k[\varepsilon,\eta]/(\varepsilon^n,\eta^n) = p(S)$ is the graph of $\tau_n(p(Z), p(Q))$. It is left to notice that the map $u \times v$ induces an isomorphism between $\operatorname{Spec} \mathcal{O}_{U\times V,Q} \otimes k[\varepsilon, \eta]/(\varepsilon^n, \eta^n)$ and $\operatorname{Spec} \mathcal{O}_{\mathbb{A}^2,p(Q)} \otimes k[\varepsilon, \eta]/(\varepsilon^n, \eta^n)$, by étaleness and the choice of local coordinate systems, and furchermore, S and $p(S)$ are isomorphic. $\square$

Lemma 4.3. *Let $Z \subset \mathbb{A}^2$ be a curve defined by an equation $h(x, y) = 0$, $Q \in Z$ a regular point such that the projection of Z on the first factor is étale. Let $f \in k[[x]]$ be the formal power series expansion of Z in the natural local coordinate system on $\mathbb{A}^2$ at Q. Then $h(x, f) = 0$.*

Proof. Without loss of generality one may assume $Q = (0, 0)$. Then the fact that n-th order slope of Z is the endomorphism f_n of $k[\varepsilon]/(\varepsilon^n)$ means that the morphism of algebras

$$k[x, y] \to k[\varepsilon]/(\varepsilon^n) \qquad x \mapsto \varepsilon,\ y \mapsto f_n(\varepsilon)$$

factors through $k[x, y]/h$, i.e. $h(x, f_n) = 0 \mod x^n$ [It may be worth giving slightly more detail why this is true. Maybe as an example following the definition of slope. This is also used below, so would be good to have explicitly, and will also clarify the definition of slope]. The conclusion of the lemma follows by passing to the limit. $\square$

Let U, V be open subsets of M, and let $u : U \to \mathbb{A}^1, v : V \to \mathbb{A}^1$ be étale maps. Let $Z \subset U \times V$ be a regular curve such that the projection of Z on U is étale, and for any point $Q = (Q_1, Q_2) \in U \times V$ the first order slope of Z at Q is well-defined (with respect to the liftings of the natural local coordinate systems at $u(Q_1), v(Q_2)$ to Q_1, Q_2) provided by the Lemma 4.2). Let $Z' \subset U \times V \times \mathbb{A}^1$ be the closed curve such that $(x, y, s) \in Z'$ if s is the first order slope of Z at (x, y).

Proposition 4.4. *In the above setting, Let $k[[u, v]] \cong \widehat{\mathcal{O}_{M^2,Q}}$ be the isomorphism given by the chosen local coordinate system at Q, and $\eta : \operatorname{Spec} k[[x]] \to Z$ the morphism given by the morphism of rings*

$$\eta^* : \mathcal{O}_{Z,Q} \to k[[x]], u \mapsto x, v \mapsto f$$

where $f \in xk[[x]]$. Consider the morphism $\eta' : \operatorname{Spec} k[[x]] \to U \times V \times \mathbb{A}^1$ defined by the morphism of rings

$$\eta'^* : \mathcal{O}_{U\times V\times\mathbb{A}^1,Q_1\times\{0\}} \to k[[x]], u \mapsto x, v \mapsto f, w \mapsto f'$$

where and f' is the formal derivative of f'. Then η' factors through Z'.

Proof. By Lemma 4.2 the slope of Z at Q is the same as the slope of $(u \times v)(Z)$ at $(u(Q_1), v(Q_2))$, in their respective local coordinate systems. So it suffices to prove the statement assuming that Z is a locally closed subset of $\mathbb{A}^2$. Without loss of generality we may assume $Q = (0, 0)$.

Let Z be defined by a polynomial $h(u, v)$. The slope of Z at a point $(x, y) \in \mathbb{A}^2$ is given by the expression

$$\frac{\frac{\partial h}{\partial u}(x, y)}{\frac{\partial h}{\partial v}(x, y)}$$

where the derivatives are formal. Let $f \in k[[x]]$ be a formal power series such that $h(x, f) = 0$ as provided by Lemma 4.3. Then

$$\frac{d}{dx} h(x, f) = \frac{\partial h}{\partial u}(x, f) + \frac{\partial h}{\partial v}(x, f) f'$$

by the chain rule (Corollaire 1, [4], IV.§6), and $\frac{d}{dx} h(x, f) = 0$ by our choice of f. Therefore

$$\frac{\frac{\partial h}{\partial u}(x, f)}{\frac{\partial h}{\partial v}(x, f)} = f'$$

and $(x, f, f') \in Z'$. □

Corollary 4.5. *Let $u : U \to \mathbb{A}^1, v : V \to \mathbb{A}^1$ be étale. Let $Q \in U \times V$ and let X be a curve incident to Q such that its projection to U is étale. Let $Z \subset U \times V \times \mathbb{A}^1$ be a closed subset that is étale over $U \times V$. Assume $X' \subset X \times \mathbb{A}^1$ is a closed subset such that $(x, y, s) \in X'$ if the first order slope of X at the point (x, y) is s, in the local coordinate system which is a lifting of the natural local coordinate system at $(u(x), v(x))$.*

Then there exist formal power series $h \in k[[x, y]]$ such that for any X as above, with a fixed slope at Q, and such that $X' \subset Z$, and $f = \varprojlim_n \tau_n(X, Q)(x) \in k[[x]]$, the formal power series expansion of X at Q,

$$f' = h(x, f)$$

where f' is the formal derivative of f.

In the statement of this corollary Z should be regarded a "multi-valued distribution", a variety that specified what slopes X is allowed to have at a particular point.

Proof. By the same reasoning as in the proof of Lemma 4.4 we can reduce the situation to U, V open subsets of $\mathbb{A}^1$ and X a locally closed subset of $\mathbb{A}^2$.

Let $s_0 = \tau_1(X, 0)$, we have the following commutative diagram

$$\begin{tikzcd}
\widehat{\mathcal{O}_{\mathbb{A}^3,(0,0,s_0)}} \cong k[[x,y,z]] \arrow[r, "f_1"] & \widehat{\mathcal{O}_{Z,0}} \arrow[r, "f_2"] \arrow[dl, "\sim"', "g"] & \widehat{O_{X',(0,0,s_0)}} \\
\widehat{\mathcal{O}_{\mathbb{A}^2,(0,0)}} \cong k[[x,y]] \arrow[u] \arrow[rr] & & \widehat{O_{X,(0,0)}} \arrow[u]
\end{tikzcd}$$

where g is an isomorphism by étaleness of the projection of Z on $\mathbb{A}^2$ and the map f_1 is given on generators as follows (identifying $\widehat{\mathcal{O}}$ via the g): $x \mapsto x, y \mapsto y, z \mapsto h$ for some $h \in k[[x, y]]$. On the other hand, by Proposition 4.4, the compostion of maps f_1 and f_2 is given on generators as follows $x \mapsto x, y \mapsto f, z \mapsto f'$. The conclusion of the Corollary follows. $\square$

4.3 Divided power structures and an ODE for $\mathbb{G}_m$

Later in Section 5 we will be looking at curves is the product of two copies of the multiplicative group $\mathbb{G}_m \times \mathbb{G}_m$, and in connection with this we will need to consider formal power series solutions to the differential equation

$$y' = a \cdot \frac{1+y}{1+x} \tag{1}$$

Below we collect some facts about solutions to this equation, especially that the situation in positive characteristic requires a somewhat subtle treatment.

Let k be a field of characteristic 0. Consider *binomial power series* for $a \in k$, defined as

$$(1+x)^a = \sum_{k=0}^{\infty} \frac{a \cdot (a-1) \dots (a-k)}{k!} x^k$$

If a is integer then this is a polynomial, but the formal power series are well-defined for any $a \in k$.

Proposition 4.6. *Let k be a field of characteristic 0. The unique solution of the differential equation (1) is the binomial power series $(1 + x)^a - 1$. These formal power series are algebraic over $k[x]$ only if a is rational.*

Proof. Observe that $\frac{d}{dx}(1+x)^a = a(1+x)^{a-1}$, then it's easy to check that $(1+x)^a$ is a solution by direct substitution.

For the second point observe that if a is not rational, then powers of $(1 + x)^a$ generate a module of infinite rank over $k[x]$, and so $(1 + x)^a$ is not algebraic. One uses the property

$$(1 + x)^a(1 + x)^b = (1 + x)^{a+b}$$

which can be easily derived oven for a not rational (see Section 2 of [13] and references therein). $\square$

Even though uniqueness of solutions of ODEs fails in positive characteristic in general, we can say something about solutions to the equation (1). Unfortunately, the binomial power series are not even well-defined in positive characteristic, because integers dividing the characteristic appear in denominators of the coefficients.

In order to remedy that, we consider solutions of differential equation (1) in a bigger ring that $k[[x]]$ maps to. This ring is the completion of the divided power polynomials ring $k\langle x\rangle$ of $k[x]$ with respect to a certain filtration. For definition and properties of divided power structures we refer to [2], see also [1].

The facts about $k\langle x\rangle$ that we will need are few. The ring $k\langle x\rangle$ is generated over k by variables $x^{[n]}, n \in \mathbb{N}$, subject to relations

$$x^{[n]}x^{[m]} = \frac{(n+m)!}{n!m!}x^{[n+m]}$$

($x^{[0]}$ customarily means 1).

Consider the homomorphism $\varphi : k[x] \to k\langle x\rangle$ that sends x to $x^{[1]}$. The kernel of this morphism is the ideal generated by x^p. Define the derivation D_x on the generators by $D_x x^{[n]} = x^{[n-1]}$. The homomorphism φ has the following property:

$$\varphi(p'(x)) = D_x\varphi(p(x))$$

Consider the completion $\widehat{k\langle x\rangle}$ of $k\langle x\rangle$ with respect to the filtration by the ideals generated by sequences of elements of the form $x, x^{[2]}, x^{[3]}, \dots, x^{[i]}$. The homomorphism φ above extends to a morphism $\bar{\varphi} : k[[x]] \to \widehat{k\langle x\rangle}$, and the derivation D_x is extended in a unique way to $\widehat{k\langle x\rangle}$. The compatibility of derivations is preserved: $\bar{\varphi}(p'(x)) = D_x\bar{\varphi}(p(x))$.

Define *divided power binomial series*

$$(1+x)^a = \sum_{n=0}^{\infty} a \cdot (a-1) \cdot \ldots \cdot (a-n+1)x^{[n]}$$

in $\widehat{k\langle x\rangle}$.

Lemma 4.7. *Let k be a field of positive characteristic $p > 0$. The differential equation* (1) *has a solution*

$$(1+x)^a - 1$$

in $\overline{k\langle x\rangle}$ if and only if $a \in \mathbb{F}_p$. Let $y_1, y_2 \in k[[x]]$ be two distinct solutions, then $\dfrac{y_1+1}{y_2+1}$ belongs to $k((x^p))$ and is non-constant.

Proof. Observe that if $f \in k[[x]]$ is a solution to (1), then $\bar{\varphi}(f)$ is a solution to (1) (with derivation interpreted as D_x).

If $a \in \mathbb{F}_p$ the binomial formal power series is a polynomial which is a solution to equation 1 by direct verification. If $a \notin \mathbb{F}_p$ then the divided powers binomial power series is a solution to (1) by direct verification.

Let y_1, y_2 be distinct solutions in $\widehat{k\langle x\rangle}$ of the equation (1). Then

$$D_x\left(\frac{y_1+1}{y_2+1}\right) = \frac{y_1'(y_2+1)-y_2'(y+1)}{(y_2+1)^2} = \frac{a(y_1+1)(y_2+1)-a(y_2+1)(y_1+1)}{(1+x)(y_2+1)^2} = 0$$

and therefore $\frac{y_1+1}{y_2+1}$ is a constant. One checks that if $\frac{y_1+1}{y_2+1}$ is a constant then it must be 1. The same argument in $k[[x]]$ yields that formal power series solutions y_1, y_2 have the property that $\frac{y_1+1}{y_2+1} \in k((x^p))$.

If $a \in \mathbb{F}_p$ then an algebraic solution y_1 must have the property that $\frac{y_1+1}{(1+x)^a}$ is algebraic.

If $a \notin \mathbb{F}_p$, one checks that $(1+x)^a \in \widehat{k\langle x\rangle}$ is not in the image of φ (since it has $x^{[n]}$ terms for $n \geq p$) and so there are no solutions in $k[[x]]$. $\square$

5 Interpretation of a field from a pure-dimensional witness family

Restricting to the regular locus of M we may assume that M is regular, which will be the standing assumption in the present section.

A reduct of M is non-locally modular if and only if there exists a *faithful* (in the terminology of [40]; the term *normal* is commonly used in the literature) definable family of curves $X \subset M^2 \times T$, i.e. a family such that $\dim T = 2$, $\dim X_t = 1$ for $t \in U$, $\dim T \setminus U = 1$ and such that $\dim(X_t \setminus X_s) \cup (X_s \setminus X_t) = 0$ for $t \neq s$. Let M be a one-dimensional algebraic curve. We consider a reduct (M, X) where $X \to T$ is a 2-dimensional family of one-dimensional subsets of M^2.

