Motivic volume of families of polarized rigid-analytic tori

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Abstract

Let k be a non-Archimedean rational valued field. We construct the moduli space of linearly rigidified polarized analytic tori over k that admit rigid-analytic uniformization by an algebraic torus and observe that it is in definable rigid subanalytic bijection with a PGL_N-bundle over a polyhedral domain in an algebraic torus. We use this observation to prove that the Hrushovski-Kazhdan motivic volume of a non-Archimedean semi-algebraic family of Abelian varieties admitting such a uniformization fibrewise vanishes. This question is motivated by the conjectural geometric interpretation of tropical refined multiplicities of Block and Goetsche proposed by Nicaise, Payne and Schroeter.

Contents

1 Introduction

Nicaise, Payne and Schroeter propose in their paper [NPS16] an approach to geometric interpretation of tropical refined Severi degrees of Block and Goettsche [BG16]. They conjecture that the refined tropical multiplicity equals the χ_y-genus of the non-Archimedean semi-algebraic subset of the universal family of compactified Jacobians over the moduli space of stable curves of fixed genus that tropicalize to the given tropical curve, and prove the conjecture in genus 1 for curves with a single node. The χ_y-genus is assigned to a semi-algebraic set with the help morphism from the Grothendieck ring of (non-Archimedean) semi-algebraic subsets to the K-ring of varieties

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over the residue field, constructed using the theory of Hrushovski and Kazhdan [HK06]. The image of a particular semi-algebraic set under this morphism is called its motivic volume.

In view of conjectures proposed in [NPS16] it is natural to try to find the contribution of the semi-algebraic families of Jacobians of smooth Mumford curves to the tropical multiplicities. More generally, one considers semi-algebraic family of totally degenerate Abelian varieties. In this note we prove that the motivic volume of the total spaces of such a family is zero.

The computation of motivic volume of a family is hindered in general due to the lack of an appropriate Fubini-type statement. In the situation of interest, totally degenerate Abelian variety is a quotient of an algebraic torus by a lattice, i.e. an analytic torus. It is therefore in a definable bijection with a domain of the form $\mathrm{trop}^{-1}(\Delta)$ where $\mathrm{trop} : \mathbb{G}_m^n(K) \to \mathbb{R}^n$ is the coordinate-wise application of the map $-\log|\cdot|$, and Δ is an polyhedron in $\mathbb{R}^n$ with some of its faces removed. The motivic volume of such semi-algebraic sets can be directly computed. Unfortunately, in order to compute the motivic volume of a family of such tori, the uniformization by an algebraic torus should be uniform.

To this end we consider the moduli space of linearly rigidified polarized analytic tori and show that the uniformization map is locally definable in the expansion of the language of algebraically closed valued fields with rigid subanalytic functions of Lipschitz and Robinson [Lip93]. We then use the invariance of motivic volume under bijections definable in this expansion to deduce the vanishing of the motivic volume.

The main result is Theorem 3.11. Section 2 collects the necessary auxiliary statements about analytic tori, polarizations, moduli of polarized Abelian varieties and Lipshitz-Robinson rigid subanalytic functions.

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2 Preliminaries

2.1 Analytic tori, Abelian varieties and polarizations

In order to establish notation we recall basic facts about polarizations on Abelian schemes; we then survey the facts about polarizations of rigid-analytic tori loosely following [Lüt16, Section 2.7], see also [FVdP12, Chapter 6]. It is helpful to keep in mind that the theory largely parallels the one in the complex case (see, for example, [BL13, Chapters 4 and 8]).

Definition 2.1 (Abelian scheme). *A group scheme $A \to S$ is called an Abelian scheme if it is smooth, proper, and its geometric fibres are connected.*

If $A \to S$ is an Abelian S-scheme, then $\mathrm{Pic}(A/S)$ is a smooth proper group S-scheme that represents the Picard functor. Let $\mathrm{Pic}^\tau(A/S)$ be its open subscheme whose geometric points correspond to invertible sheaves that are algebraically equivalent to zero. This scheme is smooth and projective over S and its geometric fibres are reduced and connected. The Abelian scheme $\mathrm{Pic}^\tau(A/S)$ is called the scheme *dual to* A and is denoted $\hat{A}$. A

The universal line bundle on $A \times \hat{A}$ is called *Poincaré line bundle* and is denoted $P_{A\times\hat{A}}$.

Let L be an arbitrary line bundle on A, and let $\mu : A \times_S A \to A$ be the multiplication map. Then the line bundle

$$\mu^* L \otimes (p_1^* L)^{-1} \otimes (p_2^* L)^{-1}$$

can be considered as a line bundle over X via the projection $p_1 : A \times_S A \to A$, and so by the definition of Pic gives rise to the mapping $\omega_L : A \to \mathrm{Pic}(A/S)$. If $e : S \to A$ is an identity then $\omega_L \circ e$ is the identity of $\mathrm{Pic}(A/S)$. Since the fibres of $\mathrm{Pic}(A/S)$ are connected, the morphism ω_L factors through $\mathrm{Pic}^\tau(A/S)$.

