Combinatorial part of the cohomology of the nearby fibre

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Abstract

Let $f : X \to S$ be a unipotent degeneration of projective complex manifolds over a disc such that the reduction of the central fibre $Y = f^{-1}(0)$ is simple normal crossings, and let X_∞ be the canonical nearby fibre. Building on the work of Kontsevich, Tschinkel, Mikhalkin and Zharkov, I introduce a sheaf of graded algebras $\Lambda^\bullet$ on the dual intersection complex of Y, denoted Δ_X. I show that there exists a map $H^q(\Delta_X, \Lambda^p) \to \operatorname{gr}^W_{2q} H^{p+q}(X_\infty, \mathbb{Q})$, where W is the monodromy weight filtration, which is injective whenever there exists a class $\omega \in H^2(Y)$ which is combinatorial and Lefschetz, a certain technical condition. When f is a Type III Kulikov degeneration of $K3$ surfaces, the sheaf Λ^1 recovers the singular affine structure of Engel and Friedman. In this case, I show that a sufficient condition for existence of such class is the existence of a positive d''-closed $(1,1)$-superform or supercurrent in the sense of Lagerberg on Δ_X.

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1. Introduction

1.1. **Overview of the results.** Given a reduced proper complex algebraic variety X such that its irreducible components are smooth and intersect normally, the combinatorics of intersections of the components determines part of the singular cohomology of X. This object appears, for example, in [ABW13] under the name "combinatorial part of the cohomology" and is identified with the top weight piece of the associated graded of the cohomology of X, with respect to the weight filtration defined by Deligne [Del71]. In a similar vein, recently [CGP21, CGP19] studied the top weight cohomology of the moduli space of curves using tools from tropical geometry, which help describe the combinatorics of the intersections of the components of the boundary divisor in the Deligne-Mumford compactification. Finally, if $f : X \to S$ is a proper morphism from a complex manifold to a disc, smooth away from the snc divisor $Y := f^{-1}(0)$, then by work of Steenbrink [Ste76, Ste95] weight 0 part of the limiting mixed Hodge structure on the cohomology of X_t for t sufficiently small is isomorphic to the singular cohomology of the *dual intersection complex* of Y, a CW-complex encoding the intersections of its irreducible components (a more functorial version of this statement was obtained by Berkovich [Ber09]).

In this paper I extend the results of Steenbrink and Berkovich, defining certain constructible sheaves Λ^p of $\mathbb{Q}$-vector spaces on the dual intersection complex for each

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integer $p \geq 0$, such that cohomology of Λ^p can be related to the cohomology of X_t, recovering parts of cohomology in even weights of the limiting mixed Hodge structure on X_t. These results are motivated by the non-archimedean approach of Kontsevich and Sobelman to the SYZ conjecture [KS06], where singular affine structures on the dual intersection complexes of Y for certain *minimal* degenerations play an important role. In fact, in the case of minimal Type III degenerations of K3 surfaces constructed by Kulikov, Persson and Pinkham [Kul77, PP81], such singular affine structures have been studied in [GHK15, Eng18, AET19], and it turns out that the sheaf Λ^1 defines the same singular affine structures for this class of degenerations. For more detailed discussion, see Section 1.3.

In the following denote $f : X \to S$ a proper morphism from a complex manifold X to a disc S, such that the reduction of the central fibre $Y = \sum_{i\in I} N_i Y_i$ is a simple normal crossings divisor. We assume $\mathbb{Q}$ coefficients for cohomology everywhere. Denote X_∞ the canonical nearby fibre and let $N = \log T$, where $T : H^\bullet(X_\infty, \mathbb{R}) \to H^\bullet(X_\infty)$ is the monodromy endomorphism, which is assumed to be unipotent. Call a class $\omega \in H^2(Y)$ *cohomologically Kähler* if $\omega|_{Y_i} \in H^2(Y_i)$ is Kähler for all $i \in I$. Call the intersections of the form $Y_{i_1} \cap \ldots \cap Y_{i_k}$ the *strata* of the divisor Y and denote $Y^{(p)}$ the union of strata of Y of codimension p in X. Denote ${}'H^\bullet(Y^{(p)})$ the subspace spanned by cycle classes of strata. The dual intersection complex of Y is a CW complex whose k-cells correspond to codimension $k+1$ strata and are glued into the boundary of those $k+1$ cells that correspond to the substrata (see Section 2.1 for precise definition). I denote the dual intersection complex of Y as Δ_X, underlining the fact that I am only interested in the situation when Y is smoothable.

I define (Section 3.1) certain sheaves of graded $\mathbb{Q}$-algebras $\Lambda^\bullet$ on Δ_X, $\Lambda^0 = \mathbb{Q}$. There always exists a sheaf A^1 which fits into the exact sequence

$$0 \to \Lambda^0 \to A^1 \to \Lambda^1 \to 0.$$

In general, the graded component Λ^1 does not determine $\Lambda^p, p > 1$, even locally. When Λ^p is determined by Λ^1 in a neighbourhood of a face σ (Definition 3.2.1), let us say that $\Lambda^\bullet$ is regular at the face σ. If $\Lambda^\bullet$ is regular at all faces σ of Δ_X then higher graded components of the sheaf $A^\bullet$ can be defined, and they fit into an exact sequence

$$0 \to \Lambda^p \to A^{p+1} \to \Lambda^{p+1} \to 0. \tag{1.1}$$

The first main result of the paper says that under a standard requirement the cohomology of sheaves Λ^p maps to the cohomology classes of X_∞ of even weight.

Theorem A. Assume that there exists a cohomologically Kähler class $\omega \in H^2(Y)$. Then for every p, q there exists a morphism

$$H^q(\Delta_X, \Lambda^p) \to \mathrm{gr}^W_{2q} H^{p+q}(X_\infty).$$

Moreover, for any class $v \in H^q(\Delta_X, \Lambda^p)$ the image of v in $H^{q+1}(\Delta_X, \Lambda^{p-1})$ under the coboundary morphism associated to the short exact sequence (1.1) coincides with Nv.

If M is a complex Kähler manifold of dimension n, a class $\omega \in H^2(M)$ is called a *Lefschetz class* if

- the Lefschetz operator $L_\omega(x) = x \cup \omega$ on the cohomology algebra induces isomorphisms
$$L_\omega^{n-i} : H^i(M) \to H^{2n-i}(M)$$
(*Lefschetz property*);

- the Hodge decomposition on $H^k(M)$ is orthogonal with respect to the form
$$\psi(x,y) = i^k \int_M x \wedge y \wedge \omega^{n-k}$$
and $i^{p-q-k}(-1)^{\frac{(p+q)(p+q-1)}{2}}\psi$ is positive definite on $\operatorname{Ker} L_\omega^{n-p-q+1} \cap H^{p,q}(M)$. (*Hodge-Riemann bilinear relations*).

A classic theorem of Hodge theory states that a class ω is a Lefschetz class if it is a Kähler class (see, for example, [Voi03, Section 6.3.2]). If $f : M \to N$ is a semi-small map to a complex variety and ω is the first Chern class of an ample line bundle, then any positive multiple of $f^*\omega$ is Lefschetz [dCM02].

We call a class $\omega \in H^{2j}(Y)$ *combinatorial* if its restriction to any irreducible component Y_i is a linear combination of cycle classes of strata.

Theorem A'. If there exists a combinatorial Lefschetz class $\omega \in H^2(Y)$ then the morphism constructed in theorem A is injective and for $p > q$
$$N^{p-q} : H^q(\Delta_X, \Lambda^p) \to H^p(\Delta_X, \Lambda^q)$$
is an isomorphism. Moreover, $\dim H^{n-q}(\Delta_X, \Lambda^{n-p}) = \dim H^q(\Delta_X, \Lambda^p)$.

For proof of Theorems A and A' see Theorems 5.1.5 and 5.1.6.

One can observe easily that central fibres of degenerations of curves or of degenerations of Abelian varieties with toric reduction admit combinatorial Lefschetz classes. In the latter case, ${}'H^i(Y^{(j)}) = H^i(Y^{(j)})$ for all i, j.

The theorem A' is intended to be applied in the case of maximally unipotent degenerations, i.e. when T has a Jordan block of size $n+1$, though I expect that the requirement on ω can be reformulated in lower unipotency rank case to still yield the same result.

It seems plausible that the cup product on cohomology of Λ^p on Δ_X is compatible with the cup product on the cohomology of X_∞, and in particular, the equality of dimensions in Theorem A' follows from a Lefschetz property for the class in $H^1(\Delta_X, \Lambda^1)$ that corresponds to a combinatorial Lefschetz class in $H^2(Y)$. The proof of this statement is at present hindered by the fact that there is no known formula for the cup product on the Steenbrink complex that computes the cohomology of the nearby fibre and that is used crucially in the proof of Theorem A.

The superforms is a generalization of differential forms on a real vector space to the context of affine geometry, introduced by Lagerberg [Lag11], that parallels the notion of (p,q)-forms in complex analysis. The definition of superforms has bees extended to the setting of tropical varieties by Chambert-Loir and Ducros, see [Gub16, CL12a]. The tropical cohomology groups can be computed with superforms [JSS19]. The cohomology of the sheaves Λ^p can be computed using superforms of Lagerberg [Lag11] using a resolution similar to the tropical Dolbeault complex of [JSS19]. If the sheaf $\Lambda^\bullet$ is regular at every face of Δ_X then furthermore the morphism N can be lifted to the level of superforms.

Recall that a unipotent snc degeneration of K3 surfaces $f : X \to S$ is called a *Kulikov degeneration* if $K_X = 0$. In this case the components of the central fibre are rational surfaces obtained from toric surfaces by a finite number of blow-ups (this number is called *charge*), which can be interpreted as operations that introduce singularities to the standard affine structure on the fan of the toric surface. The existence of a combinatorial Lefschetz class on the central fibre $Y = f^{-1}(0)$ can be characterized in this case with the help of positive superforms or supercurrents.

Theorem B. Let $f : X \to S$ be a Kulikov degeneration of K3 surfaces of Type III. Then Λ^1 is the push-forward of the sheaf of parallel 1-forms from the complement of the finite set of the singularities, with respect the affine structure defined in [GHK15], [Eng18].

Moreover,

i) If all irreducible components of Δ_X have charge 1 or 0 then
$$H^1(\Delta_X, \Lambda^1) \hookrightarrow \operatorname{gr}_2^W H^2(X_\infty)$$
is surjective, and
$$H^q(\Delta_X, \Lambda^p) \to \operatorname{gr}_{2p}^W H^{p+q}(X_\infty)$$
is an isomorphism for all other values of p, q;

ii) If Δ_X admits a virtual line bundle with a convex PL metric or, equivantly, a d''-closed positive $(1,1)$-superform or supercurrent then this morphism is injective.

The statements of Theorem B are proved as Propositions 6.1.5, 6.2.2, 6.2.4, and 6.3.1.

In the article [Sus22], I show that if the singularities of the affine structure on Δ_X have unipotent monodromy (which is the case for the affine structure on Δ_X when all components of the central fibre have charge at most 1) then there exists a positive $(1,1)$-supercurrent on Δ_X. In particular, the main result of [Sus22] implies that the map
$$H^1(\Delta_X, \Lambda^1) \hookrightarrow \operatorname{gr}_2^W H^2(X_\infty)$$
in the Theorem A is an isomorphism.

1.2. **State of the art.** Let us make a brief overview of known results about degenerations that are similar to Theorems A and A'.

If a family $X^* \to S^*$ of complex varieties over a punctured disc factors into an embedding $X \hookrightarrow \mathbb{P}^m \times S^*$ and the projection on S^* then one can associate to it yet another CW complex, its *tropical limit* $\operatorname{Trop}(X)$, see [IKMZ19, Section 3.1], [Pay08] via logarithmic and non-archimedean tropicalization maps. Geometrically, this complex can be interpreted as a dual intersection complex of the union of the central fibre of the closure of X in a certain family of toric varieties over S and the toric boundary. It is canonically embedded into a certain compactification of $\mathbb{R}^m$ homeomorphic to an m-simplex.

A tropical limit is in particular a tropical variety. A homology theory for tropical varieties is developed in [IKMZ19]. The coefficients of this theory are given by certain cosheaves $\mathscr{F}_p$, which are defined in terms of the local polyhedral structure of $\operatorname{Trop}(X)$. The main result of [IKMZ19] shows that under a certain combinatorial condition of *tropical smoothness*, spaces $\operatorname{Hom}(H_q(\operatorname{Trop}(X), \mathscr{F}_p)$ are isomorphic to $\operatorname{gr}_{2p}^W H^{p+q}(X_\infty)$ and X_∞ has no cohomology in odd weights. In particular, the monodromy is maximally unipotent. It is also clear from the proof of the theorem that all cohomology classes in $H^\bullet(X_\infty)$ have type (p,p), or that the limit mixed Hodge structure of X_∞ is of Hodge-Tate type. The latter property is much stronger than maximal unipotency (though, for example, is known to coincide with it for hyperkähler manifolds, see [Sol18]), which serves as an illustration to how restrictive the requirement of tropical smoothness is.