In this section we assume that X is a family of pure dimensional curves, namely, that X_t has only components of dimension 1 for t generic. We first prove that then (M, X) interprets a one-dimensional group. Then, assuming that there is a group structure on M, we prove that (M, X) interprets a field.

Here is some notation that we will systematically use.

For any point $Q \in M^2$, any family $Y \subset S \times M^2$ of one-dimensional subsets of M^2 we denote $Y^Q \to S^Q$ the subfamily of one-dimensional sets containing the point Q. For any correspondence $\alpha : M \vdash M$ denote $\alpha \circ X \to T$ the family such that $(\alpha \circ X)_t = \alpha \circ X_t$, similarly, $X \circ \alpha$ denotes the family $(X \circ \alpha)_t = X_t \circ \alpha$.

For two (definable) families $Y_1 \subset S_1 \times M^2, Y_2 \subset S_2 \times M^2$ of curves in M^2 we will denote $Y_1 \circ Y_2$ the (definable) family parametrized by $S_1 \times S_2$ with the curve $(Y_1)_{s_1} \circ (Y_2)_{s_2}$ corresponding to parameter s_1, s_2; similarly, the notation $Y_1 + Y_2$ will be used for families of curves in G^2 where G has a structure of a group.

We will say that two sets Y and Z of the same dimension *almost coincide* if $\dim(Y \setminus Z) \cup (Z \setminus Y) < \dim Y (= \dim Z)$. We will say that a property holds for *almost all points* of a definable set Y if it holds for a set of points Y' that almost coincides with Y.

5.1 Generically étale projections

In order to set in gear the machinery of slopes developed in the previous sections we need projections to étale, at least generically. The following lemma asserts that at least one of the "coordinate" projections of a curve $X \subset M \times M$ does satisfy this requirement. The statement of the lemma is obvious in characteristic 0, but the existence of everywhere ramified morphisms in positive characteristic makes it non-trivial in this case. Interestingly, the proof presented below does not explicitly depend on the characteristic.

Lemma 5.1. *Let M be a one-dimensional closed irreducible curve over a field of positive characteristic. Let $X \subset M \times M$ be an irreducible closed curve, and $p_1, p_2 : X \to M$ be the projections on the respective coordinates. Then there exists a dense open $O \subset M$ such that either p_1 restricted to $X \cap O \times M$ or p_2 restricted to $X \cap M \times O$ is étale.*

Proof. The lemma is clear if one of the projections p_i is not dominant. So we may assume that this is not the case.

Let $\Omega_{M/k}$, $\Omega_{X,k}$ be the sheaves of modules of Kähler differentials over k of M and of X respectively, and let $p_1, p_2 : X \to M$ be projections on the first and second factor M in $M \times M$. Consider the natural map of sheaves

$$p_1^*\Omega_{M/k} \oplus p_2^*\Omega_{M/k} \to \Omega_X$$

which is easily seen to be surjective. Localising at the generic point χ of X we get a surjective map, f, of $k(X)$-vector spaces

$$p_1^*\Omega_{M/k} \otimes k(\chi) \oplus p_2^*\Omega_{M/k} \otimes k(\chi) \to \Omega_X \otimes k(\chi).$$

Then $f = f_1 \oplus f_2$ where $f_i : p_i^*\Omega_{M/k} \otimes k(\chi) \to \Omega_X \otimes k(\chi)$ is a $K(X)$-linear map. As f is surjective and $\dim_{K(X)} \Omega_X \otimes k(\chi) = 1$ at least one of f_1 and f_2 must be an isomorphism.

Assume that for example f_1 is an isomorphism. Then the sheaf $\Omega_{X/M}$ of relative differentials on X over M with respect to the first projection, is isomorphic over a dense open set to the structure sheaf. So p_1 is generically unramified. So the lemma follows from generic flatness (Fact 5.14). □

Lemma 5.2. *Let M be a one-dimensional closed irreducible curve over a field of any characteristic. Let $X \subset T \times M \times M$ be a family of closed curves, and $p_1, p_2 : X \to M$ the projections on the first, respectively second M. Then there exists a dense open $U \subseteq T$ and a dense open $O \subseteq M$ such that either p_1 restricted to $X_t \cap O \times M$ or p_2 restricted to $X_t \cap M \times O$ is étale for all $t \in U$.*

Proof. Follows from the previous lemma applied to the generic fibre of the family $X \to T$. □

5.2 Finding enough slopes in characteristic 0

In order to obtain a one-dimensional group configuration in the reduct we will construct an M-definable family of curves incident to a point Q that lies on the diagonal variety $\Delta_M \subset M \times M$ such that for some n the n-slopes of curves at P almost coincide with a one-dimensional subgroup of $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^{n+1}))$ (viewed as a finite-dimensional algebraic group). We will present two approaches, the first uses Lemma 4.1 in a significant way, and works only in characteristic zero. It gives a family whose first order slopes are almost all of $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$. The second approach works in any characteristic but gives a family whose n-th order slopes almost coincide with a 1-dimensional subgroup of $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^{n+1}))$ where, in general, n may be greater than 1.

Lemma 5.3. *Let M be an algebraic curve (not necessarily irreducible) over an algebraically closed field of characteristic 0, and let $X \subset S \times M^2$ be a 2-dimensional faithful family of closed irreducible one-dimensional subsets of M^2. Then there exists an open subset $O \subset M^2$ such that for any point $Q \in O$ the set $\tau_1(X_s, Q)$ is infinite.*

Proof. Assume not. Then one can pick an Zariski open set $O \subset M^2$, dense in an irreducible component of M^2, such that $\tau_1(X_s, Q)$ is finite for all $Q \in O$.

Pick some étale projections $u : U \to \mathbb{A}^1, v : V \to \mathbb{A}^1$. By Lemma 4.2 the projections $(u \times v)(X_s)$ will have finitely many slopes at points of some dense open subset of $\mathbb{A}^2$, call it O'.

Let Z be the locally closed subset of $U \times V \times \mathbb{A}^1$ such that a point (x, y, s) belongs to Z if a curve from the family X incident to (x, y) has the slope s at (x, y) in the natural local coordinate system. Shrinking O' if necessary we may assume that Z is étale over O'. Pick a point $Q \in O'$ and pick one of the finitely many first-order slopes the curves from the family X take at Q, call it a. Then by Corollary 4.5 for any curve $X_t \subset U \times V$ with $\tau_1(X_t, Q) = a$, the formal power series expansion f at a point $Q \in O'$ must satisfy a differential equation

$$f' = h(x, f)$$

for some $h \in k[[x, y]]$. By Lemma 4.1 there is only one solution $f \in xk[[x]]$ per slope value at Q. But according to our assumption about X there are infinitely many curves incident to Q, which is a contradiction. □

Lemma 5.4. *Let $X \subset S \times M^2$ be a family of curves on M^2. Then for a suitable choice of $s_0 \in S$ there exists a point Q on the diagonal variety $\Delta_M \subset M \times M$ such that*

$$\{\ \tau_1(X_s \circ X_{s_0}, Q)\ \mid\ s \in S\ \}$$

the set of first order slopes of curves in the family $(X_s \circ X)^Q$ almost coincides with $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$.

Proof. Let $U \subset M^2$ be a dense open set as provided by the conclusion of the previous lemma. If $U \cap \Delta_M \neq \emptyset$, then pick a point Q in this intersection. By Lemma 5.3, the tangent spaces $T_Q X_t$ swipe a one-dimensional subset of $\mathbb{TM}^{\not\vDash}$. Since distinct $T_Q X_t$ correspond to distinct first order slopes (which are just the Plücker coordinates of $T_Q X_t$ in some fixed local coordinates), the statement of the Lemma follows.

Otherwise consider the family $X_{s_0} \circ X$ for some $s_0 \in S$ such that $(\mathrm{id} \times X_{s_0})(U) \cap \Delta_M$ is non-empty and the projections of X_{s_0} on both factors M of M^2 are generically étale. In order to satisfy the first condition $X_{s_0} \cap \Delta_M$ has to be finite, and since X is a faithful family, all but finitely many of s_0 will satisfy this requirement.

The second requirement is generically satisfied by the elements of the family X: indeed, if it were otherwise, X_t would have to be generically tangent to a "vertical" or "horizontal" vector fields, and by Lemma 4.1, it would follow that there is exactly one curve incident to a generic point, which contradicts the fact that X is a two-dimensional family. □

5.3 Finding enough slopes in positive characteristic

Let M be an algebraic curve and $X \subset M^2 \times T$ a family of subvarieties such that X_t is of dimension 1 for generic t. For any family X of curves in M^2 that we consider in this section, we may assume without loss of generality, appealing to Lemma 5.2, that for almost all elements X_t of the family the projection on the first factor M of $M \times M$ is étale.

If M is defined over a field of positive characteristic we can no longer guarantee the existence of one-dimensional families of curves that are incident to a point Q and such that first order slopes at Q constitute a one-dimensional subset of $\mathrm{End}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$. The simplest example that illustrates this phenomenon is the family of curves on $\mathbb{A}^2$ defined in the standard coordinates (x, y) by the equations $y = x + ax^p + b$. What we do instead is we find a family of curves incident to some point Q such that the set of n-order slopes at Q almost coincides with a one-dimensional subgroup of $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^{n+1}))$ for some n. Note that this approach works in characteristic 0 as well.

Lemma 5.5. *Fix local coordinate systems at $P_1, P_2, P_3 \in M$. Let $Z_1, Z_2 \subseteq M^2$ be curves with $(P_1, P_2) \in Z_1$ and $(P_2, P_3) \in Z_2$ both regular points on the respective curves. Let $f_1, f_2 \in k[[x]]$ be the associated power series expansions of Z_1, Z_2 at (P_1, P_2) and at (P_2, P_3) respectively. Assume that $f_1 = g_1^{p^n}, f_2 = g_2^{p^m}$ with $g_1, g_2 \in k[[x]]$. Then the power series expansion of $Z_1 \circ Z_2^{-1}$ at (P_3, P_1) is given by $h := (g_1 \circ g_2)^{p^{n-m}}$.*

Proof. First let us prove the statement for $M = \mathbb{A}^1$ and $P_1 = P_2 = P_3 = 0$. Since Z_1, Z_2 are regular at $(0, 0)$ only one irreducible component of each of Z_1, Z_2 passes through $(0, 0)$, so we may assume that Z_1 and Z_2 are irreducible. In that case it is enough to notice that if a formal power series expansion of a curve at $(0, 0)$ is of the form f^{p^n} then this curve is a composition (in the sense of Section 3.1) of some curve with formal power series expansion f and the graph of the n-th power of the Frobenius morphism $\mathbb{A}^1 \to \mathbb{A}^1$, and the necessary statement follows.

For the general case, find projections $p_i : M \to \mathbb{A}^1$ that map P_i to 0 and such that p_i is étale in a neighbourhood of P_i. Since étale morphisms induce isomorphisms of completed local coordinate rings, local coordinate systems at P_1, P_2, P_3 are induced by precomposing with p_1, p_2, p_3 respectively. Observe that the slope of Z_1 at (P_1, P_2) coincides with the slope of $(p_1 \times p_2)(Z_1)$ and similarly for Z_2 and $(p_2 \times p_3)(Z_2)$. Then the statement follows from the statement for $\mathbb{A}^1$ which we have already proved. $\square$

Lemma 5.6. *There exists a two-dimensional faithful family $X' \to T'$ and an open dense subset $U \subset M \times M$ such that for any point $Q \in U$ and for all $t \in T'$ such that $Q \in X'_t$ the formal power series expansion of X'_t is not in $k[[x^p]]$.*

Proof. We may assume that $M \times M$ is connected, if not, the proof applies componentwise to the connected components. We may also assume T irreducible (or otherwise run the argument for some irreducible component of T).

Fix local coordinate systems at all points of $M \times M$ (as in Subsection 4.2). Irreducibility of M and T imply that there exists a unique n (possible 0) such that for all $Q \in U$ and all generic $t \in X^Q$ such that the formal power series expansion of X_t at Q is a p^n-th power but not p^{n+1}-th power.

Consider the family $X' = X \circ X_{t_0}^{-1}$ where $Q \in X_{t_0}$ for some $Q \in U$. Write the power series expansions of X_t at Q as $(f_t)^{p^n}$ (where f_t is not a p-th power), then the formal power series expansion of $X_t \circ X_{t_0}^{-1}$ at the point Q is, by Lemma 5.5, $f_t \circ f_{t_0}^{-1}$, which is not a p-th power. $\square$

Lemma 5.7. *There exists an open set U and a family of curves $X' \to T'$ such that for any point $Q \in U$ almost all $t \in (T')^Q$ the projections of X'_t on both factors M are étale.*

Proof. That the projection on, say, the first M is étale follows from Lemma 5.2 (the second projection is étale in a neighbourhood of Q if the first-order slope is non-zero by Proposition 3.10). In view of Lemma 5.6 we may assume that there exists an open set V such that the formal power series expansion of almost all curves X_t at any point $Q \in V$, are not p-th powers.