Recall that a morphism of Abelian varieties over a field is called an isogeny if it is surjective with finite kernel; a morphism of Abelian schemes is a morphism that induces isogenies on geometric fibres.

Fact 2.2. *The construction above induces a homomorphism* $\mathrm{Pic}(A/S) \to \mathrm{Hom}_{\mathbb{Z}}(A, \hat{A})$. *The map ω_L is an isogeny if and only if L is ample.*

Definition 2.3 (Polarisation)**.** *A polarization of an Abelian S-variety A is a morphism $\varphi : A \to \hat{A}$ such that for all geometric fibres the induced morphism $\varphi_s : A_s \to \hat{A}_s$ is an isogeny of the form ω_{L_s} for some ample line bundle L on A.*

Fix a non-Archimedean valued field k, let M be a free Abelian group of rank n, denote by M' its dual $\mathrm{Hom}(M, \mathbb{Z})$, and let $T := \mathrm{Spec}\, k[M']$. Denote by $\mathrm{trop} : T \to \mathbb{R}^n$ the coordinatewise valuation map:

$$\mathrm{trop}(x_1, \dots, x_n) = (-\log|x_1|, \dots, -\log|x_n|)$$

A torsion-free subgroup $\Lambda \subset T$ is called a *lattice* if trop induces an isomorphism between Λ and a discrete subgroup $\mathrm{trop}(\Lambda)$ of the additive group $\mathbb{R}^n$.

Lattices $M \to T$ are in natural bijective correspondences with the lattices $M' \hookrightarrow T'$. Indeed, regarding M' as $\mathrm{Hom}(T, \mathbb{G}_m)$ and T' as $\mathrm{Hom}(M, \mathbb{G}_m)$ the embedding $M' \hookrightarrow T$ is given by the restriction map $\mathrm{Hom}(T, \mathbb{G}_m) \to \mathrm{Hom}(M, \mathbb{G}_m)$.

Fact 2.4 (Proposition 2.7.5 [Lüt16], Non-Archimedean Appel-Humbert theorem)**.** *The set of isomorphism classes of line bundles on T/M is in bijective* *correspondence with pairs* (λ, r)*, where* $\lambda : M \to M'$ *is a homomorphism and* $r : M \to \mathbb{G}_m$ *subject to the condition*

$$\lambda(m_1)(m_2) = \frac{r(m_1 + m_2)}{r(m_1)r(m_2)}$$

λ *is trivial if and only if* $L_{(\lambda,r)} \in \mathrm{Pic}^0$*, moreover,* Pic^0 *consists of groups of translation-invariant line bundles.*

One observes that the function $Z : M \to H^0(T, \mathcal{O}_T^\times) = r(m)\lambda(m)$, called the *automorphy factor* is a group cohomology 1-cocycle, $Z \in H^1(M, H^0(T, \mathcal{O}_T^\times))$. A function $f \in H^0(T, \mathcal{O}_T^\times)$ is called *theta function with respect to the automorphy factor* Z if

$$f(m \cdot x) = Z_m f(x)$$

Definition 2.5 (Polarization of an analytic torus)**.** *A polarization of the analytic torus* T/M *is an injective map* $\lambda : M \to M'$ *such that the bilinear map*

$$\langle \cdot, \cdot \rangle : M \times M \to K^\times \qquad \langle m_1, m_2 \rangle = \lambda(m_1)(m_2)$$

is symmetric and positive definite, that is, for any $m \in M$*,* $-\log|\langle m, m \rangle| > 0$*.*

Remark. Let $M \hookrightarrow T$ be a lattice. If $\lambda : M \to M'$ is a homomorphism of groups then it induces a morphism of tori $\varphi_\lambda : T \to T'$. If λ induces a symmetric and non-degenerate map $\langle \cdot, \cdot \rangle$ then $\varphi_\lambda(M) \subset M'$ and so the morphism $\varphi_\lambda : T/M \to T'/M'$ is well-defined. If λ defines a polarization then $\varphi_\lambda = \varphi_L$ for an ample line bundle L.

Fact 2.6 (Theorem 2.7.12, [Lüt16])**.** *A line bundle is ample if and only if* λ *defines a polarization. The global sections of a line bundle* L *are given by theta functions with respect to the automorphy factor* Z*.*

Fact 2.7 (Lemma 6.5.4, [FVdP12])**.** *If* L *is an ample line bundle on* T/M *and* $\theta_0, \ldots, \theta_n$ *is a basis of* $H^0(T/M, L^3)$ *then* $x \mapsto (\theta_0(x) : \ldots : \theta_n(x))$ *defines a closed embedding* $T/M \hookrightarrow \mathbb{P}(H^0(T/M, L^3))$*.*

Fact 2.8 (Propositions 6.10 and 6.13, [MFK94])**.** *Let* $\omega : A \to \hat{A}$ *be a polarization of an Abelian variety, let* $L = (\mathrm{id} \times \omega)^* P_{A\times}$ *and let* ω' *be the polarization induced by* L*. Then*

- $\omega' = 2\omega$
- $(\dim H^0(A, L))^2 = \deg \omega$

Denote $\mathrm{rk}\, \lambda = \#(M'/\lambda(M))$. The following fact easily follows from the automorphy equation.