The tropical homology (and associated cohomology) theory enjoys many properties that make it similar to the (co)homology of complex manifolds, for example, tropical (co)homology groups enjoy Poincaré duality [JSS19] on tropical manifolds and satisfy a version of Lefschetz theorem on $(1,1)$-classes [JRS17]. On the other hand, there are strange pathological phenomena already for curves: the group $H^1(\operatorname{Trop}(X), \mathscr{F}_1)$ can be infinite-dimensional, see [Jel19, Section 4].

Let us also mention the toric degenerations of Gross and Siebert which come by definition with singular affine structure on the dual intersection complex. In this case cohomology of the certain sheaves related to the mentioned singular affine structure coincides with the hypercohomology of the complex of log-differential forms [GS10, Section 3]. The relationship between these cohomology groups and the nearby fibre cohomology have been studied in [Rud10].

The approach presented in this paper Λ^p can be seen as interpolating between the tropical cohomology and the Gross-Siebert approach (see also [Yam21] for results that bridge the two). On the one hand, one is not restricted to the degenerations that satisfy tropical smoothness condition and the embedding of the degeneration into $\mathbb{P}^m$ is not important. On the other hand, the definition of sheaves Λ^p works on any degeneration with smooth total space and central fibre with snc support.

1.3. **Motivation and discussion.** The study of degenerations of complex varieties has recently received much interest in connection to mirror symmetry and in particular the non-archimedean approach to the SYZ conjecture proposed by Kontsevich and Soibelman. The main object of study of this research program is a family of polarized Calabi-Yau manifolds X^* of complex dimension n fibered over a punctured disc S^* and having a maximally unipotent monodromy. Such a family gives rise to a projective Calabi-Yau variety $\mathscr{X}$ over the field of germs of meromorphic fuctions $\mathbb{C}\{\{t\}\}$ and its Berkovich analytification $\mathscr{X}^{\mathrm{an}}$, a locally ringed topological space, whose points are absolute values on the residue fields of scheme-theoretic points of $\mathscr{X}$ which restrict to the natural non-archimedean absolute value on $\mathbb{C}\{\{t\}\}$. Kontsevich and Soibelman [KS06] defined a certain canonical subset $\mathrm{Sk}(\mathscr{X}^{\mathrm{an}}) \subset \mathscr{X}^{\mathrm{an}}$ and conjectured that it is homeomorphic to a manifold of dimension n, and that $\mathscr{X}^{\mathrm{an}}$ admits a retraction onto $\mathrm{Sk}(\mathscr{X}^{\mathrm{an}})$, such that the general fibre, away from a codimension 2 subset of $\mathrm{Sk}(\mathscr{X}^{\mathrm{an}})$ is isomorphic to a fibration in non-archimedean tori $\{\, |x_1| = \ldots = |x_n| = 1 \,\} \subset (\mathbb{G}_m^n)^{\mathrm{an}}$. This retraction can then be used to construct a *mirror family* $\check{X}^*$ by taking a Legendre dual of the singular integral affine structure on $\mathrm{Sk}(\mathscr{X}^{\mathrm{an}})$ induced by the torus fibration, and reconstructing the variety $\check{\mathscr{X}}$ so that $\mathrm{Sk}(\check{\mathscr{X}}^{\mathrm{an}}) = \mathrm{Sk}(\mathscr{X}^{\mathrm{an}})$ and a retraction of $\check{\mathscr{X}}$ onto $\mathrm{Sk}(\check{\mathscr{X}})$ induces the dual singular affine structure.

This singular affine structure is important in the metric version of the picture above. Let L denote the realatively ample line bundle on X^*, then by a theorem of Yau X_t admits a Ricci-flat Kähler metric with the fundamental form $\omega_t \in c_1(L_t)$. Kontsevich and Soibelman conjecture that as t tends to 0,

$$(X_t, \frac{\omega_t}{\sqrt{\mathrm{diam}(X_t)}}) \to B$$

in the sense of Gromov-Hausdorff, where B is a metric space that satisfies the following properties:

i) B is homeomorphic to Δ_X (Conjecture 3 [KS06]);
ii) there exists a dense open subset B^{sm}, such that $B \setminus B^{sm}$ has Hausdorff codimension at least 2, and such that B^{sm} is an oriented Riemannian manifold of dimension n;
iii) B^{sm} has an integral affine structure;
iv) the metric g on B^{sm} satisfies a real Monge-Ampere equation, i.e. g is given in some local affine coordinates $x_1, \ldots, x_n$ by

$$g_{ij} = \frac{\partial^2 F}{\partial x_i \partial x_j} \qquad \det(g_{ij}) \equiv const,$$

for some smooth function F.

The space B is also conjectured to be a sphere of real dimension n if $X \to S$ is a family of Calabi-Yau in the strict sense (i.e. $H^{i,0}(X_t) = 0$, $0 < i < n$), and to $\mathbb{CP}^{n/2}$ if fibres X_t are hyper-kähler.

The set $\mathrm{Sk}(\mathscr{X}^{\mathrm{an}})$ is defined as the minimum locus of a certain *weight function* on $\mathscr{X}^{\mathrm{an}}$ (see [MN15] for its properties), but in fact this set can be defined without reference to non-archimedean geometry at all. Whenever X^* admits an extension to a smooth manifold X over the disc $S \supset S^*$ so that the central fibre Y has a strictly normal crossing support, there exists a canonical embedding $\Delta_X \hookrightarrow \mathscr{X}^{\mathrm{an}}$ and retraction $\mathscr{X}^{\mathrm{an}} \to \Delta_X$ (in fact, even strong homotopy retraction) [Ber99, Thu07]. The weight function can be defined in terms of the order of vanishing of the extension of a holomorphic volume form on the irreducible components Y.

In fact, one can weaken the requirement on the pair (X, Y), namely, Nicaise and Xu have realized that if (X, Y_{red}) is a dlt pair, X is $\mathbb{Q}$-factorial and $K_X + Y_{\mathrm{red}}$ is semi-ample (*good minimal dlt degeneration*) then $\mathrm{Sk}(\mathscr{X}^{\mathrm{an}}) = \Delta_X$ [NX16] and $\mathscr{X}^{\mathrm{an}}$ has homotopy type of Δ_X, though there is no canonical retraction any more. Crepant modifications of X preserve the homeomorphism type of Δ_X but alter its subdivision into simplices. In [NXY19] the authors show that such retraction exists in codimension 1 on Δ_X for good minimal dlt degenerations. See [KX16, KLSV18, Mau20] for further results on dual intersection complexes of degenerations.

There has been considerable progress in proving properties $(ii) - (iv)$ of the Gromov-Hausdorff limit, see [Tos20] for a survey, but property (i) is only known for degenerations of Kummer surfaces [Got22]. In the case of degenerations of hyperkähler manifolds, the main technique, pioneered by Gross and Wilson is to start with a fibration of a fixed hyperkähler manifold into Abelian varieties, then obtain the degeneration $X \to S$ via hyperkähler rotation of the complex structure. In [OO18] the authors manage to extend this technique to any degeneration of K3 surfaces using approximation and careful study of the moduli space. See also [Sus18] for a description of Gromov-Hausdorff limits of curves with abelian differentials in terms of the minimum loci of the weight function.

If we assume the property (i) above, the conjectures $(ii) - (iv)$ imply that for each choice of polarisation there exists a set of affine structures on Δ_X with respect to which the limit metric satisfies the real Monge-Ampere equation. One can ask the following question:
Question A: can these structures be described algebraically, purely in terms of the polarisation L ?

The definitions and results presented in this paper are intended as a first step in the study of this question.

The cohomology of $\Lambda^p \otimes \mathbb{R}$ can be computed using the sheaves of superforms defined by Lagerberg [Lag12a], see [JSS19, CL12b, Gub16] for the construction in the tropical geometry. A Lagerberg superform on a real vector space V is a real form on $V \oplus V$ which is invariant under translations along the second summand. In particular, a (p, q)-Lagerberg form on a base of a torus fibration can be lifted to a $(p+q)$-form on the total space of the fibration. In [BJ17] Boucksom and Jonsson consider [BJ17, Section 2] for any snc degeneration $X \to S$ a map Log_X to Δ_X, defined on a neighbourhood of the special fibre. If X is maximally unipotent, the generic fibre of this map is a complex torus of the same dimension as the skeleton. In fact, this map is not canonical and depends on choices of coordinates near the strata of the central fibre, but the error resulting from different choices is $O(1/\log|t|)$. Similarly, the choice of local coordinates on Δ_X, i.e. sections of Λ^1, provides a choice of a trivialization of the torus fibration (also up to a bounded error), so Lagerberg forms can be locally lifted to forms on nearby fibres X_t, gluing using a partition of unity should allow to lift the forms globally, again, up to a controlled error.

I expect this lifting to be compatible with the morphism constructed in Theorem A in the limit as $t \to 0$. The data of a metric g from the statement of conjectures $(ii) - (iv)$ is equivalent to a $(1,1)$-superform ω on Δ_X such that $\omega^n = \mu$, where μ is the Euclidean measure on Δ_X. It is natural to expect that the lifting of ω should approach the Ricci-flat Kähler metrics in the cohomology class $c_1(L_t)$ as $t \to 0$, and so the cohomology class of ω should map under the morphism from Theorem A to the class of $c_1(L)$ in $\mathrm{gr}_2^W H^2(X_\infty)$.

I suggest therefore that tackling the following question might be useful in answering the Question A:

Question B: for a given maximally unipotent family of polarizedCalabi-Yau manifolds over a punctured disc what is the set of extensions $X \to S$ such that the first Chern class of the polarisation lies in the image of the morphism constructed in Theorem A?

1.4. **Structure of the paper.** The background information on dual intersection complexes, Steenbrink complex and Hodge-Lefschetz modules is recalled in Section 2. Section 3 introduces the sheaves $\Lambda^\bullet$ and proves that their definion is independent of the subdivision of Δ_X induced by the blow-ups of the strata. Certain complexes quasi-isomorphic to the even rows of the monodromy weight spectral sequence associated to Steenbrink complex are constructed in Section 4. They are then used to prove Theorems A and A' in Section 5. The singular affine structure on Δ_X for Type III Kulikov degenerations defined in [GHK15, Eng18] is related to A^1 in Section 6, where Theorem B is also proved.

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2. Background

Let $f : X \to S$ be a proper morphism from a complex manifold X to a disc $S \subset \mathbb{C}$. Let S^* be the punctured disc and let $\tilde{S} \xrightarrow{exp} S^*$ be its universal cover. The restriction of f to $f^{-1}(S^*)$ is a locally trivial fibration by Ehresmann's theorem and therefore the space $X_\infty = X \times_S \tilde{S}$, the *canonical nearby fibre*, is homotopy equivalent to X_t for any $t \neq 0$. The fundamental group $\pi_1(S^*) \cong \mathbb{Z}$ acts on X_∞ by deck transformations, and we denote the action of the generator represented by a counter-clockwise loop $H^\bullet(X_\infty, \mathbb{Q})$ as T. The operator T is quasi-unipotent by a theorem of Borel [Sch73, Lemma 4.5, Theorem 6.1].

Assume that f is a smooth morphism away from $Y = f^{-1}(0)$. Let k be the projection $X_\infty \to X$ and let $i : Y \to X$ be the embedding of the central fibre. Given a ring R the nearby cycles complex $\psi R := i^* Rk_* R$ has the property

$$H^\bullet(\psi R)_x \cong \varinjlim_{\varepsilon \to 0} H^\bullet(F_{x,\varepsilon}, R)$$

where $F_{x,\varepsilon} = B_{x,\varepsilon} \cap f^{-1}(0)$ and $B_{x,\varepsilon}$ is an ε-ball centered at $x \in Y$ for a sufficiently small ε ($F_{x,\varepsilon}$ is called the *Milnor fibre* of f at x). Moreover,

$$\mathbb{H}^\bullet(Y, \psi R) \cong H^\bullet(X_\infty, R)$$

If f is smooth away from the central fibre $Y = \sum_{i \in I} Y_i E_i$ and the latter is a divisor with simple normal crossings support then we will call such morphism an *snc degeneration.* If additionally the monodromy operator T is unipotent, we call it a *unipotent snc degeneration.*

2.1. **Dual intersection complexes.** Denote $\underline{\sigma}$ the set of vertices of a simplex σ. Given a face σ and a vertex $i \in \underline{\sigma}$ there exists a unique face, which we will denote $\partial_i\sigma$, such that $\partial_i\sigma \subset \sigma, \underline{\partial_i\sigma} \cup \{i\} = \underline{\sigma}$.

If σ is a face of a simplicial complex, we will denote $\mathrm{St}(\sigma)$ the *open star* of σ: the union of interiors of all simplices that contain it, or just σ if σ is a vertex. Clearly, $\mathrm{St}(\sigma) = \cap_{i \in \underline{\sigma}} \mathrm{St}(i)$. We will denote as $\overline{\mathrm{St}}(\sigma)$ the *closed star* of a simplex σ, that is, the union of simpleces that contain σ. This set has the natural structure of a simplical complex, and we will denote $\overline{\mathrm{St}}^0(\sigma)$ its set of vertices. The dual intersection complex of an snc divisor generally has a structure of a Δ-complex, see [Hat02, Ch. 0] for definition.