It is left to show that there exists U such that for all points $Q \in U$ almost all curves incident to Q have non-zero first order slopes. By an argument similar to the proof of Lemma 5.3 we conclude that were this not the case then the formal power series expansions f of any generic curve through at any generic point would satisfy the differential equation $f' = 0$. As there is a unique solution to this differential equation in $xk[[x]]$ that is not a p-th power, and – by assumption – there are infinitely many curves through every generic point, almost all of which are not p-th powers this would lead to a contradiction. $\square$

Lemma 5.8. *By passing to a family of compositions $X_{t_0} \circ X$ for suitable t_0 one can find a point Q on the diagonal variety $\Delta_M \subset M \times M$ such that the conclusion of the previous Lemma holds for Q.*

Proof. Indeed, let U be the set of points Q such that the formal power series expansion of X_t at Q is non-zero (in some, and hence any, local coordinate system) for almost all $t \in T^Q$. If $U \cap \Delta_M \neq \emptyset$, we are done. Otherwise, we need to find X_{t_0} such that $(\mathrm{id} \times X_{t_0})(U) \cap \Delta \neq \emptyset$, or, which is the same, $X_{t_0} \cap \Delta_M \neq 0$, and such that the projections from X_{t_0} on both factors M in $M \times M$ are generically étale. Most elements of the family $X' \to T'$ satisfy the latter requirement, because the formal power series expansion of X'_t for a given t is not a p-th power when it is well-defined (and it is well-defined generically by Lemma 5.1). It is clear then that one can find an X_t that satisfies the former requirement. □

Lemma 5.9. *There exists a 2-dimensional family of curves $X' \to T'$ and a natural number n, such that for some point $Q \in \Delta_M$ the set $\{\tau_n(X'_t, Q) : t \in T^Q\}$ almost coincides with a 1-dimensional subgroup of* $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^{n+1}))$.

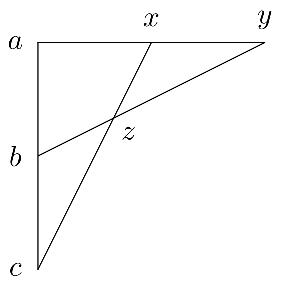
Proof. By the previous lemma we may assume that for some Q on the diagonal almost all first order slopes of T^Q are non-zero. Consider the family $X' = X_{t_0}^{-1} \circ X$ for some X_{t_0} with a non-zero first order slope. Then by Proposition 3.17 almost all curves X'_t will have n the order slope $\mathrm{id} \in \mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^{n+1}))$ for $n < n_0$ where n_0 is maximal such (possibly $n_0 = 1$). Since both projections restricted to any of our curves are étale, it follows from Lemma 5.3 that the n_0-th order slopes of X'_t form a one-dimensional subset of $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^{n_0+1}))$. By Proposition 3.10 they form a group chunk.

Note that all the constructions involved in producing the family X' produce a definable family. In case T is not irreducible, then the consructions have to be repeted successively for all one-dimenional connected components of T. □

5.4 The group and field configurations

In the strongly minimal context, certain configuration of (imaginary) elements are known to exist only in the presence of a definable group or a definable field. We will now describe this in more detail:

Definition 5.10 (Group configuration)**.** *Let M be a model of a strongly minimal theory, and let* dim *the be associated dimension function on tuples.*



The set $\{\ a, b, c, x, y, z\ \}$ of tuples is called a group configuration *if there exists an integer n such that*

- *all elements of the diagram are pairwise independent and* $\dim(a, b, c, x, y, z) = 2n + 1$*;*

- $\dim a = \dim b = \dim c = n$*,* $\dim x = \dim y = \dim z = 1$*;*

- *all triples of tuples that lie on the same line are dependent, and moreover,* $\dim(a, b, c) = 2n$*,* $\dim(a, x, y) = \dim(b, z, y) = \dim(c, x, z) = n + 1$*;*

If G is a connected group definable in a strongly minimal theory, acting transitively on a strongly minimal definable set X, then one can construct a group configuration as follows: let g, h be independent realisations of the generic type of G and let x be a realisation of a generic type of X, then $(g, h, g \cdot h, u, g \cdot u, g \cdot h \cdot u)$ is a group configuration.

Fact 5.11 (Hrushovski)**.** *Let M be a strongly minimal structure and let (a, b, c, x, y, z) be a group configuration. Then there exists a definable group G acting transitively on a strongly minimal set X with the associated group configuration $(g, h, g \cdot h, u, g \cdot u, g \cdot h \cdot u)$ such that* $\operatorname{acl}(a) = \operatorname{acl}(g), \operatorname{acl}(b) = \operatorname{acl}(h), \operatorname{acl}(g \cdot h) = \operatorname{acl}(c), \operatorname{acl}(x) = \operatorname{acl}(u), \operatorname{acl}(y) = \operatorname{acl}(g \cdot u), \operatorname{acl}(z) = \operatorname{acl}(g \cdot h \cdot u)$*. In particular,* $\dim G = \dim a$*.*

This follows from Main Theorem of [5] and the fact that infinitely definable groups in stable theories are intersections of definable groups (see, for example, Theorem 5.18[35]). The original proofs of these statements are contained in [18].

Fact 5.12. *If in the statement of Fact 5.11 one requires that the canonical base of* $\operatorname{tp}(x, y/a)$ *is interalgebraic with a and similarly for* $\operatorname{tp}(z, y/b)$ *and* $\operatorname{tp}(z, x/c)$ *then the action of G on X is faithful.*

Fact 5.13 (Hrushovski)**.** *Let G be a group of Morley rank $n > 1$ acting transitively and faithfully on a strongly minimal set X. Then there exists a definable field structure on X and either $n = 2$ and $G \cong \mathbb{G}_a \rtimes \mathbb{G}_m(K)$, or $n = 3$ and* $G = \mathrm{PSL}_2(K)$*.*

The original reference is [18], an exposition can also be found in [35] (Theorem 3.27).

Note that the crucial point in the proof of Fact 5.13 is establishing that G isomorphic to $\mathbb{G}_a \rtimes \mathbb{G}_m(K)$ or $\mathrm{PSL}_2(K)$, and in case G and X are definable in an algebraically closed field (the context in which this theorem will be applied in this article) or a Zariski structure of an algebraic curve, this statement can be proved directly and much more easily.

5.5 Flat families and intersections

As already explained, identifying $\mathcal{M}$-definably when two curves (coming from two distinct but fixed definable families) are tangent at a point $Q \in M^2$ is the key to reconstructing the multiplictative and additive groups of the field. As we will see, this approach can only work if we can show that tangency of two definable curves incident to Q is a non-generic phenomenon. In the present subsection we develop the tools allowing us to show that this can, indeed, be achieved.

Given a strongly minimal family $X \to T$ of $\mathcal{M}$-definable plane curves incident to a point $(Q, Q) \in M^2$ we form the composition family $X \circ X$ and normalise it to obtain a family $Y \to S$. Our aim is to construct an $\mathcal{M}$-definable function from S to T taking s to t if X_t has the same slope at (Q, Q) as Y_s. This will allow us to construct a group configuration based on an $\mathcal{M}$-definable function $T \times T \to T$, which by the results of subsection 3.17 corresponds to multiplication in K. The main goal of this Subsection is, therefore, given two definable families of curves (incident to a fixed point (Q, Q)), X and Y, to detect $\mathcal{M}$-definably and uniformly when a curve X_t and a curve Y_s have the same slope at (Q, Q). We obtain a good approximation of this goal in Proposition 5.22 under suitable flatness assumptions. This is one of the main technical results of the paper.

We start with recalling a few well-known geometric facts.

Fact 5.14 (Generic Flatness, Corollaire IV.6.11 in [16]). *Let Y be an integral locally Noetherian scheme and let $f : X \to Y$ be a morphism of finite type. Then there exists a dense open subset $U \subset Y$ such that the restriction of f to $f^{-1}(U)$ is flat.*

Fact 5.15 (Local flatness criterion, Proposition I.2.5 in [30]). *Let B be a flat A-algebra and consider $b \in B$. If the image of b in $B/\mathfrak{m}B$ is not a zero divisor for any maximal ideal $\mathfrak{m}$ of A then $B/(b)$ is a flat A-algebra.*

Fact 5.16 (Zariski's Main Theorem, Theorem 1.8 in [30]). *If Y is a quasi-compact scheme and $f : X \to Y$ is a separated quasi-finite morphism, then f factors as a composition $\bar{f} \circ \iota$ where $\bar{f}$ is finite and ι is an open immersion.*

For the purposes of the present paper we fix the following set the following set of conventions:

Definition 5.17. *Let M be a curve over an algebraically closed field.*

1. *a family of curves in M^2 we understand a locally closed subset $X \subseteq M^2 \times T$ for some T such that the fibres (which are not assumed irreducible) X_t are of dimension 1 for all $t \in T$;*
2. *we call a family of curves* pure-dimensional *if all irreducible components of fibres X_t are of the same dimension for all $t \in T$;*
3. *the family of scheme theoretic intersections of two families of curves $X \subseteq M^2 \times T$ and $Y \subseteq M^2 \times S$ is the surjective morphism $(X \times S) \times_{M^2 \times T \times S} (Y \times T) \to T \times S$ (that is, the family of closed subschemes of M^2 whose fibre over (s, t) is $X_t \times_{M^2} Y_s$).*

Lemma 5.18. *Suppose that M is a regular curve. Let $X \subset M^2 \times T$, $Y \subset M^2 \times S$ be two families of pure-dimensional curves in M^2, and suppose that X is flat over T. Then the family of scheme-theoretic intersections of X and Y is flat over $T \times S \setminus D$ where D is the set of pairs (t, s) such that the intersection $X_t \cap Y_s$ infinite.*

Proof. Since flatness is local on the source we may assume that all varieties involved are affine.

Consider the subvarieties $X \times S$, $Y \times T$ of $M^2 \times T \times S$. Suppose first that Y is a hypersurface in $M^2 \times S$. Since regular local rings are unique factorization domains, the scheme-theoretic intersection of $X \times S$ and $Y \times T$ is a closed subscheme of $X \times S$ that is locally the zero locus of a regular function f on $M^2 \times T \times S$ restricted to $X \times S$. By Fact 5.15, this closed subset is flat precisely over the complement of the subvariety of $T \times S$ consisting of those points (t, s) where f does not vanish on an irreducible component of $X_t \times \{s\}$. I.e., $T \times S$ is flat on the subvariety of points (s, t) where the intersecion $X_t \cap Y_s$ is finite.

Since Y is a family of pure-dimensional curves, it is a dense open subset of a hypersurface. If Y is a proper dense open subset of a hypersurface $\bar{Y}$, then the above is true for the family of scheme-theoretic intersections of X_t and $\bar{Y}_s$. The family of scheme-theoretic intersections of X_t and Y_s is a dense open subset of it, and so is flat, since flatness is local on the source. $\square$

Lemma 5.19. *Let $f : X \to Y$ be a flat quasi-finite morphism. Then the function*

$$n : Y \to \mathbb{Z} \qquad y \mapsto \#(f^{-1}(y))$$

is lower semi-continuous, i.e. the lower level sets $\{\ y \ \mid\ \#(f^{-1}(y)) \leq n\ \}$ are closed.

Proof. Follows from (i) of Proposition 15.5.1 of EGA IV.3 [14] and the fact that flat morphisms of finite type are universally open (see EGA IV [14], 2.4.6). $\square$

Lemma 5.20. *Let $f : X \to Y$ be a flat quasi-finite morphism. Then, denoting the fibre over a point $y \in Y$ as X_y, the function*

$$l : Y \to \mathbb{Z} \qquad y \mapsto \dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y})$$

is lower semi-continuous. If f is finite, then l is locally constant.

Proof. Recall that a finite morphism is projective (EGA II 6.1.11 [14]). Thus, if f is finite the lemma follows from the fact that $\dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y})$ is the constant term of the Hilbert polynomial, and the Hilbert polynomial of a flat projective family is locally constant (cf. EGA III 7.9.11).