Fact 2.9. *Theta functions have the form* $f(x) = \sum_{\chi \in M'} a_\chi \chi$*, and are determined by coefficients* $a_{\chi_1}, \ldots, a_{\chi_n}$ *where* $\chi_1, \ldots, \chi_n$ *are representatives of* $M'/\lambda(M)$*. In particular,* $\dim H^0(T/M, L) = \mathrm{rk}\, \lambda$*.*

If L is an arbitrary line bundle, define

$$\varphi_L : T/M \to T'/M' \quad \varphi_L(a) = t_a^* L \otimes L^{-1}$$

where $t_a : T/M \to T/M, t_a(x) = x + a$ for any $a \in T/M$. Clearly, the line bundle $t_a^* L \otimes L^{-1}$ is translation-invariant, so the map is well-defined. One can show that φ_L is an analytic homomorphism of groups.

Fact 2.10. *The line bundle L is ample if and only if φ_L is surjective. The degree of φ_L is of size d^2 where d is the degree of L.*

Fact 2.11 (Riemann-Roch on an Abelian variety)**.** *Let L be a positive line bundle on an Abelian variety of dimension g, then*

$$\chi(L) = L^g/g!$$
$$\chi(L)^2 = \deg \varphi_L$$

Consequently, the Hilbert polynomial of an Abelian variety endowed with polarization φ of degree d with respect to $L_\varphi^{\otimes 3}$ is $P(x) = x^g d$.

2.2 Motivic volume

Hrushovski-Kazhdan motivic integration theory [HK06] provides a way to express non-Archimedean semi-algebraic subsets of algebraic varieties over a valued field K as unions of semi-algebraic sets of two particular kinds. The first one is related to the geometry of integral polhedra, and the second one is related to algebraic varieties over the residue field.

Formally, the theory is formulated in the context of model theory of algebraically closed valued fields. Let K be such a field, then one considers several *sorts*: the valued field sort VF, the residue-value sort RV and the value group sort Γ.

Let $\mathcal{O} \subset K$ be the value ring with the maximal ideal $\mathfrak{m}$. Consider the exact sequence of groups

$$1 \to \mathcal{O}^\times/(1+\mathfrak{m}) \to K^\times/(1+\mathfrak{m}) \to \Gamma \to 0$$

The middle term is called RV and is made into a sort with the structure of the multiplicative group, and two inter-sort projection maps: $\mathrm{rv} : \mathrm{VF} \setminus \{0\} \to \mathrm{RV}$, and $v_{\mathrm{rv}} : \mathrm{RV} \to \Gamma$.

After fixing some base field K_0, one associates the following categories of definable sets to the sorts VF, RV and Γ.

Definition 2.12. *The category* $\mathrm{VF}[n]$ *is defined to be the category of definable subsets of of n-dimensional varietis over K_0.*

The category $\mathrm{RV}[n]$ *is defined to be the category of pairs (X, f) where X is a definable set and $f : X \to \mathrm{RV}^n$ is a definable map with finite fibres.*

The category $\Gamma[n]$ *is defined to be the category of Boolean combinations of subsets of* Γ^n *defined by inequalities and equalities with integral coefficients and with parameters in* $\Gamma(K_0)$.

$\Gamma^{\text{fin}}[n]$ *is the full subcategory* $\Gamma[n]$ *of definable finite subsets.*

$\mathrm{RES}[n]$ *is the full subcategory of* $\mathrm{RV}[n]$ *which consists of definable sets which project to finite definable subsets of* Γ^n *via* trop.

Definition 2.13. *If* A *is a category of definable sets then we denote by* $K_+(A)$ *the semi-ring generated by definable subsets in* A *module the relations*

- $[A] = [B]$ *if there exists a definable bijection between* A *and* B,
- $[C] = [A] + [B]$ *if* $C = A \sqcup B$.

In case $K_0 = k((t))$ there exists a canonical isomorphism between a quotient $!K(\mathrm{RES})$ of the ring $K(\mathrm{RES})$ and the equivariant Grothendieck ring $K_0^{\hat{\mu}}(\mathrm{Var}_k)$. Let $\theta : K_0(\mathrm{RES}) \to K_0 \operatorname{Var}$ be the composition of the quotient map with this canonical isomorphism and the forgetful morphism $K_0^{\hat{\mu}}(\mathrm{Var}_{k_0}) \to K_0(\mathrm{Var}_k)$ (see Seciton 4 of [HL15]).