Let X be a smooth variety and let $D = \sum_{i=1}^m N_i D_i$ be a divisor in X with snc support; the connected components of finite intersections $D_{i_1} \cap \ldots \cap D_{i_k}$ are called the *strata* of D. Let the set of k-dimensional cells of $\Delta(D)$ be in bijective correspondence with the set of the strata of D of codimension $k+1$ in X. Suppose that the k-skeleton of $\Delta(D)$ is already defined. Since the divisor D has snc support for any stratum $Z \subset D_{i_1} \cap \ldots \cap D_{i_k}$ and any $l, 1 \leq l \leq k$ there exists a unique stratum

$$Z_l \subset \bigcap_{j \neq l} D_{i_j}$$

that contains Z. The $k+1$-cell corresponding to Z is glued to the k-skeleton is such a way that the cells corresponding to Z_l are in its boundary. Thus in a dual intersection complex Δ_X the faces $\tau \in \mathrm{St}(\sigma)$ correspond to the strata Y_τ that contain Y_σ and the faces $\tau \in \overline{\mathrm{St}}(\sigma)$ correspond to the strata Y_τ that intersect non-trivially with Y_σ. It is noticed in [dFKX17, Definition 8] that if (X, Y) is a dlt pair, then the above procedure is well defined for the union $Y^{=1}$ of the irruducible components of Y with multiplicity 1.

We identify a cell σ with the simplex

$$\{(x_i)_{i \in \underline{\sigma}} \in \mathbb{R}\underline{\sigma} \mid \sum_{i \in \underline{\sigma}} N_i x_i = 1\}$$

We denote the affine space spanned by σ as $\langle \sigma \rangle$ and its tangent space as $T\langle \sigma \rangle$. In particular, each cell has an integral affine structure associated to it, but even if Δ_X is a manifold, it is not always possible to glue these affine structures to obtain an affine structure on Δ_X.

Let $f : X \to S$ be an snc degeneration with the central fibre Y, let $Z \subset Y$ be a closed connected submanifold. Let $\pi : X' \to X$ be the the blow-up of X in Z, let $Y' = (f \circ \pi)^{-1}(0) = Y'_{\neq 0} \cup Y_0$ where Y'_0 is the exceptional divisor and $Y'_{\neq 0} = \pi_*^{-1}(Y)$ is the strict transform of Y. The dual intersection complex $\Delta(Y')$ then admits a natural PL morphism to Δ_X and can be described as follows.

If Z is a stratum of Y then the dual intersection complex of $\Delta(Y')$ is obtained from Δ_X by the subdivision of the face corresponding to Z and the morphism $\Delta(Y') \to \Delta_X$ is a homeomorphism. In particular, after blowing sufficiently many strata of Y one obtains the dual intersection complex $\Delta(Y')$ of the special fibre Y' of the blow-up can be ensured to be a simplicial complex homeomorphic to Δ_X.

If Z is contained in a smallest stratum Y_σ then $\Delta(Y')$ can be described as follows [dFKX17, Paragraph 9]. The dual intersection complex $\Delta(Y'_0 \cap Y'_{\neq 0})$ is the join $\sigma *$

$\Delta(Z \cap (\cup_{i\notin\underline{\sigma}} Y'_i))$. Since each stratum of $\Delta(Y'_0 \cap Y'_{\neq 0})$ is sent by π to a stratum that intersects Z non-trivially we have a map of Δ complexes

$$\Delta(Y'_0 \cap Y'_{\neq 0}) \to \overline{\mathrm{St}}(\sigma)$$

The dual intersection complex $\Delta(Y')$ is obtained by glueing the cone over $\Delta(Y'_0 \cap Y'_{\neq 0})$ with $\overline{\mathrm{St}}(\sigma)$ via this map. The map $\Delta(Y') \to \Delta_X$ is the map above on $\Delta(Y'_0 \cap Y'_{\neq 0})$ and identity otherwise.

Following [Kas84, Sect. 1], we call the sheaves on a Δ-complex Σ such that their restrictions to the interiors of the faces of Σ are constant sheaves Σ-*constant sheaves.* They admit the following combinatorial description (p. 327 of [Kas84]):

Lemma 2.1.1. *The category of Σ-constant sheaves with values in an Abelian category $\mathscr{A}$ is equivalent to the category of functors from the poset of the cells ordered by inclusion* $\mathrm{Cells}(\Sigma)$ *to $\mathscr{A}$.*

Proof. Let $\mathscr{F}$ be a Δ-constant sheaf. Consider the correspondence $\sigma \mapsto H^0(\mathrm{St}(\sigma), \mathscr{F})$, together the restriction morphisms $H^0(\mathrm{St}(\sigma), \mathscr{F}) \to H^0(\mathrm{St}(\tau), \mathscr{F})$ for all pairs of faces $\sigma \subset \tau$ it defines a functor $\mathrm{Cells} \to \mathscr{A}$.

Conversely, if $F : \mathrm{Cells}(\Sigma) \to \mathscr{A}$ is a functor, define the presheaf $\mathscr{P}_F$ by putting $\lim_{D_U} F$ into correspondence to an open U, where D_U is the diagram of cells and inclusions that intersect U non-trivially. The sheafification of $\mathscr{P}_F(U)$ is a Δ-constant sheaf. □

We further consider simplicial complexes with orientation induced by an order on the verctices.

For any ordered set $I = \{ i_1, \dots, i_k \}, i_1 < \dots < i_k$ and any $i = i_l \in I$ we will denote

$$\mathrm{sgn}(i, I) = \frac{i_1 \wedge \dots \wedge i_k}{i_l \wedge i_1 \wedge \dots \wedge \widehat{i_l} \wedge \dots i_k} = (-1)^{l-1}.$$

The open stars $\mathrm{St}(i)$ of vertices are contractible and cover Δ_X, so given a functor $F : \mathrm{Cells}(\Sigma) \to \mathscr{A}$, the Cech complex with respecto to the cover $\{ \mathrm{St}(i) \}_{i\in\Delta^0_X}$ computes the cohomology of $\mathscr{F}$:

$$C^k(\Sigma, F) = \bigoplus_{|\underline{\sigma}|=k+1} F(\sigma), \qquad d((a_\sigma)) = \sum_{\sigma} \sum_{\sigma=\partial_i\tau} \mathrm{sgn}(i, \underline{\tau}) F(\sigma \subset \tau)(a_\sigma),$$

since

$$F(\sigma) = H^0(\mathrm{St}(\sigma), F).$$

2.2. **Gysin and residue morphisms.** For any pair of faces $\sigma \supset \tau$ denote ι^σ_τ the embeddings $Y_\sigma \to Y_\tau$.

We will use signed and unsigned Gysin and restriction maps which we denote as follows

$$\begin{array}{rcll}
\mathrm{res}^\sigma_\tau & = & (\iota^\tau_\sigma)^* & : H^\bullet(Y_\sigma) \to H^\bullet(Y_\tau) \\
\mathrm{Gys}^\sigma_\tau & = & (\iota^\sigma_\tau)_! & : H^\bullet(Y_\sigma) \to H^{\bullet+2m}(Y_\tau), m = \dim\sigma - \dim\tau \\
\gamma^\sigma_i & = & \mathrm{sgn}(i, \underline{\sigma})\, \mathrm{Gys}^\sigma_{\delta_i\sigma} & : H^\bullet(Y_\sigma) \to H^{\bullet+2}(Y_{\partial_{i_l}\sigma}) \\
\gamma^{(k)} & = & \sum\limits_{|\underline{\sigma}|=k} \sum\limits_{i\in\underline{\sigma}} \gamma^\sigma_i & : H^\bullet(Y^{(k)}) \to H^{\bullet+2}(Y^{(k-1)}) \\
\rho^{(k)} & = & \sum\limits_{|\underline{\sigma}|=k} \sum\limits_{\tau:\partial_i\tau=\sigma} \mathrm{sgn}(i, \underline{\tau})\, \mathrm{res}^\sigma_\tau & : H^\bullet(Y^{(k)}) \to H^\bullet(Y^{(k+1)})
\end{array}$$

One checks that

$$\gamma^{(k-1)} \circ \gamma^{(k)} = 0, \qquad \rho^{(k+1)} \circ \rho^{(k)} = 0.$$

We will drop the superscript in res^σ_τ when it is clear from context.

Lemma 2.2.1. *Consider a commutative diagram of manifolds*

$$\begin{tikzcd} Y \arrow[r, "f"] & X \\ W \arrow[u, "G"] \arrow[r, "F"] & Z \arrow[u, "g"'] \end{tikzcd}$$

where all morphisms are embdennings and Y and Z meet transversely. Then

$$g^* f_! \alpha = F_! G^* \alpha$$

for any $\alpha \in H^*(Y)$.

Though this statement is folklore and well-known, we sketch a proof for the benefit of the reader.

Proof. Interpreting Gysin maps as compositions of Thom isomorphisms and restriction maps of local cohomology (see, e.g. [Ive86, VIII.2.4]) we get the diagram

$$\begin{tikzcd} H^\bullet(Y) \arrow[r, "\cup\tau_Y", "\sim"'] \arrow[d, "G^*"] & H^{\bullet+l}_Y(X) \arrow[r, "r_X"] \arrow[d, "g^*"] & H^{\bullet+l}(X) \arrow[d, "g^*"] \\ H^\bullet(W) \arrow[r, "\cup\tau_W", "\sim"'] & H^{\bullet+l}_W(Z) \arrow[r, "r_Z"] & H^{\bullet+l}(Z) \end{tikzcd}$$

where τ_Y, τ_W are Thom classes of normal bundles of Y in X and W in Z, respectively, and the left square commutes by the naturality of Thom isomorphism. By interpreting local cohomology as the cohomology of the Thom space of the normal bundle and maps r_X and r_Z as Pontryagin-Thom collapse, and using the fact that the square in the statement of the theorem is Cartesian, we observe that the right square in the diagram above commutes too. □

Corollary 2.2.2. *Let $f : X \to S$ be an snc degeneration, then for any stratum Y_σ, any $i, j \in \underline{\sigma}, i \neq j$ and any class $a \in H^r(Y_{\partial_i \sigma})$,*

$$\operatorname{res}_{\partial_j \sigma} \operatorname{Gys}_j^{\partial_i \sigma} a = \operatorname{Gys}_i^{\partial_j \partial_i \sigma} \operatorname{res}_{\partial_j \partial_i \sigma} a.$$

Lemma 2.2.3. *Let $f : X \to S$ be an snc degeneration, $Y = \sum_{i \in I} N_i Y_i$ be the central fibre and assume that Δ_X is a simplicial comlpex. Let $\sigma \in \Delta_X$, then for any $a \in H^r(Y_\sigma)$*

$$\sum_{i \in \underline{\sigma}} N_i \operatorname{res}_\sigma \operatorname{Gys}^\sigma_{\partial_i \sigma} a + \sum_{\tau : \partial_j \tau = \sigma} N_j \operatorname{Gys}^\tau_\sigma \operatorname{res}_\tau a = 0$$

Proof. By Lemma 2.2.1, Projection Formula [Ive86, 7.3] and [Ive86, 7.5] we have

$$0 = a \cup \sum_{j \in I} N_j (\iota^j)^* (\iota^j)_! 1_{Y_j} = \sum_{j \in \underline{\sigma}} N_j a \cup (\iota^\sigma)^* (\iota^j)_! 1_{Y_\sigma} + \sum_{\tau : \partial_j \tau = \sigma} N_j a \cup (\iota^\tau_\sigma)_! 1_{Y_\tau} =$$

$$= \sum_{i \in \underline{\sigma}} N_j \operatorname{res}_\sigma \operatorname{Gys}^\sigma_{\partial_i \sigma} a + \sum_{\tau : \partial_j \tau = \sigma} N_j \operatorname{Gys}^\tau_\sigma \operatorname{res}_{Y_\tau} a,$$

(the first expression vanishes since the divisor $Y = \sum_{j \in I} N_j Y_j$ is principal) and therefore

$$\sum_{j \in I} N_j (\iota^j)^* (\iota^j)_! 1_{Y_j} = 0.$$

□

2.3. **Weight spectral sequence of Steenbrink.** Let $f : X \to S$ be a unipotent snc degeneration. We will recall below the definition of the limit mixed Hodge structure on the cohomology of X_∞ and a spectral sequence due to Steenbrink that computes its weight filtration.

Let V be a vector space and N be a nilpotent operator on V, $N^{m+1} = 0$, $N^m \neq 0$. Define inductively the following $2m$-step filtration on V. First put $W_0 := \operatorname{Im} N^m, W_{2m-1} := \operatorname{Ker} N^m, W_{2m} = V$. Supposing that $W_0, \dots, W_l$ and $W_{2m-l-1}, \dots, W_{2m}$ are defined and $m > l+1$, put

$$\begin{array}{lcl} W_{l+1} & := & \operatorname{Im} N^{m-l-1} \cap W_{2m-l-1}, \\ W_{2m-l-2} & := & \operatorname{Ker}(N^{m-l-1} : W_{2m-l-1}/W_l \to W_{2m-l-1}/W_l). \end{array}$$

Definition 2.3.1 (Weight monodromy filtration)**.** The filtration $W(M)_\bullet$ associated to the nilpotent operator $N = \log T : H^\bullet(X_\infty) \to H^\bullet(X_\infty)$ in the way described above is called the *weight monodromy filtration*.