In the general case, by Zariski's Main theorem f factors as a compostion $\bar{f} \circ \iota$ where $\bar{f} : \bar{X} \to Y$ is finite (and hence projective) and $\iota : X \hookrightarrow \bar{X}$ is an open immersion. Let Z_i be connected components of $\bar{X} \setminus X$. By the previous paragraph the function $y \mapsto H^0((Z_i)_y, \mathcal{O}_{(Z_i)_y})$ is constant on $f(Z_i)$. Therefore those lower level sets $\{\ y \in Y \ \mid\ l(y) \leq n\ \}$ that are properly contained in Y consist of unions of Z_i. $\square$

Lemma 5.21. *Let Y be an irreducible variety and let $f : X \to Y$ be a flat quasi-finite morphism. Let N_1, N_2 be the values of the semi-continuous functions l, n of Lemmas 5.19,5.20 on some dense open subset of Y. Then*

$$\{\ y \in Y \ \mid\ \dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y}) < N_1\ \} \subset \{\ y \in Y \ \mid\ \#f^{-1}(y) < N_2\ \}$$

Proof. Factor f according to Zariski's Main Theorem as the composition of a finite $\bar{f} : \bar{X} \to Y$ and an open immersion $\iota : X \hookrightarrow \bar{X}$. The number $\dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y})$ is constant on $\bar{X}$ by Lemma 5.20. Suppose y is such that $\dim_{k(y)} H^0(X_y, \mathcal{O}_{X_y}) < N_1$, then $f^{-1}(y) \subsetneq \bar{f}^{-1}(y)$ and hence $\# f^{-1}(y) < N_2$. $\square$

Now we can prove the main result of the present subsection describing the behaviour of intersection multiplicities in families of curves. The setting is as follows. We fix a curve M and a point $(P_1, P_2) \in M^2$, with a fixed local coordinate system associated to it. We fix two families of curves $X \subset M^2 \times N \times T$ and $Y \subset M^2 \times S$. In the applications X and Y will be $\mathcal{M}$-definable and $\mathcal{M}$-irreducible, and the first family will be a obtained as a result of "partial composition" of two families of curves (as in Proposition 3.17), or by forming an an affine combination of three families of curves (see Section 3.2). We assume, moreover, that all curves X_t are incident to a fixed point $Q \in M^2 \times N$, and that all curves Y_s are incident to $p(Q)$; in the applications this will be achieved by fixing higher dimensional families and working with subfamilies passing through the point Q. We want to identify those parameters t, s such that $\tau_n(X_t/N, Q) = \tau_n(Y_s, p(Q))$, where p is the projection $p : M^2 \times N \to M^2$. The reason to consider the relative slope and families of curves in $M^2 \times N$ is discussed in Section 3.1 after the Definition 3.15.

In this context there is a natural number, a such that $|X_t \cap Y_s \times N| = a$ for all t, s in a dense open subset of $T \times S$. We will also assume that there exists a natural number n, smallest satisfying that $\tau_n(X_t/M, P) \neq \tau_n(Y_s, P)$ on a dense open subset of $S \times T$. We point out that in the applications, at this stage, we cannot assure that this can be done $\mathcal{M}$-definably. We will circumvent this problem by applying the analysis to each open irreducible component of $S \times T$ separately. We will, therefore, assume that S and T are irreducible.

In the next proposition we show that, in the setting described above, for each pair (t, s) assumed generic in the parameter variety $T \times S$, if a curve from X is "tangent" to a curve from Y at P in the sense that $\tau_n(X_t/M, Q) = \tau_n(Y_s, p(Q))$ then, $|X_t \cap Y_s \times N| < l$. This will allow us to recover tangency $\mathcal{M}$-definably, up to finitely many false positives, which will be enough for our purposes.

Proposition 5.22. *In the setting described above, let n be the minimal number such that for generic t, s*

$$\tau_n(X_t/N, Q) \neq \tau_n(Y_s, p(Q))$$

Consider the family of intersections $X \times_{M^2 \times N} Y \times N \subset M^2 \times N \times T \times S$, and assume that $X \times_{M^2 \times N} Y \times N$ is proper over $T \times S$. Then there exist dense open $T' \subset T$ and $S' \subset S$ such that

$$\{(t, s) \in T' \times S' : \tau_n(X_t/M, Q) = \tau_n(Y_s, p(Q))\} \subseteq \{(t, s) : \#(X_t \cap Y_s \times N) < a\}.$$

Proof. By fact Fact 5.14 there exist $T' \subset T$, $S' \subseteq S$ dense open such that X is flat over T' and Y is flat over S'. Let $X \times_{M^2 \times N} Y \times$ be the family of scheme-theoretic intersections of X_t and Y_s (with possibly non-reduced structure), it has a natural morphism to $T' \times S'$, and let $U \subset T' \times S'$ be the set of (t,s) such that $(X \times_{M^2 \times N} Y)_{(t,s)}$ is 0-dimensional. Let Z denote the the preimage of U in $X \times_{M^2 \times N} Y$.

The variety Z is flat over U by Lemma 5.18 and the definition of U. Denote the projection $Z \to U$ by p_U. Let Z_0 be the irreducible component of Z supported at $\{P\} \times U$, and let W be the complement of Z_0 in Z.

Below we will use subscripts as follows to denote *scheme-theoretic* fibres: $Z_{t,s} = Z \otimes k(t,s)$ where $k(t,s)$ is the residue field of $(t,s) \in U$.

In this notation, we have to show that

$$\{ \ (t,s) \in U \ \mid \ \tau_n(X_t/N, Q) = \tau_n(Y_s, p(Q)) \ \} \subseteq \{ \ (t,s) \in U \ \mid \ \#Z_{t,s} < a \ \}$$

where $a = \#Z_{t,s}$ for $(t,s) \in U$ generic.

Let us first prove this statement when the morphism $Z \to U$ is finite. Then the number $\dim H^0(Z_{t,s}, \mathcal{O}_{Z_{t,s}})$ is constant for all $(t,s) \in U$ by Lemma 5.20. Note that

$$\begin{aligned}&\dim H^0(\{Q\}, \mathcal{O}_{Z_{t,s}}) + \dim H^0(\{Q\}, \mathcal{O}_{Z_{t,s}}) = \dim H^0(Z, \mathcal{O}_{Z_{t,s}}) = b \text{ and} \\ &\#Z_{t,s} = \#W_{t,s} - 1\end{aligned}$$

For generic t,s, if we identify $Z_{t,s}$ with a closed subscheme of $M^2 \times N$

$$\dim H^0(\{Q\}, \mathcal{O}_{Z_{t,s}}) = n$$

therefore $\dim H^0(W_{t,s}, \mathcal{O}_{W_{t,s}}) = n - b$.

If t,s is such that $\tau_n(X_t/N, Q) = \tau_n(Y_s, p(Q))$ then

$$\dim H^0(\{Q\}, \mathcal{O}_{Z_{t,s}}) > n$$

and therefore $\dim H^0(W_{t,s}, \mathcal{O}_{W_{t,s}}) < n - b$.

Applying Lemma 5.21 to W we get

$$\{ \ (t,s) \in U \ \mid \ \dim H^0(W, \mathcal{O}_{W_{t,s}} < n - b \ \} \subseteq \{ \ (t,s) \in U \ \mid \ \#W < a - 1 \ \}.$$

The latter set is the same as $\{ \ (t,s) \in U \ \mid \ \#Z < a \ \}$ which yields the statement of the Propsition.

If $Z \to U$ is not finite then compactify it using Zariski's Main Theorem: find a finite morphism $\bar{Z} \to U$ such that Z is an open subscheme of Z, let $\overline{W}$ be the complement of Z_0 in $\bar{Z}$. Note that $a = \#\bar{Z}_{t,s}, b = \dim H^0(\overline{W}, \mathcal{O}_{\overline{W}_{t,s}})$ for t,s generic, and we have just shown that

$$\{ \ (t,s) \in U \ \mid \ \dim H^0(\overline{W}, \mathcal{O}_{\overline{W}_{t,s}} < n - b \ \} \subseteq \{ \ (t,s) \in U \ \mid \ \#\bar{Z} < a \ \}.$$

Observe that

$$\begin{aligned}&\{ \ (t,s) \in U \ \mid \ \dim H^0(W, \mathcal{O}_{W_{t,s}} < n - b \ \} = \\ &\qquad = \{ \ (t,s) \in U \ \mid \ \dim H^0(\overline{W}, \mathcal{O}_{\overline{W}_{t,s}} < n - b \ \} \cup p_U(\bar{Z} \setminus Z) \\ &\{ \ (t,s) \in U \ \mid \ \#\bar{Z} < a \ \} = \{ \ (t,s) \in U \ \mid \ \#Z < a \ \} \cup p_U(\bar{Z} \setminus Z)\end{aligned}$$

which implies

$$\{ \ (t,s) \in U \ \mid \ \dim H^0(W, \mathcal{O}_{W_{t,s}} < n - b \ \} \subseteq \{ \ (t,s) \in U \ \mid \ \#Z < a \ \}.$$

which in turn implies the statement of the Proposition. $\square$

5.6 Interpreting a one-dimensional group

Our strategy is to find a point Q on the diagonal of $M \times M$ with a definable one-dimensional family of curves incident to Q such that the associated n-th order slopes at Q for some n form a one-dimensional subset of $\mathrm{Aut}(\operatorname{Spec} k[\varepsilon]/(\varepsilon^{n+1}))$, and then use this family to construct a one-dimensional group configuration.

Theorem 5.23. *Let $\mathcal{M} = (M, X)$ be a non-locally modular reduct of an algebraic curve M over an algebraically closed field, with $X \to T$ a 2-dimensional faithful family of pure-dimensional curves. Then M interprets a one-dimensional group.*

Proof. Let $P = (Q, Q) \in M^2$ be a point such that for some n the image of $\tau^P : T^P \to \mathrm{Aut}(\operatorname{Spec} k[\varepsilon]/(\varepsilon^{n+1}))$, $t \mapsto \tau_n(X_t, P)$ is one-dimensional. Such a point is guaranteed to exist by Lemmas 5.3 and 5.4 (or Lemmas 5.9 and 5.8) .

Pick a one-dimensional $\mathcal{M}$-stationary (i.e. of $\mathcal{M}$-Morley deree 1) component of T^P, call it W, and let t, s be independent realisations of the generic type of W. We may also assume that W is irreducible, or otherwise run the argument below for one of the irreducible components of W.

Let u be a point of W such that

$$\tau_n(X_u, P) = \tau_n(X_s, P)\tau(X_t, P).$$

Such u exists since t and s are generic in a one-dimensional subgroup of $\mathrm{Aut}(k[\varepsilon]/(\varepsilon^{n+1})$ by Lemma 5.3 or 5.9, so the right-hand side value is generic in the same subgroup, and therefore is in the image of τ^P.

Let us show that u is algebraic over t, s. Denote the natural projections on factors $M^3 \to M, M^3 \to M^2$ as $p_1, p_2, p_3, p_{12}, p_{23}, p_{13}$, respectively. Denote $P' = (Q, Q, Q) \in M^3$, and consider the family $X \times_{p_2, p_1} X \subset M^3 \times W \times W$ where a fibre over a point $(u, v) \in W \times W$ is $X_u \times_{p_2} X_v$. By Proposition 3.13 $\tau_n(p_{13}^{-1}(X_w)/M, P') = \tau_n(X_w, P)$ for any $w \in W$. By Proposition 3.17

$$\tau_n(X_t \times_{p_2, p_1} X_s/M, P') = \tau_n(X_t, P)\tau_n(X_s, P)$$

(the relative slope is with respect to the second factor M in M^3). By Proposition 5.22 the definable set

$$\{ \ w \in W \ \mid \ \#(X_t \times_{p_2} X_s \cap p_{13}^{-1}(X_w)) < N \ \}$$

where $N = \#(X_t \cap X_s)$, contains the set of parameters $v \in W$ such that $\tau_n(X_w, P) = \tau_n(X_t, P)\tau_n(X_s, P)$.

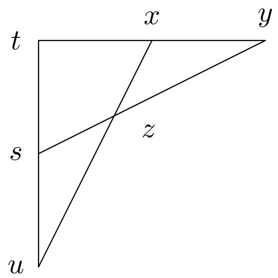
In a similar vein, let y be a point of W independent from t and let x, z be points of W such that

$$\tau_n(X_x, P) = \tau_n(X_t, P)\tau_n(X_y, P), \text{ and } \tau_n(X_z, P) = \tau_n(X_s, P)^{-1}\tau(X_y, P)$$

Then

$$\tau_n(X_u, P) = \tau_n(X_x, P)\tau(X_z, P)$$

By the same argument as above, x is algebraic over y and t, and z is algebraic over s and y, in the sense of $\mathcal{M}$. This implies that



is a group configuration.

Applying Fact 5.11 we obtain a one-dimensional group interpreted in $\mathcal{M}$. $\square$

5.7 Families of curves in reducts of one-dimensional groups

In the previous subsection we showed that a non-locally modular reduct of the full Zariski structure on an algebraic curve M over an algebraically closed field interprets a strongly minimal group, call it H. The group H is interpretable in an algebraically closed field and therefore is definably isomorphic to a 1-dimensional algebraic group by a model-theoretic version of a theorem of Weil on birational group laws (see [33], [39], [38]). Since $\mathcal{M}$ is non-locally modular, so is H (with the full structure induced from $\mathcal{M}$). Thus, by replacing M with H, we may assume that $\mathcal{M}$ expands a group. In this setting our goal is to construct a second group configuration, which will allow us to reconstruct the field in $\mathcal{M}$.