Denote $K_+ \mathrm{VF} = \oplus_n K_+(\mathrm{VF}[n]), K_+ \mathrm{RV}[\leq n] = \oplus_{l \leq n} K_+ \mathrm{RV}[l]$, and define the morphisms

$$\begin{array}{ll} \mathbb{L} : K_+(\mathrm{RV}[n]) \to K_+(\mathrm{VF}[n]), & [(X, f)] \mapsto [\mathrm{VF}^n \times_{\mathrm{rv},F} X] \\ \mathbb{L} : K_+(\Gamma[n]) \to K_+(\mathrm{VF}[n]), & [\Delta] \mapsto [\mathrm{trop}^{-1}(\Delta)] \end{array}$$

The motivic integration theory of Hrushovski and Kazhdan [HK06] (in the non-measured case) rests on two main statements: that the natural morphism

$$\oplus_{l+m=n} K_+(\Gamma[l]) \otimes_{K_+\Gamma^{\text{fin}}} K_+(\mathrm{RES}[m]) \to K_+(\mathrm{RV}[n])$$

is an isomorphism and that the morphism

$$\mathbb{L} : \oplus_n K_+(\mathrm{RV}[\leq n]) \to K_+(\mathrm{VF})$$

is surjective. The kernel I_{sp} of the latter can be explicitly described. The theory is developed in an axiomatic setting that depends only on the category RV (the corresponding notion is called V-minimality).

Consider modified Euler characteristic

$$\chi' : \oplus_n K_+(\Gamma[n]) \to \mathbb{Z}, \chi'([\Delta]) = \lim_{n \to \infty} \chi(\Delta \cap [-l, l]^n)$$

where χ is the usual o-minimal Euler characteristic (which yet again coincides with the usual Euler characteristic when $\Gamma \cong \mathbb{R}$).

Define the morphism $\mathrm{Vol} : K_0(\mathrm{VF}[n]) \to K_0(\mathrm{Var}_{k_0})$

$$\mathrm{Vol}(\mathbb{L}^{-1}([X] \otimes [\Delta]) = \theta([X]) \cdot \chi'(\Delta)(\mathbb{L} - 1)^n$$

is well-defined because $\mathrm{Id} \otimes \chi'$ is trivial on I_{sp}. The destination of the morphism can be identified with $K_+(\mathrm{Var}_k)$ if K is algebraically closed.

2.3 Lipschitz-Robinson rigid subanalytic functions

Let K be an algebraically closed, complete, non-Archimedean normed field. Let $R = \{ x \in K \mid |x| \leq 1 \}$, let $\mathfrak{m} = \{ x \in K \mid |x| < 1 \}$, and let $k = R/\mathfrak{m}$. Define the norm on the ring $K[[x, \rho]]$ as follows:

$$|\sum a_{ij} x^i \rho^j| = \sup |a_{ij}|$$

Let $R_0 \subset R$ be a maximal discrete valuation ring contained in R with prime $\pi \in \mathfrak{m}$ such that $0 < |\pi| < 1$ and $R_0/(\pi) \cong k$. For any sequence (a_i) with $a_i \in R$ and such that $|a_i| \to 0$ let $R_0[\widehat{\{a_i, i \in \mathbb{N}\}}]$ be the completion of $R_0[\{a_i, i \in \mathbb{N}\}]$ with respect to the norm on K and define

$$R_0\{a_i\}\langle x\rangle = R_0[\widehat{\{a_i, i \in \mathbb{N}\}}]\langle x\rangle$$

Let $R_0\{a_i\}\langle x\rangle[[\rho]]$ be the ring of formal power series over $R_0\{a_i\}\langle x\rangle$. Define

$$S\{a_i\}\langle x\rangle[[\rho]] = \{ \pi^{-\alpha} f \mid f \in R_0\{a_i\}\langle x\rangle[[\rho]] \}$$

and

$$K\langle x\rangle[[\rho]]_s = \cup_{\{a_i\}} S\{a_i\}\langle x\rangle[[\rho]] \subset K[[x, \rho]]$$

The elements of this ring define analytic functions $R \times \mathfrak{m} \to K$ which are well-behaved. For example, these functions have finitely many zeroes on $R \times \mathfrak{m}$.

A *rigid subanalytic function* is a function definable in the expansion of K with graphs of functions from $K\langle x\rangle[[\rho]]_s$ (Lipschitz and Robinson [Lip93]).

Proposition 2.14. *Let $S \subset X$ be a semi-algebraic subset of an algebraic variety X, and assume that S is a finite union of rational and semi-rational domains. An analytic function on a proper semi-algebraic subset of an algebraic variety is definable in the language* $\mathrm{ACVF}_{\mathrm{LR}}$.

Proof. Follows from the fact that analytifications of affine varieties can be covered by affinoid domains, that functions analytic on rational and semi-rational subdomains of affinoid domains are definable, and that semi-algebraic domains are contained in finite unions of rational and semi-rational domains. □

Corollary 2.15. *Let $T = \mathbb{G}_m^g$ be a torus, and assume that discrete group G acts on T so that fundamental domain $U \subset T$ is semi-algebraic. Let f be a meromorphic funtion on T/G, and let $p : T \to T/G$ be the quotient map. Then restriction of $p \circ f$ to the fundamental domain is definable in* $\mathrm{ACVF}_{\mathrm{LR}}$.

As was remarked in the previous section, the motivic integration theory of [HK06] can be carried out verbatim in any expansion of the theory of algebraically closed valued fields as long as the structure induced on the sort RV is unchanged. In particular the following is true.