Steenbrink [Ste76, Ste95] has defined certain complexes $A^\bullet_{\mathbb{C}}, A^\bullet_{\mathbb{Q}}$ quasi-isomorphic to $\psi\,\mathbb{C}, \psi\mathbb{Q}$, respectively, endowed with an increasing filtration $W_\bullet$ such that $A^\bullet_{\mathbb{Q}}$ is filtered quasi-isomorphic to $A^\bullet_{\mathbb{C}}$, and such that

$$W(M)_m H^\bullet(X_\infty) = \operatorname{Im}(\mathbb{H}^\bullet(Y, W_m A^\bullet_{\mathbb{Q}}) \to \mathbb{H}^\bullet(Y, A^\bullet_{\mathbb{Q}}))$$

under assumption that Y is "cohomologically Kähler", i.e. that there exists a class $\omega \in H^2(Y)$ that restricts to a Kähler class on every irreducible component of Y.

Given a double complex $(C^{\bullet,\bullet}, d', d'')$, we will denote $(sC^\bullet, d)$ the total complex:

$$sK^r = \bigoplus_{p+q=r} C^{p,q}, \qquad d = d' + d''.$$

The spectral sequence associated to the filtration W on $A^\bullet_{\mathbb{Q}}$ degenerates on E_2. Following [GA90, Section 2] we describe its first sheet ${}_W E_1^{\bullet,\bullet}$. For any $k \geq 1$ denote $Y^{(k)}$ the disjoint union of k-fold intersections of the irreducible components Y_i and denote

$$\begin{array}{rcl} K^{i,j,k}(Y) & = & \begin{cases} H^{i+j-2k+n}(Y^{(2k-i+1)}, \mathbb{Q}), & \text{if } k \geq \max\{0, i\}, \\ 0 & \text{else}, \end{cases} \\ K^{i,j}(Y) & = & \bigoplus_k K^{i,j,k}. \end{array}$$

Now denote

$$\begin{array}{ll} d' : K^{i,j,k} \to K^{i+1,j+1,k+1}, & d' = \rho^{(2k-i+1)}, \\ d'' : K^{i,j,k} \to K^{i+1,j+1,k}, & d'' = -\gamma^{(2k-i+1)}, \\ N : K^{i,j,k} \to K^{i+2,j,k}, & N = \operatorname{id}. \end{array}$$

Then

$$\begin{array}{rcl} {}_W E_1^{-r,q+r} & = & \mathbb{H}^q(X_\infty, \operatorname{gr}^W_{q+r} A^\bullet_{\mathbb{Q}}) \Longrightarrow H^q(X_\infty, \mathbb{Q}) \\ {}_W E_1^{-r,q+r} & = & K^{-r,q-n}(Y) \end{array}$$

and $d_1 = d' + d''$. Since the terms $K^{i,j,k}$ only depend on the the special fibre, we will further refer to them as $K^{i,j,k}(Y)$ and to the terms of the first page of the spectral sequence as ${}_W E_1^{i,j}(Y)$. Note that $\operatorname{Ker} N$ is a sub-spectral sequence of ${}_W E_1^{\bullet,\bullet}(Y)$ that computes (the associated graded of) the cohomology of Y.

Let $Y = \sum_{i=1}^m Y_i$ be an snc divisor in a proper complex manifold X. Recall that the weight spectral sequence of Deligne [Del71, 3.2.4.1], [PS08, 4.7]

$$E_1^{i,j} = H^{2i+j}(Y^{(-i)}) \Longrightarrow H^{i+j}(X \setminus Y),$$

where $Y^{(0)} = X$ and the differential $d_1^{i,j} : H^{2i+j}(Y^{(-i)}) \to H^{2i+j+2}(Y^{(-i-1)})$ is $-\gamma^{(-i)}$, degenerates on the first page. We will denote $M_p^\bullet(X,Y)$ the complexes standing in the rows of the first page of this spectral sequence after reindexing:

$$M_p^q(X,Y) = E_1^{q-p,2p} = H^{2q}(Y^{(p-q)}).$$

If Y is the central fibre of an snc degeneration $f : X \to S$, then for any p denote $D_p^{\bullet,\bullet}(Y)$ the double subcomplex of $K^{\bullet,\bullet,\bullet}(Y)$ with the terms

$$D_p^{q,r}(Y) = K^{-p+q+r,p+q+r-n,q}(Y) = H^{2r}(Y^{(p+q-r+1)}).$$

In particular,

$$sD_p^q = K^{-p+q,p+q-n} = {}_WE_1^{-p+q,2p}(Y) = \bigoplus_{r=0}^{\min\{p,q\}} H^{2r}(Y^{(p+q-2r+1)}).$$

The differentials d', d'' on $D_p^{\bullet,\bullet}$ have degrees $(1,0),(0,1)$, respectively. Denoting the stupid filtration on M as $\sigma_{\le m}$,

$$\sigma_{\le m} M_p^q(X,Y) = \begin{cases} M_p^q(X,Y), & \text{if } i \le m, \\ 0, & \text{otherwise,} \end{cases}$$

one can think about the double complex $D_p^{\bullet,\bullet}$ as the complex

$$0 \to \sigma_{\le p} M_p^\bullet(X,Y) \xrightarrow{\rho^{(p)}} \sigma_{\le p} M_{p+1}^\bullet(X,Y) \xrightarrow{\rho^{(p+1)}} \sigma_{\le p} M_{p+2}^\bullet(X,Y) \to \dots$$

in the category of complexes.

2.4. **Hodge-Lefschetz modules.** Recall that a rational Hodge structure of weight n is a rational vector space V together with a decomposition $V \otimes \mathbb{C} \cong \bigoplus_{p+q=n} V^{p,q}$ such that $\dim V^{p,q} = \dim V^{q,p}$ for all p, q. A polarization on V is a symmetric form $\psi : V \otimes V \to \mathbb{Q}$ such that for its linear continuation to $V \otimes \mathbb{C}$ the following holds:

i) $\psi(V^{p,q}, V^{p',q'}) = 0$ for $p \neq p', q \neq q'$;
ii) $i^{p-q}\psi(x, \bar{x}) > 0$ for all $x \in V^{p,q}$.

Let $L = \bigoplus_{i,j\in\mathbb{Z}} L^{i,j}$ be a bigraded vector space such that $L^{i,j}$ are endowed with real Hodge structures of weight $p+i+j$ and let $l_1 : L^{i,j} \to L^{i+2,j}(1), l_2 : L^{i,j} \to L^{i,j+2}(1)$ be morphisms of Hodge structures such that

$$l_1^i : L^{-i,j} \xrightarrow{\sim} L^{i,j} \qquad l_1^j : L^{i,-j} \xrightarrow{\sim} L^{i,j}$$

are isomorphisms. The tuple (V, l_1, l_2) is called a *bigraded Hodge-Lefschetz module of weight* p. A form $\psi : L \otimes L \to \mathbb{Q}(-p)$ is called *polarization* of V if

i) ψ restricts to morphisms of Hodge structures on $L^{-i,-j} \otimes L^{i,j}$;
ii) $\psi(l_1x, y) + \psi(x, l_1y) = \psi(l_2x, y) + \psi(x, l_2y) = 0$,
iii) the restriction of $\psi(-, l_1^i l_2^j -)$ to $L^{-i,-j} \cap \operatorname{Ker} l_1 \cap Kerl_2$ is a polarization of a Hodge structure.

Remark 2.4.1. We define Hodge-Lefschetz modules over reals following the convention of [GA90], though we will only apply the results of [GA90] to rational modules (tensoring with $\mathbb{R}$).

Fact 2.4.2 (Proposition 3.6, [GA90])**.** *If $\omega \in H^2(Y)$ restricts to a Lefschetz class on all strata Y_σ then $(K^{\bullet,\bullet} \otimes \mathbb{R}, N, L_\omega)$ is a bigraded Hodge-Lefschetz module.*

A map $d : V^{\bullet,\bullet} \to V^{\bullet+1,\bullet+1}$ is a called *differential* on a polarized Hodge-Lefschetz module if

i) $d \circ d = 0$;

ii) $[d, l_1] = [d, l_2] = 0$;
iii) $\psi(dx, y) = \psi(x, dy)$.

Fact 2.4.3 (Proposition 3.5, [GA90])**.** *If ω restricts to Lefschetz classes on all strata Y_σ then $d = d' + d''$ is a differential on the Hodge-Lefschetz module $({}'K^{\bullet,\bullet}(Y) \otimes \mathbb{R}, N, L_\omega)$.*

A real Hodge-lefschetz module L is naturally endowed with an action of a Lie group

$$\sigma : \mathrm{SL}(2,R) \times \mathrm{SL}(2,R) \to \mathrm{GL}(L) \qquad d\sigma\left(\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), 0\right) = l_1 \text{ and } d\sigma\left(0, \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)\right) = l_2$$

Let

$$w = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right), \qquad w_2 = (w, w) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$$

and define operators

$$d^t = w_2^{-1} d w_2 \qquad \Box = dd^t + d^t d$$

Fact 2.4.4 (Theorème 4.5, [GA90])**.** $\mathrm{Ker}\,\Box$ *is naturally isomorphic to $H^\bullet(V, d)$.*

Facts 2.4.3 and 2.4.4 together imply that every cohomology class of $sD^\bullet(Y)$ contains a unique representative that belongs to $\mathrm{Ker}\,\Box$.

2.5. **Superforms.** We recall here the definitions of the superforms on a real vector space introduced by Lagerberg following the ideas of Berndtsson. We refer to the articles [Lag11, Lag12b] for the basic facts about superforms and their integrals. See also [Gub16, CL12a] for the adaptation of the theory of superforms to the setting of tropical and non-archimedean geometry.

Let V be a real vector space. Consider the direct sum of two copies of V, which we will denote $\bar{V} = V^0 \oplus V^1$ and define differential superforms on V to be the space of differential forms on $\bar{V}$ which are invariant under translations along vectors from V^1. Denote the sheaf of n-superforms on V as $\mathscr{A}^n$.

The direct sum decompositionof $\bar{V}$ lifts to the tangent space, denote

$$J : TV^0 \oplus TV^1 \to TV^1 \oplus TV^0$$

the morphism that permutes the direct summands. The morphism J then also acts on the superforms. Denote the subsheaf of the superforms on V that are pull-backs of the real forms on V^0 as $\mathscr{A}^{1,0}$ and define sheaves $\mathscr{A}^{0,1}$ and $\mathscr{A}^{p,q}$ as follows: for every open $U \subset V$ set

$$\mathscr{A}^{0,1}(U) = J\,\mathscr{A}^{1,0}(U) \qquad \mathscr{A}^{p,q}(U) = \bigwedge^{p} \mathscr{A}^{1,0}(U) \otimes_{\mathscr{C}^\infty(U)} \bigwedge^{q} \mathscr{A}^{0,1}(U) \subset \mathscr{A}^{p+q}(U).$$

The sections of $\mathscr{A}^{p,q}$ are called (p,q)-*superforms.* We have

$$\mathscr{A}^{p,q}(U) \cong \mathscr{C}^\infty(U) \otimes_{\mathbb{R}} \bigwedge^{p} T^*V^0 \otimes_{\mathbb{R}} \bigwedge^{q} T^*V^1,$$

where the isomorphism is defined as follows. Let $x_1, \ldots, x_n$ be a set of affine functions on U such that $d'x_1, \ldots, d'x_n$ define a basis of $T^*_xV \cong T^*V$ for any $x \in U$. Then any (p,q)-superform α on U is given by an expression

$$\alpha = \sum_{|I|=p, |J|=q} f_{IJ}(x_1, \ldots, x_n) d'x_I \wedge d''x_J$$

where $d'x_I = d'x_{i_1} \wedge \ldots \wedge d'x_{i_k}$ for any index set $I = \{i_1, \ldots, i_k\}, i_1 < \ldots < i_k$, and similarly for $d''x_J$. Here, $f_{IJ} \in \mathscr{C}^\infty(U), d'x_I \in T^*V, d''x_J \in T^*V$.

By definition, J is an isomorphism of sheaves between $\mathscr{A}^{p,q}$ and $\mathscr{A}^{q,p}$. Denote the map of sheaves induced by the de Rham differential as

$$d' : \mathscr{A}^{p,q} \to \mathscr{A}^{p+1,q}$$

and define $d'' = J \circ d' \circ J$. Then d'' maps $\mathscr{A}^{p,q}$ to $\mathscr{A}^{p,q+1}$. One easily checks that d', d'', J are given by the following formulas

$$\begin{aligned}
d'\alpha &= \sum_{|I|=p,|J|=q} \frac{\partial f_{IJ}}{\partial x_i} d'x_i i \wedge d'x_I \wedge d''x_J \\
d''\alpha &= (-1)^p \sum_{|I|=p,|J|=q} \frac{\partial f_{IJ}}{\partial x_i} d'x_i i \wedge d'x_I \wedge d''x_j \wedge d''x_J \\
J\alpha &= (-1)^{pq} \sum_{|I|=p,|J|=q} f_{IJ} d'x_J \wedge d''x_I
\end{aligned}$$

Denote $\mathrm{ev} : TV \otimes T^*V \to \mathbb{R}$ the natural pairing between tangent vectors and covectors. The coevaluation morphism $\mathrm{coev} : \mathbb{R} \to TV \otimes T^*V$ is the unique morphism that makes the compositions of the following maps

$$\begin{aligned}
&T^*V \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} T^*V \otimes (TV \otimes T^*V) \cong (T^*V \otimes TV) \otimes T^*V \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} T^*V \\
&TV \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} (TV \otimes T^*V) \otimes TV \cong TV \otimes (T^*V \otimes TV) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} TV
\end{aligned}$$

identities.