Let H be a connected one-dimensional algerbaic group, and let $Z \subset H^2$ be an irreducible one-dimensional subset that is not a coset (such sets are components of definable sets that witness non-local modularity in reducts of groups by a theorem of Hrushovski and Pillay [24]). We prove that shifts of Z incident to the identity of H^2 have infinitely many distinct first order slopes at the identity. This will be necessary to construct a two-dimensional group configuration. In characteristic 0, an alternative way to obtain a family of curves with enough distinct first-order slopes is to consider a two-dimensional family of curves in H^2 and apply Lemma 5.3, then shift the family obtained to the identity. Note that the proof of following lemma is characteristic-free.

Lemma 5.24. *Let A be an elliptic curve and let Z be a closed one-dimensional irreducible subset of $G = A^2$. The tangent spaces to G at any point g can be identified with T_0H via the isomorphism $d\lambda_g : T_0G \to T_gG$, where $\lambda_g(x) = g \cdot x$. Suppose that for any $z \in Z$ the tangent space $T_zZ \subset T_0G$ is constant. Then Z is a coset of a closed subgroup of G.*

Proof. Since Z is a projective curve with a trivial tangent bundle, it is an elliptic curve itself. Since any algebraic variety morphism between Abelian varieties with finite fibres and preserving the identity automatically preserves the group structure by Rigidity Theorem, Z is a coset of an Abelian subvariety of G. $\square$

Let $H = \mathbb{G}_a$ or $H = \mathbb{G}_m$. Let Z be a locally closed one-dimensional irreducible subset of $G = H^2$ such that the restriction to Z of the projection on the first factor H is étale (possibly, after removing finitely many points). For any $x \in Z$ denote the translate

$$Y_x := Z \cdot x^{-1} = \{ \ u \cdot x^{-1} \in G \ \mid \ u \in Z \ \}$$

This defines a family $Y \to Z$ of curves incident to (e, e), parametrized by Z.

Lemma 5.25. *In the setting as above, assume H is defined over a field of characteristic 0. Fix a local coordinate system at $e \in H$ so that the slope of any curve incident to $O = (e, e)$ is well-defined. Suppose that the slope $\tau_1(Y_x, O)$ is constant for x in an open neighbourhood of O. Then Z coincides with an open subset of a closed subgroup of G.*

Proof. Without loss of generality we may assume that $O \in Z$, .

Let $H = \mathbb{G}_m$, and let Z be cut out by an equation $h(x, y) = 0$. Then $Y_{(x_0,y_0)}$ is cut out by the equation $h(x \cdot x_0, y \cdot y_0)$, and

$$\tau_1(Y_{(x_0,y_0)}, O) = \left. \frac{\partial_x h(x \cdot x_0, y \cdot y_0)}{\partial_y (x \cdot x_0, y \cdot y_0)} \right|_{x=1,y=1} = \frac{\partial_x h(x_0, y_0) \cdot x_0}{\partial_y h(x_0, y_0) \cdot y_0}$$

Therefore, if f is the expansion of Z into formal power series at the identity then by Proposition 4.4 $\dfrac{\partial_x h(x, f)}{\partial_y h(x, f)}$ is f', the formal derivative of f. If $\tau_1(Y_x, O)$ is constant for x is the neighbourhood of (e, e), then for some $a \in k$ the formal power series f satisfies the differential equation

$$f' \cdot \frac{x + 1}{f + 1} = a$$

Similarly, for the additive group ($H = \mathbb{G}_a$)

$$\tau_1(Y_{(x_0,y_0)}, O) = \left. \frac{\partial_x h(x + x_0, y + y_0)}{\partial_y h(x + x_0, y + y_0)} \right|_{x=0,y=0} = \frac{\partial h_x(x_0, y_0)}{\partial_y h(x_0, y_0)}$$

and the corresponding differential equation is

$$f' = a$$

The series $f = ax$ satisfies the second equation, and by Lemma 4.1 this is the only solution with zero constant term. It follows that Z is defined by the same equation, and so is a subgroup of $\mathbb{G}_a \times \mathbb{G}_a$.

In case $H = \mathbb{G}_m$ it follows from Lemma 4.6 that $f = (x + 1)^a - 1$ is the unique formal power series solution of the differential equation

$$y' = a \frac{f + 1}{x + 1}$$

and is only integral over $k[x]$ a is rational, in which case f is a formal power series expansion at $(1, 1)$ of an irreducible component of a curve defined by an equation $y^n = x^m$. But all such curves are subgroups of $\mathbb{G}_m \times \mathbb{G}_m$. $\square$

Lemma 5.26. *Let f, g be formal power series expansions at $(0,0)$ of curves X, Y in $\mathbb{G}_a \times \mathbb{G}_a$. Then the formal power series expansion of $X + Y$ (where $+$ is understood in the sense of Definion 3.18) is $f + g$.*

Proof. Follows from Proposition 3.9 and the definition the group law on $\mathbb{G}_a$. $\square$

Lemma 5.27. *In the same setting, suppose that $H = \mathbb{G}_a$ is defined over a field of positive characteristic. Suppose that Z is not a coset of a subgroup of G and that the projection of Z on the first factor H is étale on some dense open set. Then there exists a family of curves incident to a point $Q \in \Delta_M$ such that the set of first order slopes at Q almost coincides with* $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$.

Proof. Without loss of generality we may assume that $(0,0) \in Z$. Let f be the formal power series expansion of Z at $(0,0)$. Then by the same argument as in the proof of Lemma 5.25 f satisfies the differential equation

$$f' = a$$

The solutions of the differential equation are of the form $ax + g$, where g belongs to $\mathfrak{m}^p$ where $\mathfrak{m}$ is the maximal ideal $\mathfrak{m}$ of $k[[x]]$, by direct observation. If $f = ax$ then Z is a coset, contradicting our assumption. The since Z is not a coset the formal power series expansion of some of its shifts, $Z' = Z \cdot (x_0, y_0)$ would have a formal power series expansion of the same form, let us say $ax + g'$, where $g \in \mathfrak{m}^p$.

By Proposition 3.22 (see also Remark 3.2) the formal power series expansion of $Z - Z'$ is $g - g'$. Let n be the largest integer such that $g - h \in \mathfrak{m}^{p^n}$. In a similar manner consider two other shifts of Z, call then W and W', and obtain $W - W'$ with formal power series expansion $h - h'$ in $\mathfrak{m}^{p^n}$

Then by Lemma 5.5 $(Z - Z') \circ (W - W')^{-1}$ has power series expansion in $\mathfrak{m}^{p^l}$, where $l < n$. Repeating this procedure several times we will obtain a definable curve that has a power series expansion in $\mathfrak{m}$ but not in $\mathfrak{m}^p$. Since it will not be of the form ax, the slopes at $(0,0)$ of its shifts will take infinitely many values. $\square$

Lemma 5.28. *Let f, g be formal power series expansions at $(1,1)$ of curves X, Y in $\mathbb{G}_m \times \mathbb{G}_m$. Then the formal power series expansion of $X + Y$ (where $+$ is understood in the sense of Definion 3.18) is $f + g + fg$.*

Proof. Follows from Proposition 3.9 and the definition the group law on $\mathbb{G}_m$. $\square$

Lemma 5.29. *Let f be formal power series expansion at $(1,1)$ of a curve X in $\mathbb{G}_m \times \mathbb{G}_m$. Then for any $x_0 \in \mathbb{G}_m$ there exists y_0 such that $(x_0, y_0) \in X$ and the formal power series expansion of a shift $X \cdot (x_0, y_0)^{-1}$ is of the form $f(x + x_0)$.*

Proof. Follows from Proposition 3.9. $\square$

Lemma 5.30. *In the same setting, suppose that $H = \mathbb{G}_m$ is defined over a field of positive characteristic. Suppose that Z is not a coset of a subgroup of G and that the projection of Z on the first factor H is étale on some dense open set. Then there* *exists a family of curves incident to a point $Q \in \Delta_M$ such that the set of first order slopes at Q almost coincides with* $\mathrm{Aut}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$.

Proof. Without loss of generality we may assume that $(0,0) \in Z$. Let f be the formal power series expansion of Z at $(0,0)$. Then by the same argument as in the proof of Lemma 5.25 f satisfies the differential equation

$$(x+1)^a = \sum_{k=1}^{\infty} \binom{a}{k} x^k$$

for some element $a \in k$.

By Lemma 4.7 the solutions of the differential equation are of the form $a(1+x)^a + g$ where $g \in k[[x^{p^n}]]$. If $f = a(1+x)^a$ for a rational then Z is the curve defined by $y^m = x^n$ where $a = \dfrac{m}{n}$. But this is a coset, contradicting our assumptions.

The solutions of the differential equation are of the form $ax+g$, where g belongs to $\mathfrak{m}^p$ where $\mathfrak{m}$ is the maximal ideal $\mathfrak{m}$ of $k[[x]]$, by direct observation. If $f = ax$ then Z is a coset, contradicting our assumption. The since Z is not a coset the formal power series expansion of some of its shifts, $Z' = Z \cdot (x_0, y_0)$ would have a formal power series expansion of the same form, let us say $ax + g'$, where $g \in \mathfrak{m}^p$.

By Proposition 3.22 (see also Remark 3.2) the formal power series expansion of $Z - Z'$ is $g - g'$. Let n be the largest integer such that $g - h \in \mathfrak{m}^{p^n}$. In a similar manner consider two other shifts of Z, call then W and W', and obtain $W - W'$ with formal power series expansion $h - h'$ in $\mathfrak{m}^{p^n}$

Then by Lemma 5.5 $(Z - Z') \circ (W - W')^{-1}$ has power series exponsion in $\mathfrak{m}^{p^l}$, where $l < n$. Repeating this procedure several times we will obtain a definable curve that has a power series expansion in $\mathfrak{m}$ but not in $\mathfrak{m}^p$. Since it will not be of the form ax, the slopes at $(0,0)$ of its shifts will take infinitely many values. □

5.8 Interpreting the field

With the preparations made in the previous section we can now show that a non-locally modular reduct of one-dimensional algebraic group interprets a field. This strengthens the main result of [28].

Lemma 5.31. *Let $M = (G, \cdot, \dots)$ be a reduct of a one-dimensional connected algebraic group $(G, \cdot)$ with identity e. Suppose that $Z \subset M^2$ is a one-dimensional definable subset which is not a Boolean combination of closed subgroups of G^2. Fix some local coordinate system at the identity of G. Consider the map*

$$\tau : Z \to \mathrm{End}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)),\ t \mapsto \tau_1(Z \cdot t^{-1}, (e,e))$$

Then the image of τ in $\mathrm{End}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2)) \cong \mathbb{A}^1$ *is one-dimensional.*

Proof. By quantifier elimination, Z is of the form $\bigcup Z_i \setminus W_i$ where Z_i, W_i are closed subsets of G. It follows from Lemma 5.25, 5.24, 5.27 or 5.30 (depending on G and the field characteristic) that τ is non-constant on Z_i, therefore the image of any of Z_i is one-dimensional in $\mathrm{End}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$. Therefore $\mathrm{Im}\,\tau$ is one-dimensional in $\mathrm{End}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$. □

Theorem 5.32. *Let $M = (G, \cdot, \ldots)$ be a reduct of one-dimensional algebraic group $(G, \cdot)$ over an algebraically closed field of any characteristic. Suppose that $Z \subset M^2$ is a one-dimensional definable subset which is not a union of cosets of subgroups of G^2. Then M interprets a field.*

Proof. For any $x \in Z$ denote the translate

$$Y_x := Z \cdot x^{-1} = \{\ u \cdot x^{-1} \in G \ \mid\ u \in Z\ \}$$

and denote $P = (e, e)$. We may without loss of generality assume that Z is irreducible, or otherwise run the argument below for one of the irreducible components of Z.

Take $a_1, a_2, b_1, b_1, u \in Z$ generic and pairwise independent. Let c_1, c_2 be such that (all slopes are taken at P)

$$\begin{aligned}\tau_1(Y_{c_1}) &= \tau_1(Y_{a_1})\tau_1(Y_{b_1})\\ \tau_1(Y_{c_2}) &= \tau_1(Y_{a_2})\tau_1(Y_{b_1}) + \tau(Y_{b_2})\end{aligned}$$

Since the image of the function $x \mapsto \tau_1(Y_x, P)$ for x ranging in Z is one-dimensional by Lemma 5.31, the values of expressions in the right-hand-side are generic in $\mathrm{End}(\mathrm{Spec}\, k[\varepsilon]/(\varepsilon^2))$ as parameters vary. Therefore $\tau_1(Y_{a_1})\tau_1(Y_{b_1})$ and $\tau_1(Y_{a_2})\tau_1(Y_{b_1})+\tau(Y_{b_2})$ are generic, and c_1, c_2 with such slopes can be picked in Z.