Fact 2.16 (Lemma 3.33, [HK06]). *Let X, X' be semi-algebraic subsets algebraic varieties over an algebraically closed valued field. If there exists a bijection $X \xrightarrow{\sim} X'$ definable in* $\mathrm{ACVF}_{\mathrm{LR}}$ *then $[X] = [X']$ in $K_0(\mathrm{VF})$.*

3 Motivic volume of a family of polarized analytic tori

In this section k is a complete rational valued field, i.e. a field complete with respect to a non-Archimedean norm and such that the image of the map $\log|\cdot|\colon k^\times \to \mathbb{R}$ is contained in $\mathbb{Q}$, for example, k can be a discretely valued field, or its algebraic closure, such as the field of Laurent series $\mathbb{C}((t))$ or the field of Puiseux series $\mathbb{C}((t))^{\mathrm{alg}}$. Denote the residue field $\bar{k}$.

3.1 The moduli space of linearly rigidified polarized analytic tori

Definition 3.1 (Uniformized analytic tori). *By a family of uniformized analytic tori we will understand*

- *flat morphism of rigid analytic spaces $\pi : A \to S$*
- *an action of $\mathbb{Z}^g$ on $\mathbb{G}_m^g \times S$ by shifts so that $\mathbb{Z}^g \hookrightarrow (\mathbb{G}_m^g)_s$ is a lattice for all $s \in S$, and an S-isomorphism $\mathbb{G}_m^g \times S \cong A$.*

Two families $A_1 \to S$ and $A_2 \to S$ are isomorphic if there exists an S-isomorphism $A_1 \xrightarrow{\sim} A_2$ that can be lifted to a $\mathbb{Z}^g$-equivariant isomorphism of respective covers by $\mathbb{G}_m^g \times S$.

Definition 3.2 (Linear rigidification). *Let S be a scheme or an analytic space, let $p : A \to S$ be an analytic torus or an Abelian scheme over S and let $\varphi : A \to \hat{A}$ be a polarization of degree d. Then for any $s \in S$*

$$\dim(p_*\mathcal{L}_\varphi^3)_s = m := 6^g \cdot d$$

An isomorphism $\mathbb{P}(p_*\mathcal{L}_\varphi^3) \cong \mathbb{P}(\mathcal{O}_S^m)$ is called a linear rigidification of (A, φ).

Let M, M' be free rank g Abelian groups, and let $T = \operatorname{Spec} k[M'], T' = \operatorname{Spec} k[M]$. Let $\lambda : M \to M'$ be an injective homomorphism and S be a rigid analytic space. Define $\mathcal{A}^u_{g,\lambda}(S)$ to be the set of isomorphism classes of uniformized polarized analytic tori with polarization of type λ and define $\mathcal{H}^u_{g,\lambda}(S)$ to be the set of isomorphism classes of uniformized polarized analytic tori together with a linear rigidification. This defines two functors

$$\mathcal{A}^u_{g,\lambda} : \mathrm{RigSp}_k \to \mathrm{Sets} \qquad \mathcal{H}^u_{g,\lambda} : \mathrm{RigSp}_k \to \mathrm{Sets}$$

Pick a distinguished basis $\varepsilon_1, \ldots, \varepsilon_g$ in M, then the space B_g of all embeddings $M \hookrightarrow T$ with the distinguished basis be identified with the space of matrices $E = (e_{ij})$ where $e_{ij} \in K^\times$ is the j-th coordinate of the image of i-th basis vector. Define the space of lattices with a distinguished basis.

$$\tilde{B}_g = \{\ \iota(\varepsilon_1), \ldots, \iota(\varepsilon_g)\ \mid\ \iota : M \hookrightarrow T\ \}$$

The group $\mathrm{GL}(M)$ acts on the on B_g: if $\Omega = (\omega_{ij}) \in \mathrm{GL}(M)$ then

$$\Omega \cdot E = (\prod_i e_{ij}^{\omega_{ji}})$$

and the quotient $\tilde{B}_g$ is the space of embeddings $M \hookrightarrow T$.

We call a domain $A \subset \mathbb{G}_m^n$ *polyhedral* if $A = \mathrm{trop}^{-1}(\Delta)$ for some integral polytope Δ.

Proposition 3.3. *The fundamental domain of the monomial free action of* $\mathrm{GL}(M)$ *on* $\tilde{B}_g$ *is polyhedral.*

Proof. This is easily deduced from the fact that $\mathrm{GL}_n(\mathbb{Z})$ is generated by diagonal matrices which have $+1$ and -1 entries and matrices of the form $\mathrm{Id} + E_{ij}$ where E_{ij} is the elementary matrix that has 1 as the ij entry and otherwise 0. $\square$

Fix an isomorphism $i : M \cong M'$, then for any embedding $M \hookrightarrow T$ given by the matrix E the corresponding embedding $M' \hookrightarrow T'$ is represented by E^*.

Define the universal tori

$$\tilde{Z}_g = (T \times \tilde{B}_g)/M \qquad \tilde{Z}'_g = (T' \times \tilde{B}_g)/M'$$

over $\tilde{B}_g$. Since the map $\varphi_\lambda : T \to T'$ is M-equivariant, it descends to the quotients. Furthermore, the action of $GL(M)$ naturally lifts from B_g to $T \times B_g$ and sends T_s to $T_{\Omega s}$ is such a way that $\Omega(M_s) = M_{\Omega s}$.