Definition 2.5.1 (Monodromy morphism)**.** Let V be a real vector space. For any open set $U \subset V$ define the morphism $N : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p-1,q+1}(U)$ to be the composition of the morphisms

$$\begin{aligned}
C^\infty(U) \otimes \wedge^p T^*V \otimes \wedge^q T^*V &\to C^\infty(U) \otimes \wedge^p T^*V \otimes (TV \otimes T^*V) \otimes \wedge^q T^*V \\
&\to C^\infty(U) \otimes (\wedge^p T^*V \otimes TV) \otimes (T^*V \otimes \wedge^q T^*V) \\
&\to C^\infty(U) \otimes \wedge^{p-1} T^*V \otimes \wedge^{q+1} T^*V
\end{aligned}$$

where the first map is given by the coevaluation map.

The definition of the morphism N is due to Yifeng Liu [Liu17]).

The explicit formula for the morphism N is the following:

$$N(\sum_{|I|=p,|J|=q} f_{IJ} d'x_I \wedge d''x_J) = \sum_{|I|=p,|J|=q} \sum_{i \in I} (-1)^{p-1} \operatorname{sgn}(i, I) d'x_{I \setminus \{i\}} \wedge d''x_i \wedge d''x_J.$$

3. Sheaves Λ^p

In this section we fix a unipotent snc degeneration $f : X \to S$ with the central fibre $Y = \sum_{i \in I} N_i Y_i$, where I is a finite ordered set. For any face τ of the dual intersection complex of Y we will denote $N_\tau = \prod_{i \in \mathcal{T}} N_i$.

To simplify the exposition, we will assume that all intersections $Y_{i_1} \cap \ldots \cap Y_{i_k}$ are connected and smooth, and so Δ_X is a simplicial complex. With a little bit of effort the definitions in this section can be generalized to Δ-complexes. We justify this choice by the fact that any Δ-complex can be turned into a simplical complex after a subdivision corresponding to some blow-ups of the strata of Y, and the the definition of the sheaves $\Lambda^\bullet$ is invariant under subdivisions by Proposition 3.3.2.

3.1. **Definition of Λ^p.** For any face σ denote $\lambda^\bullet(\sigma) = \bigwedge^\bullet \mathbb{Q}\underline{\sigma}/\mathscr{J}^\bullet(\sigma)$, where $\mathscr{J}^\bullet(\sigma)$ is the homogeneous ideal generated by $\sum_{i\in\underline{\sigma}} i$. For each p the vector space $\lambda^p(\sigma)$ is naturally isomorphic to the vector space of translation invariant differential p-forms with rational coefficients on the affine space $\langle\sigma\rangle$, and the space of derivations of $\bigwedge^\bullet \mathbb{Q}\underline{\sigma}$ that preserves $\mathscr{J}^\bullet(\sigma)$ is identified with the tangent space $T\langle\sigma\rangle$. Under this interpretation, for any pair of faces $\sigma \subset \tau$ there exists a natural restriction of differential forms map $a \in \lambda^\bullet(\tau) \mapsto a|_\sigma \in \lambda^\bullet(\sigma)$. In concrete terms, it can be defined on the primitive tensors as follows:

$$(i_1 \wedge \ldots \wedge i_l)|_\sigma = \begin{cases} i_1 \wedge \ldots \wedge i_l & \text{, if } \forall l\ i_l \in \underline{\sigma} \\ 0 & \text{otherwise.} \end{cases}$$

Now for any face σ define

$$\bar{\Lambda}^\bullet(\sigma) = \{\ (f_\tau) \in \bigoplus_{\tau\supset\sigma} \lambda^\bullet(\tau)\ \mid\ \forall \tau \supset \alpha \supset \sigma,\ f_\tau|_\alpha = f_\alpha\ \}$$

This defines a functor: the morphism associated to an inclusion $\sigma \supset \tau$ is the natural projection $\bar{\Lambda}^\bullet(\sigma) \hookrightarrow \bar{\Lambda}^\bullet(\tau)$ coming from the inclusion $\overline{\mathrm{St}}(\tau) \subset \overline{\mathrm{St}}(\sigma)$. We will denote it $a \mapsto a|_{\overline{\mathrm{St}}(\tau)}$.

The space $\Lambda^p(\sigma)$ is spanned by primitive tensors of the form $i_1 \wedge \ldots \wedge i_p$

For any face σ, $|\underline{\sigma}| \geq p$ define the map $c^p_\sigma : \bar{\Lambda}^p(\sigma) \to H^2(Y_\sigma) \otimes \lambda^{p-1}(\sigma)$ on the primitive tensors by the formula

$$c^p_\sigma(i_1 \wedge \ldots \wedge i_p) = \sum_{l=1}^{p} (-1)^{l-1} c_1(\mathcal{O}(Y_{i_l}))|_{Y_\sigma} \otimes (i_1 \wedge \ldots \wedge \widehat{i_l} \wedge \ldots \wedge i_p)|_\sigma.$$

The maps c^p_σ are coordinate components of the differential of the complex $K^0(\sigma, \Lambda^p)$ that will be introduced in Section 4.2. They are well-defined by Lemma 4.2.3.

Lemma 3.1.1. *For any face σ the map $c^\bullet_\sigma$ is a derivation, that is, for any $a \in \bar{\Lambda}^p, b \in \bar{\Lambda}^q$*

$$c^{p+q}_\sigma(a \wedge b) = c^p_\sigma(a) \wedge b + (-1)^p a \wedge c^q_\sigma(b).$$

Proof. Immediate from the definition. □

Definition 3.1.2. For any $p \geq 1$ define functors Λ^p as follows. For all faces $\sigma, |\sigma| \geq k$ let

$$\Lambda^p(\sigma) = \{\ \alpha \in \bar{\Lambda}^p\ \mid\ \forall \tau \supset \sigma, \alpha|_\tau \in \operatorname{Ker} c^p_\tau\ \},$$

and denote the sheaves on Δ corresponding to these functors by Lemma 2.1.1 as Λ^p. We also agree to denote the constant sheaf $\mathbb{Q}$ as Λ^0.

Lemma 3.1.3. *For any $p > 1$ let*

$$U_p = \bigcup_{|\underline{\tau}|\geq p} \mathrm{St}(\tau).$$

and $j : U_p \hookrightarrow \Delta$ be the open embeddings. Then

$$\Lambda^p = j_* j^* \Lambda^k.$$

Proof. Follows immediately from Definition 3.1.2, as

$$\operatorname{Ker} c^p_\sigma = \bar{\Lambda}^p(\sigma)$$

for all σ such that $|\underline{\sigma}| < p$, since $\lambda^{p-1}(\sigma) = 0$ for such faces σ. □

Lemma 3.1.4. *For all p, q for which the corresponding sheaves are defined*

$$\Lambda^p \wedge \Lambda^q \subset \Lambda^{p+q}.$$

Proof. This amounts to showing that for any face α and any faces $\sigma, \tau \subset \alpha$,

$$a \in \Lambda^p(\sigma), b \in \Lambda^q(\tau) \Rightarrow a|_{\overline{\mathrm{St}}(\alpha)} \wedge b|_{\overline{\mathrm{St}}(\alpha)} \in \Lambda^{p+q}(\alpha),$$

which follows immediately from Lemma 3.1.1. □

3.2. **Regularity and sheaves A^p.** Consider the sheaf of piece-wise affine functions on Δ_X that restrict to affine functions on the faces $\tau \supset \sigma$ — such functions are determined by their values at the vertices:

$$\bar{A}^1(\sigma) = \{\, f : \overline{\mathrm{St}}^0(\sigma) \to \mathbb{Q} \,\}$$

Consider further a subsheaf of $\bar{A}^1$ that corresponds to the functor

$$A^1(\sigma) = \{\ f \in \bar{A}^1(\sigma) \ \mid \ \sum_{i \in \overline{\mathrm{St}}^0(\sigma)} N_i f(i) \cdot c_1(\mathcal{O}(Y_i))|_{Y_\sigma} = 0 \ \}$$

This sheaf have been defined in an unpublished note [KT02a] of Konstevich and Tschinkel (where it is called the sheaf of "linear functions", Sect. 6.2), see also [Yu15, SRJ18]. One can observe easily that for any face σ we have a functorial exact sequence

$$0 \to \Lambda^0(\sigma) \to A^1(\sigma) \to \Lambda^1(\sigma) \to 0.$$

that gives rise to the exact sequence of sheaves

$$0 \to \Lambda^0 \to A^1 \to \Lambda^1 \to 0.$$

One can wonder under which circumstances an analogous sequence exists in higher degrees.

Take a face $\sigma \subset \Delta_X$. There exists a natural map given by the evaluation function

$$e_\sigma : \mathrm{St}(\sigma) \to H^0(\mathrm{St}(\sigma), A^1)^* \qquad x \mapsto [f \mapsto f(x)]$$

which factors through the subspace

$$T(\sigma) := \{\ F \in H^0(\mathrm{St}(\sigma), A^1)^* \ \mid \ \forall c \in \mathbb{R}\, F(c) = c \ \}.$$

The latter is clearly a torsor under the vector subspace

$$H^0(\mathrm{St}(\sigma), \Lambda^1)^* \cong \{\ F \in H^0(\mathrm{St}(\sigma), A^1)^* \ \mid \ F(c) = 0 \ \} \subset H^0(\mathrm{St}(\sigma), A^1)^*.$$

The elements of $A^1(\sigma)$ are tautologically functions on $T(\sigma)$. The function e_σ is affine on the simplices of $\overline{\mathrm{St}}(\sigma)$, therefore $e_\sigma(\overline{\mathrm{St}}(\sigma))$ is a simplicial complex in $T(\sigma)$ and $e_\sigma(\mathrm{St}(\sigma))$ is the star of the simplex $e_\sigma(\sigma)$.

Definition 3.2.1 (Regularity)**.** Let Δ be a simplicial complex endowed with a constructible sheaf $\Lambda^\bullet$ of graded algebras and a subsheaf A^1 of the sheaf of piece-wise affine functions on Δ such that $A^1/\operatorname{const} = \Lambda^1$. We say that $\Lambda^\bullet$ is *regular* at a face σ if

$$\Lambda^p(\sigma) = e_\sigma^* \bigwedge^p \Lambda^1_{T(\sigma)}.$$

If $\Lambda^\bullet$ is regular at a face σ we can define

$$A^p(\sigma) = e_\sigma^* \bigwedge^p A^1(\sigma)$$

for any $p \geq 0$, and the sequence

$$0 \to \Lambda^p(\sigma) \to A^{p+1}(\sigma) \to \Lambda^{p+1}(\sigma) \to 0$$

is exact. If $\Lambda^\bullet$ is regular at every face $\sigma \subset \Delta_X$ then A^p defines a functor and a sheaf on Δ_X, and we have an exact sequence of sheaves

$$0 \to \Lambda^p \to A^{p+1} \to \Lambda^p \to 0. \tag{3.1}$$

3.3. **Invariance under subdivision.** Let $f : X' \to X$ be a blow-up of a stratum Y_σ. The dual intersection complex Δ_X is naturally homeomorphic to $\Delta(Y')$ and we will identify the underlying topological spaces from now on; let ξ denote the homeomorphism. The simplicial complex $\Delta(Y')$ is obtained from Δ_X by adding the vertex e corresponding to the exceptional divisor to σ and subdividing all faces that contain σ. For every face $\tau \not\supset \sigma$ we denote the corresponding face of $\Delta(Y')$ as τ too. If τ is a face of $\overline{\mathrm{St}}(\sigma)$ that does not contain σ then there exists a unique face of $\Delta(Y')$, that we will denote τ', that contains τ and the vertex e. We will allow τ to be empty for uniformity, in which case we agree that $\tau' = e$.

The face σ is identified with the intersection of the positive octant in the vector space $\mathbb{R}^m$, $m = |\underline{\sigma}|, \sigma = \{ i_1, \dots, i_m \}$, with the affine subspace cut out by the equation

$$\sum_{k=1}^{\infty} N_{i_k} x_k = 1$$

and the vertex e with the point $\left(\dfrac{1}{N_\sigma}, \dots, \dfrac{1}{N_\sigma}\right)$.