Let z, v be such that

$$\begin{aligned}\tau_1(Y_z) &= \tau_1(Y_{a_1})\tau_1(Y_u) + \tau_1(Y_{a_2})\\ \tau_1(Y_v) &= \tau_1(Y_{b_1})^{-1}\tau_1(Y_u) - \tau_1(Y_{b_2})\end{aligned}$$

By a similar reasoning, z, v are generic. It also follows form the way c_1, c_2, z, v were defined that

$$\tau_1(Y_z) = \tau_1(Y_{c_1})\tau_1(Y_v) + \tau_1(Y_{c_2})$$

We will now show that (c_1, c_2) is algebraic over (a_1, a_2) and (b_1, b_2) in the sense of $\mathcal{M}$. Denote $P' = (e, e, e)$, $P'' = (e, e, e, e, e)$. By Propositions 3.17, 3.24, 3.20 and Lemma 3.13

$$\begin{aligned}\tau_1(Y_{a_1} \times_{p_2} Y_{b_1}/G, P') &= \tau_1(Y_{a_1}, P)\tau_1(Y_{b_1}, P),\\ \tau_1(\mathrm{Aff}(Y_{a_2}, Y_{b_1}, Y_{b_2})/G^3, P'') &= \tau_1(Y_{a_2}, P)\tau_1(Y_{b_1}, P) + \tau_1(Y_{b_2}, P)\end{aligned}$$

Let $N_1 = \#(Y_{c_1} \cap Y_{a_1} \circ Y_{b_1}), N_2 = \#(Y_{c_2} \cap \mathrm{Aff}(Y_{a_2}, Y_{b_1}, Y_{b_2}))$ for $a_1, a_2,\ b_1, b_2,$ $c_1, c_2 \in Z$ generic and independent. Since the number of intersections is a first-order property, it does not matter what particular parameters a_i, b_i, c_i we take as long as they are generic and independent. By Proposition 5.22 the $\mathcal{M}$-definable set

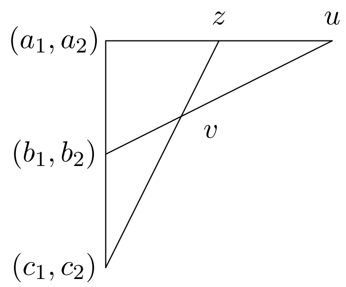
$$\{\ w \in Z \ \mid\ \#(p_{13}^{-1}(Y_w) \cap Y_{a_1} \times_{p_2} Y_{b_1}) < N_1\ \}$$

contains c_1 and by definition of N_1 is finite. By Proposition 5.22 again the $\mathcal{M}$-definable set

$$\{\ w_2 \in Z \ \mid\ \#(p_{15}^{-1}(Y_w) \cap \mathrm{Aff}(Y_{a_2}, Y_{b_1}, Y_{b_2})) < N_2\ \}$$

contains c_2 and by definition of N_2 is finite. Arguing in a similar fashion, by application of Proposition 5.22, we deduce that c_1 and c_2 are algebraic over z, v.

It follows from the discussion above that



constitutes a group configuration. Therefore, by Fact 5.11 there exists a two-dimensional group definable in $(G, \cdot, Z)$ that acts transitively on a one-dimensional set. One checks that the conditions of the Fact 5.12 are verified as well. By Fact 5.13, the group G is isomorphic to the affine group $\mathbb{G}_a(k) \rtimes \mathbb{G}_m(k)$ of an infinite definable field k. $\square$

Theorem 5.33. *Let (M, X) be a non-locally modular reduct of an algebraic curve M over an algebraically closed field M of characteristic 0, with $X \to T$ a normal family of pure-dimensional curves. Then M interprets a field.*

Proof. Conjunction of Theorem 5.23 and Theorem 5.32. $\square$

6 Getting rid of zero-dimensional components

In the previous section we defined a group and then a field in $\mathcal{M}$ under the assumption that the generic fibre of X has no 0-dimensional connected components. The goal of this section is to justify this assumption. We do so by constructing in $\mathcal{M}$ a 2-dimensional faithful family with a generic elemnt of the family a pure-dimensional one-dimensional set.

In this section, as it is important to distinguish between genericity in the sense of the reduct, $\mathcal{M}$, and in the sense of the full Zariski structure on M, we use *reduct generic* for the former, and *generic* or *field generic* for the latter. Note that being reduct generic implies being generic but not, a priori, vice versa. Similarly, $\mathrm{acl}_{\mathcal{M}}(\cdot)$ denotes the (model theoretic) algebraic-closure operator in the sense of $\mathcal{M}$ while $\mathrm{acl}(\cdot)$ will denote the field theoretic algebraic closure.

Drawing upon a tradition in model-theoretic literature, one-dimensional definable sets in M^2 will be refereed to as "plane curves".

6.1 Preliminaries

For a 1-dimensional definable Z set denote Z^1 the union of its 1-dimensional connected components, Z^0 the union of its 0-dimensional components, $\overline{Z^1}$ the closure of Z^1 and $\mathrm{Fr}(Z) = \overline{Z^1} \setminus Z^1$. The same notation will apply for families: if $Y \to S$ is a family of one-dimensional sets, then e.g. $\mathrm{Fr}(Y) \to S$ is the family that consists of frontiers of elements of the family Y.

The results of this section use only basic intersection theory and are, to a large extend, independent from previous sections. Our main results (stated in somewhat greater generality than we actually need) is:

Theorem 6.1. *Let M be an algebraic curve over an algebraically closed field k. Let $\mathcal{M}$ be a strongly minimal non-locally modular reduct of the k-induced structure on M. Let $S \subseteq M^2$ be an $\mathcal{M}$-definable strongly minimal set. Then $\mathrm{Fr}(S) \subseteq \mathrm{acl}_{\mathcal{M}}([S])$, where Fr is taken with respect to the Zariski topology and $[S]$ is a canonical parameter for S.*

This theorem follows from the following, somewhat more technical result:

Proposition 6.2. *Let $\mathcal{M}$ be as above. Let $X \to T$ be a faithful $\mathcal{M}$-definable family of curves with $\dim(T) \geq 3$. Assume, moreover, that if $t \in T$ is generic and P is a $\emptyset$-dimensional component of X_t then P is generic over $\emptyset$ and $S \notin \mathrm{acl}(t)$. Then there exists an $\mathcal{M}$-definable family of plane curves $\widetilde{X} \to T$ such that for all $t \in T$:*

1. *$X_t \sim \widetilde{X}_t$.*
2. *$\widetilde{X}_t$ is pure-dimensional.*

Our strategy is as follows. First, we show the existence of an $\mathcal{M}$-definable family of plane curves $X \to T$ satisfying all the technical assumptions of the previous proposition. Fixing $s \in T$ field generic and $P \in X_s^0$ our assumptions assure that any generic independent $t, u \in T^P$ are, in fact, generic independent over $\emptyset$. Assuming, as we may, that T is $\mathcal{M}$-irreducible it follows that $\#(X_t \cap X_u)$ is independent of the choice of t, u. Moreover, we show that $\#(X_t \cap X_u) = \#(\overline{X_t^1} \cap \overline{X_u^1})$. Assuming towards a contradiction that $P \notin \mathrm{acl}_{\mathcal{M}}(s)$ we get immediately that $\#(X_t \cap X_u) = \#(X_t \cap X_s)$, implying – as P is isolated in X_s – that $\#(\overline{X_t^1} \cap \overline{X_u^1}) > \#(\overline{X_t^1} \cap \overline{X_s^1})$. We then apply basic intersection theory to show that, as t was arbitrary, this leads to a contradiction.

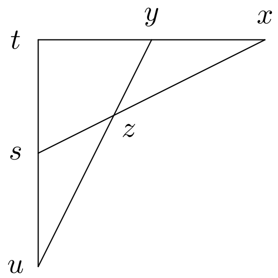
We start by addressing the technical requirements Proposition 6.2. This will require a few steps. The first result we need is well known to the experts, and goes back to Hrushovski's PhD thesis and Buechler's works from the early 1980s (see e.g., [19, p.88]). As we were unable to find an explicit reference, we give a brief overview of the proof:

Lemma 6.3. *If $\mathcal{M}$ is a non-locally modular strongly minimal set there is no bound on the dimension of definable families of plane curves.*

Proof. Clearly, if a field is interpretable in $\mathcal{M}$ then the lemma is true. It will suffice, therefore, to prove that if Z is an n-dimensional ($n \geq 2$) faithful family then either $\dim(Z \circ Z) > n$ or a field is interpretable in $\mathcal{M}$.

Let S' be a parameter set for the normalisation of $Z \circ Z$. We prove that if $\dim S' = n$ then $\mathcal{M}$ interprets an infinite field. Our assumption implies that there exists an $\mathcal{M}$-definable finite-to-finite correspondence $\mu : S' \vdash S$. By definition, there is also an $\mathcal{M}$-definable function $p : S \times S \to S'$ (defined by the requirement that $Z_t \circ Z_s$ is – up to a finite set – the curve defined by $p(t,s)$).

Let t, s be reduct generic independent elements of S and $u \in \mu(p(t,s))$. Let x be a reduct generic point of M, and $y \in Z_t(x)$, $z \in Z_s^{-1}(x)$. Construct the following configuration:



As $y \in Z_t \circ Z_s(z)$ it follows that y is inter-$\mathcal{M}$-algebraic with x over u, implying that the above is, in fact, a group configuration. By Fact 5.11 there exists an $\mathcal{M}$-definable group G of dimension n acting definably on a definable set X of dimension 1. The canonical base of $\mathrm{tp}(y, x/t)$ is inter-algebraic with t by faithfulness of the family Z, and similarly for the canonical bases of $\mathrm{tp}(z, x/s)$ and $\mathrm{tp}(z, x, /u)$. Therefore by Fact 5.12 the action of G on X is faithful. By Fact 5.13 there exists a field definable in $\mathcal{M}$. $\square$

In the discussion that follows canonical parameters (see Sub-section 2.2) will play an important role. To simplify the discussion, we will denote, given a definable set S, its canonical parameter $[S]$. As explained in the introduction $[S]$ is not uniquely determined, but $\mathrm{acl}([S])$ is, which will suffice for our purposes. Formally, we have to distinguish between $\mathcal{M}$-canonical parameters and field-canonical parameters. In practice, and in order to overload the notation, we will always use $\mathcal{M}$-canonical parameters (as long as the definable sets in questions are $\mathcal{M}$-definable, of course). Note, and this will be used implicitly throughout, that if $X \to T$ is a faithful $\emptyset$-definable family (of plane curves) then t is a canonical parameter for X_t.

Now we turn to the genericity of isolated points (in the sense of the assumptions of Proposition 6.2):

Lemma 6.4. *Let $S \subseteq M^2$ be an $\mathcal{M}$-definable curve with $[S] \notin \mathrm{acl}(\emptyset)$. Let $Z \to T$ be an $\mathcal{M}$-irreducible n-dimensional $\mathcal{M}$-definable faithful family of plane curves for some $n \geq 2$. Then there exists an $\mathcal{M}$-definable family of plane curves $Z' \to T$ with $Z'_t \sim Z_t$ for all $t \in T$ and such that for any $t \in T$ generic over $[S]$ any point in $(S \circ Z'_t)^0$ is field generic.*

Proof. We may assume that S is $\mathcal{M}$-irreducible (otherwise, repeat the argument for each $\mathcal{M}$-component of S separately) and projecting dominantly on both coordinates. We may also assume that there is no point $P \in M^2$ incident to Z_t for all generic t. We may assume that if $(p,q) \in S^0$ then $p,q \notin \operatorname{acl}(\emptyset)$, as if that is not the case we can replace Z with $Z \setminus (\{p\} \times M \times T \cup M \times \{q\} \times T)$.

Now note that if $(p,q) \in S^0$ then $p,q \in \operatorname{acl}([S])$. Thus, if $t \in T$ is generic over $[S]$ then any point of the form (p,r), where $(q,r) \in Z_t$, is generic over $\emptyset$. Indeed, since q is generic over $\emptyset$ and t is generic over q any point $(q,r) \in Z_t$ is generic in Z_t, therefore generic over $\emptyset$.

If for some $(p,q) \in Z_t^0$, say, $p \in \operatorname{acl}(\emptyset)$ then, by genericity of t and the $\mathcal{M}$-irreducibility of $Z \to T$, for all $t' \in T$ generic we have that $(p,q') \in Z_{t'}^0$ for some q'. Similarly if $q \in \operatorname{acl}(0)$. Let $\{p_1,\dots,p_k\} \subseteq \operatorname{acl}(\emptyset)$ such that $(p_i,q) \in Z_t^0$ for some $q \in M$. Similarly, define $\{q_1,\dots,q_r\}$. Setting $Z' \to T$ by defining

$$Z' = Z_t \setminus \bigcup_{i=1}^{k} \{p_i\} \times M \times T \cup \bigcup_{i=1}^{r} M \times \{q_i\} \times T$$

we may assume that for generic $t \in T$ and any $(p,q) \in Z_t^0$ both p and q are generic (not necessarily independent) over $\emptyset$.