Any homomorphism $\lambda : M \to M'$ is of the form $\Lambda \circ i$; if λ is a polarization then $\Lambda \in \mathrm{End}(M)$ is injective.

The morphism λ induces a surjective morphism of algebraic tori $\varphi_\lambda : T \to T'$ and for any embedding $M \hookrightarrow T$, $\varphi_\lambda(M) \subset M' \subset T'$ and $\varphi_\lambda | M = \lambda$. It therefore descends to the quotients: $\varphi_\lambda : Z_g \to Z'_g$.

For any matrix $E \in B_g$, $E = (e_{ij})$ denote by $\bar{E}$ the matrix $(-\log|e_{ij}|)$. Define

$$\tilde{A}_{g,\lambda} = \{\ E \in B_g\ \mid\ (\Lambda E) = (\Lambda E)^*, \bar{E} > 0\ \}$$

Put $\tilde{Z}^u_{g,\lambda} = Z_g \times_{\tilde{B}_g} \tilde{A}^u_{g,\lambda}$.

By construction, each $x \in A^u_{g,\lambda}$ defines a lattice and the fibre $(Z^u_{g,\lambda})_x$ carries the structure of a uniformized analytic torus, and the restriction of φ to it is a polarization.

Proposition 3.4. *If* $\iota : M \hookrightarrow T$ *is represented by a matrix* $E \in \tilde{A}_{g,\lambda}$ *then the map* $\varphi_\lambda : T/M \to T'/M'$ *is a polarization.*

Proof. We need to check that the form $\langle -,-\rangle : M\times M \to K^\times, \langle m_1, m_2\rangle = \lambda(a)(b)$ is symmetric and positive definite. Indeed,

$$\begin{aligned}\left\langle \sum_{i=1}^g a_i\varepsilon_i, \sum_{j=1}^g b_j\varepsilon_j \right\rangle &= \lambda(\sum_i a_i\varepsilon_i)(\prod_j \iota(\varepsilon_j)^{b_j}) \\ &= \prod_k (\prod_j e_{kj}^{b_j})^{\sum_i \lambda_{ik}a_i} \\ &= \prod_i\prod_j(\prod_k e_{kj}^{\lambda_{ik}})^{a_ib_j}\end{aligned}$$

which is clearly symmetric, given $\prod_k e_{kj}^{\lambda_{ik}} = \prod_k e_{ki}^{\lambda_{jk}}$. Further,

$$-\log|\langle \sum a_i\varepsilon_i, \sum_{j=1}^g b_j\varepsilon_j\rangle| = \sum_i\sum_j a_ib_j(\sum_k \lambda_{ik}\bar{e}_{kj}) = (\Lambda\bar{E}a, b)$$

where $(-,-)$ is the Euclidean scalar product on $\mathbb{R}^n$. Therefore, since the matrix $\Lambda\bar{E}$ is strictly positive definite, the bilinear symmetric form $-\log|\langle a,b\rangle|$ is also positive definite. □

Proposition 3.5. *The space $\tilde{A}^u_{g,\lambda}$ is isomorphic to a union of polyhedral domains. For any polyherdal domain $S\subset \tilde{A}^u_{g,\lambda}$, there is a bijective morphism from a polyhedral domain onto $\tilde{Z}_{g,\lambda}\times_{\tilde{A}^u_{g,\lambda}} S$.*

Proof. The first statement clearly holds for $\tilde{A}^u_{g,\mathrm{id}}$: the symmetry condition is intersection of some diagonal varieties, and positivity condition means that the coefficients of $\bar{e}_{ij}$ belong to some open subset of R^{g^2}, which is a union of integral polyherdra. Notice that

$$\tilde{A}^u_{g,\lambda} = \tilde{A}_{g,\mathrm{id}}\times_{B_g,\lambda} B_g$$

For any polyhedral domain $\mathrm{trop}^{-1}(\Delta)\subset A^u_{u,\mathrm{id}}$ the set $\mathrm{trop}^{-1}(\Delta)\times_{B_g,\lambda} B_g$ is polyhedral since λ is a monomial morphism.

The second statement follows from the fact that the fundamental domain of the action of M on each fibre $(T\times \tilde{A}^u_{g,\lambda})_s$, is a polyhedral domain, and that it only depends on $\mathrm{trop}(s)$. □

Let T/M be an analytic torus with a polarization $\varphi_\lambda : T/M \to T'/M'$ and let $L = (\mathrm{id}\times\varphi_\lambda)^*P$ where P is the Poincaré bundle on $T/M\times T'/M'$. Then linear rigidifications of L^3 are a PGL_N torsor where $N = 6^gd$ and $d = \mathrm{rk}\,\lambda$ (by Facts 2.8 and 2.7). Indeed, by Fact 2.9 the sections of L_λ are determined by coefficients $a_{w_1},\dots,a_{w_N}$ of theta functions $\sum_{\chi\in M'} a_\chi\chi$, where $w_1,\dots,w_N$ are some representatives of $M'/6\cdot\lambda(M)$. A linear rigidification is uniquely determined by a choice of basis in the space of these coefficients, up to scalar multiplication.