Let β be a face in $\overline{\mathrm{St}}(\sigma)$ and let $\eta \subseteq \beta$ be a face of $\Delta(Y')$ such that $f(Y_\eta) = Y_\beta$. In particular, $\mathrm{St}(\eta) \subset \mathrm{St}(\beta)$; denote this open embedding $\xi_{\beta,\eta}$. There are two options: either $\beta \not\supset \sigma$ and $\eta = \beta$, or $\beta \supset \sigma$ and $\eta = \alpha'$ for some $\alpha \subset \beta$. In both cases $\xi^*_{\beta,\eta} : \bar{A}^1_X(\beta) = H^0(\mathrm{St}(\beta), \bar{A}^\bullet_X) \hookrightarrow \bar{A}^1_{X'}(\eta) = H^0(U_\eta, \bar{A}^\bullet_{X'})$ is injective. Indeed, if an element of $\bar{A}^1_X(\beta)$ is represented by a piece-wise affine function $f : \overline{\mathrm{St}}^0(\beta) \to \mathbb{R}$, then to describe its restriction to $\overline{\mathrm{St}}^0_{X'}(\eta)$ it suffices to calculate the value of f in e, which is

$$f(e) = \frac{1}{\sum_{i \in \underline{\sigma}} N_i} \sum_{i \in \underline{\sigma}} f(i) = \frac{1}{N_e} \sum_{i \in \underline{\sigma}} f(i)$$

Lemma 3.3.1. *For any stratum $Y'_\eta \subset Y'_e$,*

$$H^2(Y'_\eta) \cong H^2(Y_\sigma) \oplus \mathbb{Q} c_1(\mathcal{O}(Y'_e))|_{Y'_\eta}$$

Proof. The exceptional divisor Y'_e is a projective bundle over the stratum Y_σ. Since Y is an snc divisor, the stratum Y'_τ is a projective subbundle of Y'_e. The statement of the Lemma then follows from [Voi03, Lemma 7.32]. $\square$

Proposition 3.3.2. *Let β beta a face in $\overline{\mathrm{St}}(\sigma)$ and let $\eta \subseteq \beta$ be a face of $\Delta(Y')$ such that $\mathrm{St}(\eta) \subset \mathrm{St}(\beta)$. Denote $r_{\beta,\eta} : \lambda^\bullet(\beta) \to \lambda^\bullet(\eta)$. Then the diagram*

$$\begin{array}{ccc} \bar{\Lambda}^p_X(\beta) & \xrightarrow{c^p_\beta} & H^2(Y_\beta) \otimes \lambda^{p-1}(\beta) \\ \downarrow{\scriptstyle \xi^*_{\beta,\eta}} & & \downarrow{\scriptstyle f^* \otimes r_{\beta,\eta}} \\ \bar{\Lambda}^p_{X'}(\eta) & \xrightarrow{c^p_\eta} & H^2(Y_\eta) \otimes \lambda^{p-1}(\eta) \end{array}$$

is commutative, and $\mathrm{Ker}\, c^p_\eta = \xi^*_{\beta,\eta} \mathrm{Ker}\, c^p_\beta$. In particular, the definition of sheaves A^p and Λ^p is invariant under the subdivisions induced by blow-ups of the strata of Y.

Proof. Take a basis element $i \in \bar{\Lambda}^1(\beta)$, then

$$\xi^*_{\beta,\eta}(i) = i + \frac{N_i}{N_e}.$$

The diagram is commutative for $p = 1$ since

$$c^1_\beta(\xi^*_{\beta,\eta}(i)) = c_1(i + \frac{N_i}{N_e} e) = N_i(c_1(\mathcal{O}_{X'}(Y'_i)) + c_1(\mathcal{O}_{X'}(Y'_e))) = f^* N_i c_1(\mathcal{O}_X(Y_i)) = f^* c^1_\eta(i).$$

The commutativity of the diagram for $p > 1$ follows by induction from Lemma 3.1.1 and commutativity of the diagram

$$\begin{tikzcd} \bar\Lambda^p(\beta) \arrow[r] \arrow[d, "\xi^*_{\beta,\eta}"] & \lambda^p(\beta) \arrow[d, "r_{\beta,\eta}"] \\ \bar\Lambda^p(\eta) \arrow[r] & \lambda^p(\beta) \end{tikzcd}$$

where the horizontal morphisms are restriction maps, for all $p \geq 1$.

The second statement follows from the fact that $\bar\Lambda^1(\eta) = \xi^*_{\beta,\eta}\bar\Lambda^1(\beta) \oplus \mathbb{Q}e$, from Lemma 3.3.1 and from the inclusion

$$c^1_\beta(\xi^*_{\beta,\eta}\bar\Lambda^1(\beta)) \subset f^*H^2(Y_\beta) \subset H^2(Y_\eta),$$

that has just been established. □

In view of Proposition 3.3.2 the following definition is consistent.

Definition 3.3.3 (Sheaves $\Lambda^\bullet$ in the normal crossings case)**.** For any degeneration $f : X \to S$ such that reduction of its central fibre is normal crossings but not necessarily simple normal crossings, define the sheaves $\Lambda^\bullet$ on Δ_X to be those of a modification X' of X obtained by a sequence of blow-ups of the strata, and such that the central fibre is strictly normal crossings and $\Delta_{X'}$ is a simplicial complex.

3.4. **Toric strata.** Let X be a toric variety acted upon by a torus $T = \mathbb{C}[M], M \cong \mathbb{Z}^n$, let $N = \mathrm{Hom}(M, \mathbb{Z})$ and let $\Sigma \subset N \otimes \mathbb{R}$ be the toric fan of X. Let $t : X \to \mathbb{A}^1$ be an equivariant morphism, assume that $Y = t^{-1}(0)$ is an snc divisor and denote the corresponding linear function on N as $\tilde t$. The dual intersection complex of $t^{-1}(0)$ can be identified with the polyhedral complex $\tilde t^{-1}(1)$. We will denote the intersections of cones σ with $\tilde t^{-1}(1)$ as σ too.

Proposition 3.4.1. *For any toric stratum $X_\sigma \subset Y$, $\Lambda^\bullet(\sigma)$ is the algebra of translation-invariant differential forms on $\overline{\mathrm{St}}(\sigma) \subset \tilde t^{-1}(1)$ with rational coefficients.*

Proof. The group of Cartier $\mathbb{Q}$-divisors on X_Σ is isomorphic to the group of piece-wise linear functions with rational coefficients on Σ that restrict to a linear function on each cone $\eta \subset \Sigma$; the linear functions among them correspond to the principal divisors. The stratum X_σ is a toric variety in its own right, with the dense torus $T_\sigma = \mathbb{C}[\sigma^\perp]$. We may identify $\mathrm{Hom}((\sigma \cap \mathbb{Q})^\perp, \mathbb{Q})$ with $N/\langle\sigma\rangle$, where $\langle\sigma\rangle$ is the vector space spanned by by $\sigma \cap \mathbb{Q}$. Given a $\mathbb{Q}$-divisor D on X_Σ corresponding to a piece-wise linear function $\tilde t_D : N \otimes \mathbb{R} \to \mathbb{R}$, the restriction of its linear equivalence class X_σ can be decribed as follows. There exists a linear function $\tilde t'_D : N \otimes \mathbb{R} \to \mathbb{R}$ such that $\tilde t_D - \tilde t'_D$ vanishes on $\langle\sigma\rangle$, this function is therefore a pull-back of a piece-wise linear function with rational coefficients $\eta : N/\langle\sigma\rangle \to \mathbb{R}$; the function η corresponds to the linear equivalence class of the restriction of the linear equivalence class of D to X_σ.

The elements of $\bar A^1(\sigma)$ by definition correspond to piece-wise linear functions with rational coefficients on the cone over $\overline{\mathrm{St}}(\sigma)$, or to affine functions with rational coefficients on $\tilde t^{-1}(1)$. It then immediately follows from the description of the restriction of toric divisors to X_σ that

$$\Lambda^1(\sigma) = \mathrm{Aff}(\overline{\mathrm{St}}(\sigma))/\,\mathrm{const}\,.$$

By Lemma 3.1.1 we have

$$\bigwedge^p \Lambda^1 \subset \Lambda^p$$

To conclude, it will suffice show for any codimension 1 face $\tau \subset \overline{\mathrm{St}}(\sigma)$ that

$$\Lambda^p(\tau) \subset \bigwedge^p \Lambda^1(\tau)$$

Since τ is a codimension 1 face, it has two faces of maximal dimension contianing it, call them η_1 and η_2, $\underline{\eta}_1 = \underline{\tau} \cup \{i\}$ and $\underline{\eta}_2 = \underline{\tau} \cup \{j\}$. Any function $f \in A^1(\tau)$ detemines an affine function on η_1, and since η is of maximal dimension, this linear function extends uniquely to an affine function on $\overline{\mathrm{St}}(\tau)$. It follows that

$$\bar{\Lambda}^1(\tau) = \Lambda^1(\tau) \oplus \mathbb{Q}j, \qquad \bar{\Lambda}^p(\tau) = \bigwedge^p \Lambda^1(\tau) \oplus \mathbb{Q}j \wedge \left(\bigwedge^{p-1} \Lambda^1(\tau) \right)$$

By Lemma 3.1.1 we have for any $a \in \bar{\Lambda}^{p-1}(\tau)$

$$c^p_\tau(j \wedge a) = c^1_\tau(j) \wedge a + (-1)^{p-1} c^{p-1}_\tau(a) \wedge j = c^1_\tau(j) \wedge a,$$

from which it is clear that that c^p_τ cannot vanish on the elements of the second direct summand, and we can conclude. □

Corollary 3.4.2. *Let $f : X \to S$ be an snc degeneration, $Y = f^{-1}(0)$, and let $g : X_\sigma \to \mathbb{A}^1$ be a morphism of toric varities as in the statement of Proposition 3.4.1. Assume that a tubular neighbourhood of a stratum Y_σ of Y is isomorphic to a tubular neighbourhood of a stratum of $X_\tau \subset g^{-1}(0)$. Then $\Lambda^\bullet(\sigma)$ is isomorphic to the algebra of translation-invariant rational differential forms on $\overline{\mathrm{St}}(\tau)$.*

4. The complex $K^\bullet(\Delta_X, \Lambda^p)$

In this section we will study a certain complex that will play the crucial role in the proof of Theorem A. Let us outline our strategy, which is inspired by [IKMZ19, Section 5] (though since Theorem A is about cohomology and not homology, our set up is dual; we also do not have to deal with sedentarity and infinite faces). We construct resolutions $K^\bullet(\sigma, \Lambda^p)$ of vector spaces $\Lambda^p(\sigma)$, functorial in σ, and use them to define a complex $K^\bullet(\Delta_X, \Lambda^p)$ quasi-isomorphic to the complex $C^\bullet(\Delta_X, \Lambda^p)$ of simplicial cochains with coefficients in Λ^p. We then show that $C^\bullet(\Delta_X, \Lambda^p)$ has the same cohomology as a certain subcomplex $s' D^\bullet_p(Y)$ of $sD^\bullet_p(Y)$, the $2p$-th line of the Steenbrink spectral sequence.

For the benefit of a reader familiar with the proof of the main result of [IKMZ19] we discuss technical peculiarities that pertain to our set up. Firstly, the definition of the complex $K^\bullet(\Delta_X, \Lambda^p)$ uses only cohomology classes that lie in the span of cycle classes of the strata of Y. Its differential takes into account the multiplicities of the components of the central fibre as opposed to [IKMZ19], where a unimodular triangulation of the tropical limit is chosen first. This complex plays a role similar to that of the complex $K^{(p)}_{\bullet,\bullet}$ of [IKMZ19, 5.3-4], but in order to show that it is quasi-isomorphic to $C^\bullet(\Delta_X, \Lambda^p)$ we use an inductive argument in Lemma 4.1.2 instead of the purity of the weight filtration on the cohomology of the complement of a hyperplane arrangement that was used in [IKMZ19]. Secondly, the treatment of signs in the computations of the differentials of the zeroth and first pages of the spectral sequence associated to the filtration F^Δ on the cochain complex $K^{(p)}_\bullet$ in Section 5.4 of [IKMZ19] contains gaps. In particular, the components of the differential of the residue complex should have signs that depend on the destination face. Filtration G on the cochain complex $K^\bullet(\Delta_X, \Lambda^\bullet)$ is the analogue of F^Δ in our set up, and in our Lemma 5.1.2 the correct signs for the differential in the complex $S^\bullet_n(\sigma)$, the dual of the residue complex of [IKMZ19], are computed. These signs are then used to derive correct signs for the differentials and for the coordinate components of the map N in the proofs of Lemmas 5.1.1, 5.1.3.