Under these assumptions, if $(p,q) \in Z_t^0$ for $t \in T$ generic over $[S]$ then any point (r,q) for $r \in M$ such that $(r,p) \in S$ is generic over $\emptyset$. Indeed, as $[S] \notin \operatorname{acl}(\emptyset)$ and $[S]$ is independent from t over $\emptyset$ (by symmetry) we get that $r \notin \operatorname{acl}(p,q)$ and by exchange $q \notin \operatorname{acl}(r,p)$. So $\dim(p,q,r) = 3$.

As $(S \circ Z_t)^0 \subseteq S^0 \circ Z_t \cup S \circ Z_t^0$, the conclusion of the lemma follows. □

We may now conclude:

Corollary 6.5. *There exists an $\mathcal{M}$-definable family of plane curves $X \to T$ satisfying the assumptions of Proposition 6.2.*

Proof. Fix $Z \to T$ a faithful $\mathcal{M}$-definable family of plane curves of dimension at least 3, as provided by non local modularity. Fix a generic Z_{t_0} in that family. By the previous lemma, we can find $Z' \to T$ of the same dimension such that $X := Z_{t_0} \circ Z'$ has the desired properties. □

Notation From now on we fix, once and for all, a family $X \to T$ satisfying the assumptions of Proposition 6.2. We will assume that there is no $P \in M^2$ such that $P \in X_t$ for all generic $t \in T$.

Our aim is to use intersection theory in order to identify the isolated components of X_t for $t \in T$ generic. Our setting, however, only allows us direct access to global intersection properties (such as the number of geometric intersection points of two curves), and for such global phenomena the existence of isolated points, frontier points and other local obstructions of similar flavour, may interfere with the geometric argument. We now turn to studying the nature of these possible obstructions.

Definition 6.6. *Let $X \to T$ be a definable family of plane curves, $P \in M^2$ any point. A point $Q \neq P$ is* P-indistinguishable *(with respect to X) if $T^P \sim T^Q$. The point Q is* frontier-P-indistinguishable *(with respect to a field generic type p extending X) if $Q \in \mathrm{Fr}(X_t)$ for all (field) generic $t \in T^P$ such that $t \models p$.*

Note that for an $\mathcal{M}$-definable family Y of plane curves the property of being indistinguishable with respect to Y is $\mathcal{M}$-definable, while the property of being frontier indistinguishable is, a priori, only definable in the full Zariski structure on M.

Remark. In the definition of frontier indistinguishable points (and in all further references to frontier point in the present section) we have intentionally omitted any clear reference to the topological space where this frontier is computed. In the algebro-geometric context of the present text this has no importance. In other contexts where one may consider generalising the results of this section the main requirement to keep in mind is that the frontier of a plane curve be finite.

Lemma 6.7. *Let $X \to T$ be as above. Let $t, u \in T$ be field generic independent over $\emptyset$ (satisfying the same field-type over $\emptyset$). Then*

$$X_t \cap X_u = \overline{X_t^1} \cap \overline{X_u^1} \setminus C$$

Where C is the set of frontier P-indistinguishable points for some (equivalently, any) $P \in X_t \cap X_u$.

Proof. Let $P \in X_t \cap X_u$ be any point. By faithfulness of the family, $P \in \mathrm{acl}_{\mathcal{M}}(t, u)$. Since u is independent of t over $\emptyset$ we immediately get that P is field generic in X_t (otherwise $P \in X_s$ for all generic $s \in T$, and by choice of $X \to T$ no such points exist) and by symmetry P is also generic in X_u. Since P was arbitrary $X_t \cap X_s \subseteq \overline{X_t^1} \cap \overline{X_u^1}$, with the desired conclusion. More precisely,

$$2\dim(T) = \dim(P, t, u) = \dim(u) + \dim(P/u) + \dim(t/P, u) = \dim(T) + 1 + \dim(t/P, u)$$

implying that u, t are independent generics in T^P, so that – by definition of frontier indistinguishable points – $C \cap X_t \cap X_s = \emptyset$. $\square$

In the above lemma we analysed the intersection of two generic independent curves. In the application, however, our main concern will be in the situation where $s \in T$ is generic but $X_u \in T^P$ for some $P \in X_s^0$. We prove:

Lemma 6.8. *Let $s \in T$ be generic, $P \in X_s^0$ and $(t, u) \in T^P \times T^P$ generic independent from s over P. Then either $P \in \mathrm{acl}_{\mathcal{M}}(s)$ or $\#(\overline{X_t^1} \cap \overline{X_s^1}) < \#(\overline{X_t^1} \cap \overline{X_u^1})$.*

Proof. We assume that $s \notin \mathrm{acl}_{\mathcal{M}}(P)$ (as we will see later on, this assumption will ultimately lead to a contradiction). So $P \in X_s$ is $\mathcal{M}$-generic.

Claim I: $\mathrm{tp}_{\mathcal{M}}(s,t) = \mathrm{tp}_{\mathcal{M}}(t,u)$.

Proof. To see this, note that, by assumption T is $\mathcal{M}$-irreducible (i.e., has a unique generic type). By the choice of X the point P is (field) generic (over $\emptyset$) and therefore T^P has a unique $\mathcal{M}$-generic type p. Thus $T^P \times T^P$ has a unique $\mathcal{M}$-generic type, denoted $p \otimes p$ and by construction $(t,u) \models p \otimes p$. It will now suffice to show that $(s,t) \models p \otimes p$. Indeed, as t is generic in T^P over s this reduces to proving that $s \models p$. Our assumption that $P \notin \mathrm{acl}_{\mathcal{M}}(s)$ implies that $\dim_{\mathcal{M}}(P,s) = \dim_{\mathcal{M}}(s) + 1$. So $\dim_{\mathcal{M}}(s/P) = \dim_{\mathcal{M}}(T) - 1 = \dim_{\mathcal{M}}(T^P)$. As p is the unique type in T^p of maximal dimension, the claim is proved.

$\square_{\text{Claim I}}$

It follows that $\#(X_t \cap X_s) = \#(X_t \cap X_u)$.

By definition both $X_t \cap X_s$ and $X_t \cap X_u$ contain all P-indistinguishable points. By the previous lemma all P-indistinguishable points are, in fact, in $\overline{X_t^1} \cap \overline{X_u^1}$ so there is no harm assuming that $X_s^0 \cap X_t = \{P\}$, as any other point in that set is P-indistinguishable (because X_s^0 is finite) and therefore will contribute to $\overline{X_t^1} \cap \overline{X_u^1}$ but not to $\overline{X_t^1} \cap \overline{X_s^1}$, making our task easier.

For exactly the same reason $\mathrm{Fr}(X_s) \cap X_t$ contains only P-indistinguishable points. But as $s \in T^P$ is $\mathcal{M}$-generic (as follows from Claim I) all P-indistinguishable points are, in fact, contained in X_s, so cannot be in $\mathrm{Fr}(X_s)$. Thus, $\mathrm{Fr}(X_s) \cap X_t = \emptyset$.

Claim II: Let $Q \in X_s \cap \mathrm{acl}(t)$ then $Q \in \mathrm{acl}(s)$.

Proof. First, observe that as $\dim(T) > 2$ and P is field generic over $\emptyset$ we get

$$\dim\{s' \in S : P \in X_{s'}^0\} > 0,$$

i.e., $s \notin \mathrm{acl}(P)$. Let $\phi(x,t)$ isolate $\mathrm{tp}(Q/t)$. By compactness we may assume that for any $t' \equiv_P t$ (in the full structure) the formula $\phi(x,t')$ is algebraic. Consider

$$F := \{Q' : \dim\{t \in T^p : Q' \models \phi(x,t)\} \geq \dim(T) - 2\}.$$

Then $\dim(F) \leq 1$ and $Q \in F$. Therefore, as F is definable over P, we get that $F \cap X_s$ is finite, so $Q \in \mathrm{acl}(s)$. $\square_{\text{Claim II}}$

It follows from the above claim that $X_s \cap X_t^0 = \emptyset$. Indeed, the claim implies that any $Q \in X_s \cap X_t^0$ is algebraic over s, and therefore P-indistinguishable. But the previous lemma all P-indistinguishable points are generic in X_t, so in particular not in X_t^0.

Using the same claim again we see that if $Q \in X_s \cap \mathrm{Fr}(X_t)$ then Q is frontier P-indistinguishable. So we get that $B := X_s \cap \mathrm{Fr}(X_t) \subseteq C$, where C is the set of all frontier P-indistinguishable points (in the notation of the previous lemma).

Summing up all of the above, together with the previous lemma, we get:

$$\#(X_t \cap X_s) = \#(X_t \cap X_u).$$

But

$$\#(X_t \cap X_s) = \#(\overline{X_t^1} \cap \overline{X_s^1}) - |B| + 1$$

where P accounts for the extra point on the left hand side. On the other hand

$$\#(X_t \cap X_u) = \#(\overline{X_t^1} \cap \overline{X_u^1}) - |C|$$

and as $B \subseteq C$ the desired conclusion follows. $\square$

The previous lemma gives us the advantage of working with families of closed curves, allowing us to use intersection theory. The fact that the family we will be working with is (a priori) only definable in the full structure, and not necessarily in $\mathcal{M}$ will not be of importance, as we will show that the conclusion of the previous lemma leads to a contradiction, unless for generic $s \in T$ and $P \in X_s^0$ we have that $P \in \mathrm{acl}_{\mathcal{M}}(s)$.

6.2 Multiplicities

We remind that if $X, Y \subset M^2$ are curves and $Q \in X \cap Y$ is a regular point on both then the intersection multiplicity of X and Y at P is defined

$$\mathrm{mult}(X, Y; P) = \dim_k \mathcal{O}_{M^2,Q}/I_X I_Y$$

where I_X and I_Y are the ideals cutting out the germs of X and Y around Q. If X_a and Y_b are fibres of families of curves over parameters a, b which are generic in definable sets A, B, we regard them as curves over fields $k(a), k(b)$ which are function fields of $\mathrm{locus}(a), \mathrm{locus}(b)$ respectively. We understand by intersection multiplicity of X_a, Y_b at a point P defined over $\mathrm{acl}(a, b)$

$$\mathrm{mult}(X_a, Y_b; P) = \dim_{k(a,b)^{alg}} \mathcal{O}_{M^2,P}/I_{X_a} I_{Y_b}$$

where I_{X_a}, I_{Y_b} are ideals that cut out the germs of $X_a \otimes k(a, b)^{alg}$ and $Y \otimes k(a, b)^{alg}$, and $k(a, b)$ is the function field of $\mathrm{locus}(a, b)$.

The key local property of the intersection multiplicity used further is given by the following lemma:

Lemma 6.9. *Let R be a regular local ring over a field k, and let I_1, I_2, I_3 be three ideals such that $R/I_1, R/I_2, R/I_3$ are regular. Assume that $R/(I_1I_2)$,$R/(I_2I_3)$ and $R/(I_1I_3)$ are finite-dimensional k-vector spaces. Then*

$$\dim_k R/(I_1I_2) \geq \min\{\dim_k R/(I_1I_3), \dim_k R/(I_2I_3)\}$$

Proof. By symmetry of the statement it suffices to show that if

$$\dim_k R/(I_1I_2) \geq \dim_k R/(I_1I_3),$$
$$\dim_k R/(I_1I_2) \geq \dim_k R/(I_2I_3),$$

then $\dim_k R/(I_1I_3) = \dim_k R/(I_2I_3)$.

By regularity of R/I_2, all 0-dimensional quotient algebras are of the form $k[a]/(a^n)$ for some generator; for two such algebras $k[a]/(a^n), k[a]/(a^m), n > m$ there exists a natural reduction morphism $k[a]/(a^n) \to k[a]/(a^m)$. It follows from the first inequality above that there exists a morphism of this form $f : R/(I_1I_2) \to R/(I_3I_2) \cong R/I_3 \otimes R/I_2$. We have the following diagram:

$$\begin{tikzcd}
R/I_2 \arrow[r] \arrow[d, "p"] & R/I_3 \otimes R/I_2 \arrow[d, shift left, "\mathrm{id} \otimes p"] \\
R/(I_1I_2) \arrow[r, "i_{12}"'] \arrow[ur, "f"] & R/I_3 \otimes R/(I_1I_2) \arrow[u, shift left, "h"]
\end{tikzcd}$$

where h is defined by $a \otimes b \mapsto a \cdot f(b)$.

One observes that both morphisms h and $\mathrm{id} \otimes p$ are surjective, and since $R/I_3 \otimes R/I_2$ and $R/I_3 \otimes R/(I_1I_2)$ are finite-dimensional vector spaces, they are bijective and so isomorphisms. By a similar argument, $R/I_3 \otimes R/(I_1I_2)$ is isomorphic to $R/I_3 \otimes R/I_1$, and therefore $R/I_3 \otimes R/I_1$ is isomorphic to $R/I_3 \otimes R/I_2$. $\square$

Geometrically, the above lemma expresses the fact that if X, Y and Z are curves in M^2 all meeting at a common point, Q, regular on all three, and if $\mathrm{mult}(X, Y; Q) = \mathrm{mult}(X, Z; Q)$ then $\mathrm{mult}(Y, Z; Q) \geq \mathrm{mult}(X, Y; Q)$.