Acting by automorphism of the torus on the argument sends characters of T to characters. Let $\tilde{H}^u_{g,\lambda} = \tilde{A}^u_{g,\lambda} \times \mathrm{PGL}_N$ and extend the action of $\mathrm{GL}_g(\mathbb{Z})$ from $\tilde{A}^u_{g,\lambda}$ to $\tilde{H}^u_{g,\lambda}$ via the action of $\mathrm{GL}_g(\mathbb{Z})$ on the basis of theta functions by substitution:

$$\Omega \cdot \sum_{\chi \in M'} a_\chi \chi(x) = \sum_{\chi \in M'} a_{\Omega\chi} \chi(x)$$

Define

$$Z^u_{g,\Lambda} = \tilde{Z}_{g,\lambda}/\mathrm{GL}_g(\mathbb{Z}) \qquad H^u_{g,\Lambda} = \tilde{H}_{g,\lambda}/\mathrm{GL}_g(\mathbb{Z}) \qquad A^u_{g,\Lambda} = \tilde{A}_{g,\lambda}/\mathrm{GL}_g(\mathbb{Z})$$

The obvious map that forgets linearization makes $H^u_{g,\lambda}$ into a PGL_N-bundle over $A^u_{g,\lambda}$.

Lemma 3.6. *Let $A \in M_g(\mathbb{R})$ be a real matrix, and assume that there is a neighbourhood U of A in $M_g(\mathbb{R})$ such that all matrices $A' \in U$ define bilinear forms that are strictly positive definite on $\mathbb{Z}^n \subset \mathbb{R}^n$. Then A is positive definite.*

Proof. First note that $x \mapsto (Ax, x)$ is positive on $\mathbb{Z}^n$ if and only if it is positive on $\mathbb{Q}^n$.

Suppose A is not positive definite. It cannot have negative eigenvalues, so assume it has an eigenvector x with eigenvalue 0. By assumption this vector has irrational coordinates. As A' varies in U, the eigenspace $\mathbb{R} \cdot x$ varies too. Clearly there exists an A' with arbitrarily close eigenspace V with eigenvalue zero with $V \cap \mathbb{Q}^n \neq \{0\}$. For such A' the assumption is not true, and we have arrived at a contradiction. $\square$

Proposition 3.7. *For any polarization type $\lambda : M \to M'$ and any rationally valued field k, the set of analytic tori $(Z_{g,\lambda})_x$ as x ranges in $H^u_g(k)$ coincides with the set $\mathcal{H}^u_{g,\lambda}(k)$.*

Proof. Follows from construction of $H^u_{g,\lambda}$,$Z_{g,\lambda}$ Propositon 3.4 and Lemma 3.6. $\square$

Recall that the functor $\mathcal{H}_{g,d,n} : \mathrm{Sch}/S \to \mathrm{Sets}$, defined in Section 6 of [MFK94], associates to a scheme S the set of isomorphism classes of linearly rigidified degree d polarized Abelian schemes over S with level n structures. For our purposes we do not need to deal with the level structure and we will only consider the functor $\mathcal{H}_{g,d,1}$ which we will denote $\mathcal{H}_{g,d}$. Let $H_{g,d}$ be the k-scheme that represents the functor $\mathcal{H}_{g,d}$.

Proposition 3.8. *For any polarisation λ of degree d there exists a rigid-analytic embedding $\mathcal{H}^u_{g,\Lambda} \hookrightarrow (\mathcal{H}_{g,d})^{an}$ and a rigid-analytic embedding $Z_{g,\Lambda} \to Z^{an}_{g,d}$ compatible with projection to $H_{g,d}$*

Proof. By [Con06, Theorem 4.1.3] there exists an analytic embedding of $\mathcal{H}_g^u$ into $((\mathrm{Hilb}_{\mathbb{P}^N}/k)^{P(x)})^{an}$, where $P(X) = 6^g \cdot d \cdot x^g$, and of $Z_{g,\lambda}^u$ into $(Z_{g,d})^{an}$. For any $s \in \mathcal{H}_g^u$ the fibre $(Z_{g,\lambda})_s$ is a polarized Abelian variety and hence, by Proposition 7.3 of [MFK94], $s \in H_{g,d}^{an} \subset ((\mathrm{Hilb}_{\mathbb{P}^N}/k)^{P(x)})^{an}$. □

Corollary 3.9. *Let k be rationally valued. For any k-variety S, any polarized Abelian scheme $A \to S$ and any semi-algebraic subset $U \subset S$ such that A_s is mulitplicatively uniformized for all $s \in U$ there exists a map $U \to H_g^u$ such that $Z \times_{H_g^u} U \cong A \times_S U$.*

3.2 Integration

We are going to use the tropical motivic Fubini theorem of Nicaise and Payne which we now recall.