4.1. **Complexes $'M_q^\bullet(Y)$ and $'D_p^{\bullet,\bullet}(Y)$.** For any k denote $Y_{\neq k} = \sum_{i\neq k} Y_i$ and denote Y_k° the divisor $Y_k \cap Y_{\neq k}$ in Y_k. More generally, if $Y_\sigma \subset Y$ is a stratum of Y, denote

$$Y_\sigma^\circ = \bigcup_{j\notin \underline{\sigma}} Y_j \cap Y_\sigma.$$

Definition 4.1.1. For any stratum $Y_\sigma \subset Y$ denote

$$'H^\bullet(Y_\sigma) = \mathbb{Q}\{\ \mathrm{Gys}_\sigma^\tau(1_{Y_\tau})\ \mid\ \tau \in \mathrm{St}(\sigma)\ \} \subset H^\bullet(Y_\sigma).$$

More generally, denote

$$'H^\bullet(Y^{(p)}) = \bigoplus_{|\underline{\sigma}|=p+1} {'H^\bullet(Y_\sigma)}$$

Note that $'H^\bullet(Y^{(p)})$ is a subring of $H^\bullet(Y^{(p)})$. Denote $'M_p^\bullet(X,Y)$ the subcomplex of $M_p^\bullet(X,Y)$:

$$0 \to {'H^0(Y^{(p)})} \xrightarrow{\gamma^{(p)}} {'H^2(Y^{(p-1)})} \to \ldots \to {'H^{2p-2}(Y^{(1)})} \xrightarrow{\gamma^{(1)}} {'H^{2p}(X)} \to 0$$

Similarly, denote $'D_p^{\bullet,\bullet}(Y)$ the complex

$$0 \to \sigma_{\leq p}\, {'M_p^\bullet(X,Y)} \xrightarrow{\rho^{(p)}} \sigma_{\leq p}\, {'M_{p+1}^\bullet(X,Y)} \xrightarrow{\rho^{(p+1)}} \sigma_{\leq p}\, {'M_{p+2}^\bullet(X,Y)} \to \ldots$$

Lemma 4.1.2. *For any variety X and an snc divisor $Y \subset X$, for all $p \geq 0$ the complex $'M^\bullet(X,Y)$ doesn't have cohomology in degree > 0.*

Proof. The proof is by induction on p. Indeed, $'M_0^\bullet(X,Y)$ consists a single term and so the statement of the Lemma is true for it. It follows from the definition of $'M_p^\bullet(X,Y)$ that the following sequence is exact

$$0 \to {'M_p^\bullet(X, Y_{\neq k})} \to {'M_p^\bullet(X,Y)} \to {'M_{p-1}^\bullet(Y_k, Y_k^\circ)} \to 0.$$

By induction hypothesis the cohomology of the first and third terms is concentrated in degree 0, therefore, the same is true for $'M_p^\bullet(X,Y)$. ☐

4.2. **Complex $K^\bullet(\sigma,\Lambda^p)$: differential.** Consider a face $\sigma \subset \Delta_X$. The space $\lambda^1(\sigma)$ has a number of distinguished bases: for each $k \in \underline{\sigma}$, the vectors of the form $dx_{ik} = i - k \in \lambda^1(\sigma), i \neq k$ are linearly independent. These bases have dual ones in $T\langle\sigma\rangle$: for any $i \in \underline{\sigma}$ let $\delta_i \in T\langle\sigma\rangle$ be the derivation determined by

$$\delta_i(i) = 1 \qquad \delta_i(j) = 0, j \neq i$$

and let $\delta_{ik} = \delta_i - \delta_k$. In other words, δ_{ik} is the contraction with the vector going from vertex k to the vertex i.

Lemma 4.2.1. *If $\tau \supset \sigma, i, k \in \underline{\sigma}$ then for all $a \in \lambda^\bullet(\tau)$, $(\delta_{ik}a)|_\sigma = \delta_{ik}(a|_\sigma)$.*

Proof. Straightforward. ☐

Definition 4.2.2. For any face σ define let the term $K^r(\sigma,\Lambda^p)$ to be the subspace of

$$\bigoplus_{\tau\supset\sigma} {'H^{2r}(Y_\tau)} \otimes \lambda^{p-r}(\tau)$$

spanned by tuples of primitive tensors $\sum_{\tau\supset\sigma} a_\tau \otimes b_\tau$ that satisfy the equations

$$\mathrm{res}_\beta(a_\alpha) = a_\beta, \qquad b_\beta|_\alpha = b_\alpha,$$

for all $\beta \supset^1 \alpha$. Pick a vertex $k \in \underline{\sigma}$ and define the differential

$$d'_k : \bigoplus_{\tau \supset \sigma}{}' H^{2r}(Y_\tau) \otimes \lambda^{p-r}(\tau) \to \bigoplus_{\tau \supset \sigma}{}' H^{2r+2}(Y_\tau) \otimes \lambda^{p-r-1}(\tau)$$

on the primitive tensors by the formula

$$d'_k(a_\tau \otimes b_\tau) = \sum_{i \in \underline{\tau} \setminus \underline{\sigma}} N_i \operatorname{Gys}^{\tau}_{\partial_i \tau}(a_\tau) \otimes (\delta_{ik} b_\tau)|_{\partial_i \tau} + \sum_{i \in \underline{\sigma}} N_i \operatorname{res}_\tau(\operatorname{Gys}^{\tau}_{\partial_i \tau}(a_\tau)) \otimes \delta_{ik} b_\tau$$

where the first term belongs to

$$\bigoplus_{i \in \underline{\tau}}{}' H^{2r+2}(Y_{\partial_i \tau}) \otimes \lambda^{p-r}(\partial_i \tau)$$

and the second belongs to

$${}' H^{2r+2}(Y_\tau) \otimes \lambda^{p-r}(\tau).$$

Lemma 4.2.3. *For any $k, k' \in \underline{\sigma}$ and any $v \in K^r(\sigma, \Lambda^p)$, $d'_k(v) = d'_{k'}(v)$.*

Proof. Consider a sum of primitive tensors

$$v = \sum_{\tau \supset \sigma} a_\tau \otimes b_\tau \in K^r(\sigma, \Lambda^p),$$

then we have

$$(d'_k - d'_{k'})(v)_\alpha = \sum_{\eta : \alpha = \partial_i \eta} N_i \operatorname{Gys}^{\eta}_{\alpha}(a_\eta) \otimes (\delta_{k'k} b_\eta)|_\alpha + \sum_{i \in \underline{\alpha}} N_i \operatorname{res}_\alpha(\operatorname{Gys}^{\alpha}_{\partial_i \alpha}(a_\alpha)) \otimes \delta_{k'k} b_\alpha$$

$$= \left(\sum_{\eta : \alpha = \partial_i \eta} N_i \operatorname{Gys}^{\eta}_{\alpha}(\operatorname{res}_\eta(a_\alpha)) + \sum_{i \in \underline{\alpha}} N_i \operatorname{res}_\alpha(\operatorname{Gys}^{\alpha}_{\partial_i \alpha}(a_\alpha)) \right) \otimes \delta_{k'k} b_\alpha,$$

which vanishes by Corollary 2.2.3. □

In view of the Lemma, we will denote any of the differentials d'_k as simply d'.

Lemma 4.2.4. $d'(K^r(\sigma, \Lambda^p)) \subset K^{r+1}(\sigma, \Lambda^p)$.

Proof. To perform the computations below let us fix a vertex $k \in \underline{\sigma}$. It suffices to prove that $d'_k v \in K^{r+1}(\sigma, \Lambda^p)$ for all sums of primitive tensors

$$v = \sum_{\tau \supset \sigma} a_\tau \otimes b_\tau \in K^r(\sigma, \Lambda^p).$$

Take two faces $\alpha, \beta \supset \sigma, \alpha = \partial_l \beta$. Then

$$(d'v)_\alpha = \sum_{\eta : \alpha = \partial_i \eta, \eta \neq \beta} N_i \operatorname{Gys}^{\eta}_{\alpha}(a_\eta) \otimes (\delta_{ik} b_\eta)|_\alpha +$$
$$+ N_l \operatorname{Gys}^{\beta}_{\alpha}(a_\beta) \otimes (\delta_{lk} b_\beta)|_\alpha + \sum_{j \in \underline{\alpha}} N_j \operatorname{res}_\alpha(\operatorname{Gys}^{\alpha}_{\partial_j \alpha}(a_\alpha)) \otimes \delta_{jk} b_\alpha,$$

$$(d'v)_\beta = \sum_{\varepsilon : \beta = \partial_i \varepsilon} N_i \operatorname{Gys}^{\varepsilon}_{\beta}(a_\varepsilon) \otimes (\delta_{ik} b_\varepsilon)|_\beta +$$
$$+ \sum_{j \in \underline{\alpha}} N_j \operatorname{res}_\beta(\operatorname{Gys}^{\beta}_{\partial_j \beta}(a_\beta)) \otimes \delta_{jk} b_\beta + N_l \operatorname{res}_\beta(\operatorname{Gys}^{\beta}_{\alpha}(a_\beta)) \otimes \delta_{lk} b_\beta.$$

Since Δ_X is assumed to be a simplicial complex, there is a one-to-one correspondence between faces enumerating the summands of the first terms of $d(v)_\beta$ and $d(v)_\alpha$. All such pairs of faces ε, η satisfy

$$\eta = \partial_l \varepsilon, \qquad \partial_i \varepsilon = \beta, \qquad \partial_i \eta = \alpha.$$

We have

$$\mathrm{res}_\beta(\mathrm{Gys}^\eta_\alpha(a_\eta)) = \mathrm{Gys}^\varepsilon_\beta(\mathrm{res}_\varepsilon(a_\eta))) = \mathrm{Gys}^\varepsilon_\beta(a_\varepsilon),$$

since $v \in K^r(\sigma, \Lambda^p)$ and by Corollary 2.2.2, and we also have

$$(\delta_{ik} b_\varepsilon)|_\alpha = (\delta_{ik}(b_\varepsilon|_\eta))|_\alpha = (\delta_{ik} b_\eta)|_\alpha$$

by Lemma 4.2.1.

Again by Corollary 2.2.2, for all $j \in \underline{\alpha}$

$$\mathrm{res}_\beta \, \mathrm{res}_\alpha(\mathrm{Gys}^\alpha_{\delta_j \alpha}(a_\alpha)) = \mathrm{res}_\beta(\mathrm{Gys}^\beta_{\partial_j \beta}(\mathrm{res}_\beta(a_\alpha))) = \mathrm{res}_\beta(\mathrm{Gys}^\beta_{\partial_j \beta}(a_\beta)),$$

and by Lemma 4.2.1,

$$(\delta_{jk} b_\beta)|_\alpha = \delta_{jk}(b_\beta|_\alpha) = \delta_{jk} b_\alpha.$$

Finally, the terms

$$\mathrm{Gys}^\beta_\alpha(a_\beta) \otimes (\delta_{lk} b_\beta)|_\alpha \text{ and } \mathrm{res}_\beta \, \mathrm{Gys}^\beta_\alpha(a_\beta) \otimes \delta_{lk} b_\beta$$

clearly satisfy the identities from the definition of $K^r(\sigma, \Lambda^p)$. We conclude that $d'_k(v) \in K^{r+1}(\sigma, \Lambda^p)$. □

Lemma 4.2.5. *$d' \circ d' = 0$.*

Proof. Let $v \in K^r(\sigma, \Lambda^p)$ be a sum of primitive tensors as before.

$$\begin{aligned} d'_k(d'_k v) = \sum_{\tau \supset \sigma} \Bigg(& \sum_{j \in \underline{\partial_i \tau}} \sum_{i \in \underline{\tau}} N_i N_j \, \mathrm{Gys}^{\partial_i \tau}_{\partial_i \partial_j \tau} \mathrm{Gys}^\tau_{\partial_i \tau} a_\tau \otimes (\delta_{jk} \delta_{ik} b_\tau)|_{\partial_j \partial_i \tau} + \\ & + \sum_{j \in \underline{\tau}} \sum_{i \in \underline{\tau}} N_i N_j \, \mathrm{Gys}^\tau_{\partial_j \tau} \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_i \tau} a_\tau \otimes (\delta_{jk} \delta_{ik} b_\tau)|_{\partial_j \tau} + \\ & + \sum_{j \in \underline{\partial_i \tau}} \sum_{i \in \underline{\tau}} N_i N_j \, \mathrm{res}_{\partial_i \tau} \mathrm{Gys}^{\partial_i \tau}_{\partial_j \partial_i \tau} \mathrm{Gys}^\tau_{\partial_i \tau} a_\tau \otimes \delta_{jk}(\delta_{ik} b_\tau)|_{\partial_i \tau} + \\ & + \sum_{j \in \underline{\tau}} \sum_{i \in \underline{\tau}} N_i N_j \, \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_j \tau} \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_i \tau} a_\tau \otimes \delta_{jk} \delta_{ik} b_\tau \Bigg). \end{aligned}$$

Then by Corollary 2.2.2

$$\mathrm{Gys}^\tau_{\partial_j \tau} \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_i \tau} = \mathrm{res}_{\partial_j \tau} \mathrm{Gys}^{\partial_i \tau}_{\partial_j \partial_i \tau} \mathrm{Gys}^\tau_{\partial_i \tau} = \mathrm{res}_{\partial_j \tau} \mathrm{Gys}^{\partial_j \tau}_{\partial_i \partial_j \tau} \mathrm{Gys}^\tau_{\partial_j \tau}$$

and

$$\begin{aligned} \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_j \tau} \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_i \tau} a_\tau &= \mathrm{res}_\tau \mathrm{res}_{\partial_j \tau} \mathrm{Gys}^{\partial_i \tau}_{\partial_j \partial_i \tau} \mathrm{Gys}^\tau_{\partial_i \tau} a_\tau = \\ &= \mathrm{res}_\tau \mathrm{res}_{\partial_i \tau} \mathrm{Gys}^{\partial_j \tau}_{\partial_j \partial_i \tau} \mathrm{Gys}^\tau_{\partial_j \tau} a_\tau = \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_i \tau} \mathrm{res}_\tau \mathrm{Gys}^\tau_{\partial_j \tau} a_\tau, \end{aligned}$$

and therefore by Lemma 4.2.1 the sum in parenthesis vanishes for each $\tau \supset \sigma$.