Lemma 6.10. *Let s be a generic point in T, $P \in X_s^0$ and t be a generic point in T^P. Let $Q \in X_t \cap X_s$. Assume that all geometric intersection points (that is, defined over* $\mathrm{acl}(t, s)$*) are regular. Assume that all geometrically irreducible components of T^P are definable over s and let Y be the irreducible t belongs to. Then there exists a number m such that for all t', Q' such that* $\mathrm{tp}(s, Q) = \mathrm{tp}(s, Q')$ *and t' is generic in Y, the intersection multiplicity of $X_{t'}$ and X_s at Q' is m.*

Proof. We first show the following: let W be an irreducible (over t, s) component of $X_t \cap X_s$, then there exists a number m such that the multiplicity of intersection of X_t and X_s at any geometric point in W is m.

In algebro-geometric terms we are looking at a regular point W on the scheme $X_t \otimes k(t, s) \cap X_s \otimes k(t, s)$ with a residue field which is an algebraic extension of $k(t, s)$. The fiber product $Z = X_t \times_{M^2} X_s = \mathrm{Spec}\, \mathcal{O}_{M^2,W}/I_{X_t}I_{X_s}$ is then a spectrum of an algebra of the form $k(t, s)[\epsilon]/(\epsilon^{m+1})$, since W is regular.

For any geometric point $\eta : \mathrm{Spec}\, k(t, s)^{alg} \to Z$, since localization commutes with base change, $\mathcal{O}_{M^2,\eta}/I_{X_t}I_{X_s} \cong \mathcal{O}_{M^2,W}/I_{X_t}I_{X_s} \otimes k(t, s)^{alg}$. Therefore, the multiplicity at η is $\dim_{k(t,s)^{alg}} \mathcal{O}_{M^2,W}/I_{X_t}I_{X_s} \otimes k(t, s)^{alg} = \dim \mathrm{Spec}\, k(t, s)^{alg}[\epsilon]/(\epsilon^{m+1}) = m$.

Now to prove the statement of the Lemma, observe that $\mathrm{mult}(X_{t'}, X_s; Q')$ by definition depends only on the type $\mathrm{tp}(t', s, Q')$, and for t' is generic in an s-definable set Y this type is determined by the type $\mathrm{tp}(s, Q')$ $\square$

We can now show:

Lemma 6.11. *Let $s \in T$ be generic, $P \in X_s^0$. Let $Q \in X_s$ be generic over P and $t, u \in T^P \cap T^Q$ independent generics. Assume that $\mathrm{tp}(t/P, Q) = \mathrm{tp}(u/P, Q)$ and $\mathrm{tp}(s) = \mathrm{tp}(t)$ where all types are taken with respect to the full structure. Then $\mathrm{mult}(X_t, X_s, Q) \leq \mathrm{mult}(X_u, X_t, Q)$.*

Proof. By our choice of $X \to T$ we know that $\dim(P) = 2$ and as $\dim(T) > 2$ we get $\dim(T^P) \geq 2$. Moreover, as in the proof of Claim II of Lemma 6.8, $s \notin \mathrm{acl}(P)$. This implies that $Q \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} P$, whence $\dim(T^P \cap T^Q) = \dim(T) - 2 \geq 1$. Thus, if $t \in T^P \cap T^Q$ is generic we have

$$\dim(T) + 2 = \dim(t, P, Q) = \dim(Q/t, P) + \dim(t/P) + \dim(P).$$

Since $\dim(Q/t, P) = 1$ this implies that $\dim(t/P) = \dim(T) - 1$, so t is generic in T^P. Similarly, if $t, u \in T^P \cap T^Q$ we have:

$$2\dim(T) = \dim(P) + \dim(t, u/P) + \dim(Q/t, u, P)$$

and as $Q \in \mathrm{acl}(t, u, P)$ this implies that $\dim(t, u/P) = 2\dim(T) - 2$, i.e., (t, u) are independent generics in T^P. Since P is generic over $\emptyset$ this implies that t, u are independent generic over $\emptyset$ as well.

Thus, $\mathrm{mult}(X_u, X_t, Q)$ is well defined (i.e., Q is regular on both curves). Assume towards a contradiction that $\mathrm{mult}(X_s, X_t, Q) > \mathrm{mult}(X_t, X_u, Q)$. By Lemma 6.10, since $\mathrm{tp}(t/P, Q) = \mathrm{tp}(u/P, Q)$, also $\mathrm{mult}(X_s, X_u, Q) = \mathrm{mult}(X_s, X_t, Q)$, and so $\mathrm{mult}(X_u, X_s, Q) > \mathrm{mult}(X_t, X_u, Q)$. But by Lemma 6.9, $\mathrm{mult}(X_t, X_u, Q) \geq \mathrm{mult}(X_u, X_s, Q)$, which constitutes a contradiction. $\square$

The global implication of the previous (local) lemma is:

Lemma 6.12. *Let $s \in T$ be generic $P \in X_s^0$. Let $t, u \in T^P$ be independent generic over all the data (satisfying the same type in the full structure). Then*

$$\sum_{Q \in \overline{X_s^1} \cap \overline{X_t^1}} \mathrm{mult}(\overline{X_s^1}, \overline{X_t^1}, Q) < \sum_{Q \in \overline{X_u^1} \cap \overline{X_t^1}} \mathrm{mult}(\overline{X_u^1}, \overline{X_t^1}, Q)$$

Proof. For simplicity we will assume that all curves in question have a unique 1-dimensional component (with respect to the full structure). Otherwise we repeat the argument component by component.

Denote $\mathcal{Q} := \overline{X_s^1} \cap \overline{X_t^1}$ and $\mathcal{Q}_1$ the subset consisting of the points generic in $\bar{X}_s^1$. Observe that if $Q \in \mathcal{Q} \setminus \mathcal{Q}_1$ then $Q \in \overline{X_{t'}^1}$ for all $\mathrm{tp}(u/P, s) = \mathrm{tp}(t/P, s)$, in particular,

$\mathcal{Q} \setminus \mathcal{Q}_1$ does not depend on the choice of t (only on its field-theoretic type). So by Lemma 6.9 we get

$$\operatorname{mult}(\overline{X_s^1}, \overline{X_t^1}, Q) \leq \operatorname{mult}(\overline{X_u^1}, \overline{X_t^1}, Q).$$

Thus,

$$\sum_{Q \in \mathcal{Q} \setminus \mathcal{Q}_1} \operatorname{mult}(\overline{X_s^1}, \overline{X_t^1}, Q) \leq \sum_{Q \in \mathcal{Q} \setminus \mathcal{Q}_1} \operatorname{mult}(\overline{X_u^1}, \overline{X_t^1}, Q).$$

On the other hand, as we assumed that X_s and X_t are irreducible, and – restricting to an irreducible component of T^P, we get by Lemma 6.10 that there exists a number m such that whenever $t \in T^P$ is field generic (of a fixed type) $\operatorname{mult}(\overline{X_s^1}, \overline{X_t^1}, Q) = m$ for any $Q \in \overline{X_s^1} \cap \overline{X_t^1}$ generic in X_s. Similarly, there exists n such that whenever $t, u \in T^P$ are independent generics, satisfying the same field theoretic type $\operatorname{mult}(\overline{X_u^1}, \overline{X_t^1}, Q) = n$ for any $Q \in \overline{X_s^1} \cap \overline{X_t^1}$ generic in X_t.

By Lemma 6.11 we get $m \leq n$. So the conclusion of the lemma would follow if we showed that $\#\mathcal{Q}_1$ is strictly smaller than the number of generic intersection points in $\overline{X_u^1} \cap \overline{X_t^1}$. But by Lemma 6.8 we know that

$$\#(\overline{X_s^1} \cap \overline{X_t^1}) < \#(\overline{X_u^1} \cap \overline{X_t^1})$$

and as we have just shown that the two sets share the same set $\mathcal{Q} \setminus \mathcal{Q}_1$ of non-generic points, the desired conclusion follows. $\square$

Proof of Proposition 6.2. Fix a family $X \to T$ as provided by Corollary 6.5. If for $s \in T$ generic $X_s^0 \subseteq \operatorname{acl}_{\mathcal{M}}(s)$ then, by compactness (and induction on $\dim(T)$) we have nothing to prove. So assume that this is not the case. We will derive a contradiction.

Let M^* be a regular proper curve that contains M. Consider the first order structure with the universe M^* which contains the family X, interpreted as a subset of $(M^*)^l$ for some l via the embedding $M \hookrightarrow M^*$ that has been just picked. Note that any definable subsets of M^l in M^* is definable in the original structure $\mathcal{M}$, and vice versa.

Let $s \in T$ be field generic and $P \in X_s^0$ such that $P \notin \operatorname{acl}_{\mathcal{M}}(s)$, $Q \in S$ a field generic point. Let $t, u \in T^P \cap T^Q$ be field independent generic over all the data. By our assumption it follows from Lemma 6.8 that $\#(\overline{X_s^1} \cap \overline{X_t^1}) < \#(\overline{X_u^1} \cap \overline{X_t^1})$, where the closure is taken in (some Cartesian power of) M^*.

As is well-known from intersection theory, intersection number of curves on proper regular varieties is stable in algebraic families ([12], Section 10.2). Therefore the sum of local intersection multiplicities over all intersection points should be the same for pairs X_t, X_s and X_t, X_u. This is in direct contradiction with the previous lemma. $\square$

Though Proposition 6.2 suffices for our needs in the current paper, we give the proof of Theorem 6.1, whose statement is cleaner, and may be of interest on its own right.

Proof of Theorem 6.1. Let S be any $\mathcal{M}$-definable curve. Absorbing the parameters required to define S, we may assume that S is $\emptyset$-definable. Let $X \to T$ be any $\mathcal{M}$-definable family of plane curves satisfying the assumptions of Proposition 6.2. For simplicity, we may also assume that X satisfies the conclusion of the proposition. Consider the family $S \circ X \to T$. Our assumption implies that for a generic $t \in T$ the only isolated points of $S \circ X_t$ are of the form $S^0 \circ \overline{X_t^1}$ (as $\overline{X_t^1} = \overline{X_t}$). Applying Proposition 6.2 to $S \circ X_t$ (for some generic $t \in T$) we get a curve $Z_t \sim S \circ X_t$ such that $Z_t^0 = \emptyset$ (and Z_t is definable over t). So $S^0 \subseteq \{P \in S : P \circ X_t \setminus Z_t \neq \emptyset\}$. Note that the right hand side is t-definable (and finite). So

$$S_0 := \bigcap_{t \in T \text{ generic}} \{P \circ X_t \setminus Z_t \neq \emptyset\}$$

and by definabilty of Morley rank the right hand side is $\emptyset$-$\mathcal{M}$-definable. $\square$

Note that the proof of Theorem 6.1 follows almost formally from Proposition 6.2, and has little to do with the topological definition of the set of 0-dimensional components. The only property of 0-dimensional components used in the proof is that if D has no 0-dimensional components then $(S \circ D)^0 \subseteq S^0 \circ D$.

Also the proof of Proposition 6.2 does not seem endemic to algebraic geometry. The only algebro-geometric ingredients used in the proof are

1. Finiteness of the frontier of (plane) curves.
2. Lower semi-continuity of the intersection number in flat families.
3. The multiplicity inequality of Lemma 6.9

The last two of these three properties seem to have satisfactory analogues in a variety of analytic and topological settings. E.g., the lower semi-continuity of the intersection number in flat families may be replaced in certain contexts with the invariance of the topological degree under homotopy (and see, e.g., [17, Lemma 4.19, Lemma 4.20]). The multiplicity inequality is a refinement of ideology that tangency should be an equivalence relation. In many respect this is the cornerstone upon which Zilber's Trichotomy – suggesting the construction of a field from purely geometric, even combinatorial, data – relies. It is therefore reasonable to expect to have natural analogues in any context in which one can reasonably hope to prove this trichotomy.

6.3 The main theorem

Theorem 6.13. *Let M be an algebraic variety of dimension 1 defined over an algebraically closed field k. Let $X \subset T \times M^2$ be a family of constructible subsets of M^2 generically of dimension and Morley degree 1. Then the structure $\mathcal{M} = (M, X)$ interprets an infinite field.*

Proof. By Lemma 6.8 and Proposition 6.2 there exists a two-dimensional family of curves definable in (M, X) with generic fibre of pure dimension 1. By Theorem 5.23 a one-dimensional group G is definable in $\mathcal{M}$. The structure induced on G by $\mathcal{M}$ is non-locally modular, so the theorem is reduced to the case when there is a group structure on M.

A non-locally modular strongly minimal group G has a definable set $Z \subset G^2$ which is not a coset. Use it to produce the definition of a field by Theorem 5.33. □

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