Theorem 3.10 ([NP17])**.** *Let $A \subset Y \times \mathbb{G}_m^n$ be a semi-algebraic subset, and let $\pi : Y \times \mathbb{G}_m^n \to \mathbb{G}_m^n$ be the projection map. Then there definable subsets $\Delta_1, \ldots, \Delta_m \subset \mathbb{R}^n$ and classes $X_1, \ldots, X_m \in K_0(\mathrm{Var}_{\bar{k}})$ such that for any integer $i, 1 \leq i \leq n$ and for any $\xi \in \Delta_i$, $\mathrm{Vol}((\mathrm{trop} \circ \pi)^{-1}(\xi)) = X_i \in K_0(\mathrm{Var}_{\bar{k}})$ and*

$$\mathrm{Vol}(A) = \sum_{i=1}^{m} \chi'(\Delta_i)(\mathbb{L} - 1)^n \cdot X_i$$

We finally put together all the ingredients prepared so far.

Theorem 3.11. *Let T be a k-variety, let $\pi : A \to S$ be a Abelian scheme of relative dimension g over S and let $U \subset S$ be a semi-algebraic set such that A_s can be uniformized by a torus for any $s \in U$. Then $\mathrm{Vol}(A \times_S U) = 0$.*

Proof. By Fact 2.16 we may use maps definable in $\mathrm{ACVF}_{\mathrm{LR}}$.

We may assume that S and T are connected. Pick some polarization on A, then by Corollary 3.9 there exists a map $T \to H_{g,d}$ for some d and such that the image of U lies in $H_{g,\lambda}^u$ for some λ, $\mathrm{rk}\,\lambda = d$.

Using Corollary 2.15 and Proposition 3.5 we will identify $Z_{g,\lambda}^u$ and $A_{g,\lambda}^u$ with unions of polyhedral domains. It follows from Proposition 3.3 that there exists a decompostion $U = \sqcup U_i$ with U_i semi-algebraic such that restirictions of $A \times_S U_i$ to the fibres of the projection $H_{g,\lambda}^u \to A_{g,\lambda}^u$ are trivial families of tori. Consequently, $A \times_S U$ is in a definable bijection with a semi-algebraic set $Z_{g,\lambda}^u \times_{A_{g,\lambda}^u, \psi} U$ for some definable map $\psi : U \to A_{g,\lambda}^u$.

Let $\Sigma = \mathrm{trop}(\psi(U)) \subset \mathrm{trop}(A_{g,\lambda}^u)$. Then $Z^u \times_{A_{g,\lambda}^u} \mathrm{trop}^{-1}(\Sigma) = \mathrm{trop}^{-1}(\Delta)$ for some definable subset $\Delta \subset \mathbb{R}^n$ for some n. Let ψ' be the definable bijection $A \to Z_{g,\lambda}^u \times U$ induced by ψ. Then $A \times_T U$ is in definable bijection with the graph $\Gamma_{\psi'} \subset (A \times_T U) \times \mathrm{trop}^{-1}(\Delta)$ of the map ψ'.

Denote $\pi : \Delta \to \Sigma$ the natural projection, and denote

$$P = \{ \ x \in k^\times \ \mid \ |x| = 1 \ \}$$

the unit annulus. One observes that

$$(\mathrm{trop} \circ \psi')^{-1}(\xi) = (\mathrm{trop} \circ \psi)^{-1}(\xi) \times P^g$$

for $\xi \in \Delta$.

Finally, by Theorem 3.10 there exists a decomposition $\Sigma = \sqcup_{i=1}^{n} \Sigma_i$ into definable subsets such that $\mathrm{Vol}((\mathrm{trop} \circ \psi)^{-1}(\xi))$ is constant for all $\xi \in \Sigma_i$, for each i, and so

$$\begin{aligned}\mathrm{Vol}(\Gamma_{\psi'}) &= \sum_{i=1, \xi \in \Sigma_i}^{n} \mathrm{Vol}(P^g \times (\mathrm{trop} \circ \psi')^{-1}(\xi)) \chi'(\pi^{-1}(\Sigma_i))(\mathbb{L} - 1)^{\dim \pi^{-1}(\Sigma_i)} \\ &= \sum_{i=1, \xi \in \Sigma_i}^{n} \mathrm{Vol}((\mathrm{trop} \circ \psi')^{-1}(\xi))) \chi'(\pi^{-1}(\Sigma_i))(\mathbb{L} - 1)^{\dim \pi^{-1}(\Sigma_i)}\end{aligned}$$

Here, $\chi'(\pi^{-1}(\Sigma_i)) = 0$ since χ' is multiplicative and fibres of π are fundamental domains of a lattice, so χ' vanishes on them. □

Corollary 3.12. *Let $C \to T$ be a family of smooth projective curves. Let $S \subset T$ be a semi-algebraic subset of T such that C_s is a Mumford curve for all $s \in S$, and let $J(C/T) \to T$ be the relative Jacobian. Then* $\mathrm{Vol}(J(C/T) \times_T S) = 0$.

Proof. The family $J(C/T) \to T$ is a projective Abelian scheme, and its restriction to S can be uniformized by a torus fibrewise, therefore, Theorem 3.11 applies. □

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