□

4.3. **Complex $K^\bullet(\sigma, \Lambda^p)$: resolution.** It is clear from the Definition 3.1.2 that

$$\Lambda^p(\sigma) \cong K^0(\sigma, \Lambda^p)$$

For any pair of faces $\sigma, \tau, \sigma \subset \tau$ define the map

$$\kappa^\tau_\sigma : \lambda^\bullet(\tau) \to \lambda^{\bullet - l}(\sigma) \qquad a \mapsto (\delta_{i_l,\tau} \circ \ldots \circ \delta_{i_1,\tau})(a)|_\sigma,$$

where $\underline{\tau} \setminus \underline{\sigma} = \{\, i_1, \ldots, i_l \,\}, i_1 < \ldots < i_l$.

Denote $\lambda^\bullet(\tau;\sigma)$ the subspace of differential forms in $\lambda^\bullet(\tau)$ such that their restriction to $\langle\sigma\rangle$ vanishes, in particular, for any such form a and any derivation $\delta \in T\langle\sigma\rangle$, $\delta a = 0$. Define

$$F^m\lambda^p(\tau) = \bigoplus_{l=0}^{m-1} \lambda^{p-l}(\tau;\sigma) \wedge \lambda^l(\sigma), \qquad F^m K^r(\sigma, \Lambda^p) = {\bigoplus_{\tau\supset\sigma}}' H^{2r}(Y_\tau) \otimes F^m\lambda^{p-r}(\tau).$$

If $a \in \lambda^{p-l}(\tau;\sigma) \wedge \lambda^l(\sigma)$ and $k \in \underline{\sigma}, i \in \underline{\tau} \setminus \underline{\sigma}$ then $\delta_{ik}a \in \lambda^{p-l-1}(\tau;\sigma) \wedge \lambda^l(\tau)$, and therefore the differential d' respects the filtration $F^\bullet$, so it is an increasing filtration of $(K^\bullet(\sigma,\Lambda^p), d')$ by subcomplexes. Since $\bar\Lambda^\bullet(\sigma)$ and $\Lambda^\bullet(\sigma)$ embed into $\bigoplus_{\tau\supset\sigma} \lambda^\bullet(\sigma)$, both graded vector spaces inherit the filtration.

Lemma 4.3.1. *Let $J' \subset J$ and let $i \notin J'$. We have*

- $\operatorname{sgn}(i, J' \cup \{i\}) \operatorname{sgn}(i, J \setminus J') = \operatorname{sgn}(i, J)$,
- $\dfrac{\prod_{j\in J'}(-1)^{|J|}\operatorname{sgn}(j,J)}{\prod_{j\in J'}(-1)^{|J|-1}\operatorname{sgn}(j,J\setminus\{i\})} = \operatorname{sgn}(i, J'\cup\{i\}).$
-
- $\dfrac{\prod_{j\in J'}(-1)^{|J|-1}\operatorname{sgn}(j,J\setminus\{i\})}{\prod_{j\in J'\cup\{i\}}(-1)^{|J|}\operatorname{sgn}(j,J)} = \operatorname{sgn}(i, J\setminus J').$

Proof. The first statement is straigtforward.

To prove the second statement, notice that

$$\begin{aligned}(-1)^{|J|}\operatorname{sgn}(j,J) &= -(-1)^{|J|-1}\operatorname{sgn}(j,J\setminus\{i\} &&\text{for } j<i,\\ (-1)^{|J|}\operatorname{sgn}(j,J) &= (-1)^{|J|-1}\operatorname{sgn}(j,J\setminus\{i\}) &&\text{for } j>i.\end{aligned}$$

Therefore, the left hand side expression is equal $(-1)^l$, where l is the number of elements of J' less than i. The statement follows.

The left hand side in the third statement differs from left hand side in the second statement by $\operatorname{sgn}(i, J)$. So we can conclude by combining second and the first statement. □

Lemma 4.3.2. *The cohomology of the complex $K^\bullet(\sigma, \Lambda^p)$ is concentrated in degree 0 for all $p \geq 0$.*

Proof. Clearly $\operatorname{gr}^m_F \lambda^p(\tau) \cong \lambda^{p-m+1}(\tau;\sigma) \wedge \lambda^{m-1}(\sigma)$. If $|\underline{\tau}| = |\underline{\sigma}| + p - m + 1$, the space $\lambda^{p-m+1}(\tau;\sigma)$ is of dimension 1 and the map κ^τ_σ establishes an isomorphism between $\operatorname{gr}^m_F \lambda^p(\tau)$ and $\lambda^{m-1}(\sigma)$. Observe that if $|\underline{\tau}| < |\underline{\sigma}| + p - m + 1$ then $\operatorname{gr}^m_F \lambda^p(\tau) = 0$, and if $|\underline{\tau}| > |\underline{\sigma}| + p - m + 1$ then any element $a_\tau \otimes b_\tau \in F^m\lambda^p(\tau)$ is a sum of elements of the form $a_\eta \otimes b_\tau|_\eta \in F^m\lambda^{p-m+1}(\eta), a_\eta|_\tau = a_\tau$, where $\eta \subset \tau$ is a face such that $|\underline{\eta}| = |\underline{\sigma}| + p - m + 1$.

Therefore, the maps

$$\pi_{m,r} : \mathrm{gr}^m_F K^r(\sigma,\Lambda^p) \to \bigoplus_{\tau\supset^{p-r-m+1}\sigma} {}'H^{2r}(Y_\tau)\otimes\lambda^{m-1}(\sigma) \cong {}'M^r_{p-m+1}(Y_\sigma,Y^\circ_\sigma)\otimes\lambda^{m-1}(\sigma),$$
$$\sum_{\tau\supset\sigma} a_\tau\otimes b_\tau \mapsto \sum_{\tau\supset^{p-r-m+1}\sigma} N_\tau a_\tau\otimes\kappa^\tau_\sigma(b_\tau)$$

are isomorphisms. In fact, for a fixed m the maps $\pi_{m,r}$ define a morphism of complexes, and hence a quasi-isomorphism: for any $v=\sum_{\tau\supset\sigma} a_\tau\otimes b_\tau \in K^r(\sigma,\Lambda^p)$,

$$\pi_{m,r+1}(d'v) = \sum_{\tau\supset^{p-r-m}\sigma}\sum_{i\in\underline{\tau}\setminus\underline{\sigma}} N_i N_{\partial_i\tau}\,\mathrm{Gys}^\tau_{\partial_i\tau}(a_\tau)\otimes\kappa^{\partial_i\tau}_\sigma((\delta_{ik}b_\tau)|_{\partial_i\tau}) =$$
$$= \sum_{\tau\supset^{p-m-r}\sigma}\sum_{i\in\underline{\tau}\setminus\underline{\sigma}} N_\tau\,\mathrm{sgn}(i,\underline{\tau}\setminus\underline{\sigma})\,\mathrm{Gys}^\tau_{\partial_i\tau} a_\tau\otimes\kappa^\tau_\sigma(b_\tau),$$

Since $M^\bullet_{p-m+1}(Y_\sigma,Y^\circ_\sigma)$ has cohomology concentrated in degree 0, by Lemma 4.1.2 then so does $\mathrm{gr}^m_F K^r(\sigma,\Lambda^p)$. □

Definition 4.3.3. Let

$$K^{q,r}(\Delta_X,\Lambda^p) = \bigoplus_{|\underline{\sigma}|=q+1} K^r(\sigma,\Lambda^p)$$

and for any $\alpha\supset\sigma$ let π^σ_α be the natural projection

$$\pi^\sigma_\alpha : \bigoplus_{\tau\supset\sigma} {}'H^{2r}(Y_\tau)\otimes\lambda^{p-r}(\tau) \to \bigoplus_{\beta\supset\alpha} {}'H^{2r}(Y_\beta)\otimes\lambda^{p-r}(\beta).$$

Define the horizontal differential $d'' : K^{q,r}(\Delta_X,\Lambda^p)\to K^{q+1,r}(\Delta_X,\Lambda^p)$

$$d''(x) = \sum_{\alpha:\partial_i\alpha=\sigma} \mathrm{sgn}(i,\underline{\alpha})\pi^\sigma_\alpha.$$

The inclusions $\Lambda^\bullet(\sigma)\hookrightarrow K^0(\sigma,\Lambda^\bullet)$ extend to the inclusions

$$\bigoplus_{|\underline{\sigma}|=q}\Lambda^\bullet(\sigma) = C^q(\Delta_X,\Lambda^\bullet)\hookrightarrow K^{q,0}(\Delta_X,\Lambda^\bullet)$$

for each $q\geq 0$ and it is clear from the definition of d'' that they define a morphism of complexes. Moreover, by Lemma 4.3.2

$$C^0(\Delta_X,\Lambda^p) = \mathrm{Ker}\, d'.$$

It follows that the induced morphism $C^\bullet(\Delta_X,\Lambda^\bullet)\to sK^\bullet(\Delta_X,\Lambda^\bullet)$ is a quasi-isomorphism.

4.4. **Logarithm of monodromy.** Let σ be a face of Δ_X, $\underline{\sigma}=\{i_1,\dots,i_l\}, l\geq 2$, and assume that $j,k\in\underline{\sigma}$ are top two vertices of σ in the orientation order. Define $N_\sigma : K^r(\partial_k\sigma,\Lambda^p)\to K^r(\sigma,\Lambda^{p-1})$ to be the map

$$N_\sigma(v) = \sum_{\tau\supset\sigma}(-1)^{|\underline{\tau}|} a_\tau\otimes\delta_{jk}b_\tau,$$

where v is a sum of primitive tensors $\sum_{\tau\supset\partial_k\sigma} a_\tau\otimes b_\tau$ (the meaning of the sign will become apparent in the proof of Lemma 5.1.4). This map is always well-defined when

$p = 1$ and is well-defined for $p > 1$ if $\Lambda^\bullet$ is regular at σ. Define the map

$$N : K^{q,r}(\Delta_X, \Lambda^\bullet) \to K^{q+1,r}(\Delta_X, \Lambda^{\bullet-1}), \qquad N(v) = (-1)^{q+1} \sum_{|\underline{\sigma}|=q+2} N_\sigma,$$

which in turn induces the map $N : sK^\bullet(\Delta_X, \Lambda^\bullet) \to sK^{\bullet+1}(\Delta_X, \Lambda^{\bullet-1})$, when it is well-defined.

Lemma 4.4.1. *The map on the cohomology*

$$H^\bullet(\Delta_X, \Lambda^1) \to H^{\bullet+1}(\Delta_X, \Lambda^0)$$

induced by the map N coincides with the coboundary map associated with the short exact sequence of sheaves (3.1). If $\Lambda^\bullet$ is regular at every face σ of Δ_X then the statement is also true about the map

$$H^\bullet(\Delta_X, \Lambda^{p+1}) \to H^{\bullet+1}(\Delta_X, \Lambda^p)$$

induced by N for all $p > 0$.

Proof. The proof uses an idea similar to the one used in the proof of [SRJ18, Proposition 3.5].

Since the inclusion $C^\bullet(\Delta_X, \Lambda^\bullet) \to sK^\bullet(\Delta_X, \Lambda^\bullet)$ is a quasi-isomorphism, it suffices to check the statement on the complex $C^\bullet(\Delta_X, \Lambda^\bullet)$.

Take a cocycle $a = (a_\sigma) \in C^q(\Delta_X, \Lambda^p)$, where $a_\sigma \in \Lambda^p(\sigma)$. If $\tilde{a} = (\tilde{a}_\sigma) \in C^q(\Delta_X, A^p)$ is some lifting of the cocycle a then

$$d\tilde{a} \in \mathrm{Ker}(C^{q+1}(\Delta_X, A^p) \to C^{q+1}(\Delta_X, \Lambda^p)) = \mathrm{Im}(h : C^{q+1}(\Delta, \Lambda^{p-1}) \to C^{q+1}(\Delta_X, A^p)).$$

We will pick $\tilde{a}$ in such a way that $d\tilde{a} = h(Na)$.

Recall that for any face τ elements of $\lambda^1(\tau)$ are translation-invariant differential forms with rational coefficients on $\tau \subset T(\tau) \otimes \mathbb{R} \subset H^0(\mathrm{St}(\tau), A^1)^* \otimes \mathbb{R}$. The elements of $A^1(\tau)$ are tautologically identified with linear functions on $H^0(\mathrm{St}(\tau), A^1)^*$ or with affine functions on $e(\tau) \subset H^0(\mathrm{St}(\tau), A^1)^*$. More generally, elements of $\bigwedge^p A^1(\tau)$ can be identified with translation-invariant differential forms with rational coefficients on the linear subspace spanned by $e(\tau) \subset H^0(\mathrm{St}(\tau), A^1)^* \otimes \mathbb{R}$. If $p > 1$ we assume from now on that $\Lambda^\bullet$ regular at any face of Δ_X: $\Lambda^p(\tau) = \bigwedge^p \Lambda^1(\tau)$. Recall that 1_τ is the function that is constantly 1 on τ, we will use the same notation for the corresponding differential form. The natural inclusion

$$\Lambda^p(\tau) \hookrightarrow A^p(\tau)$$

is induced by wedging with 1_τ. Note that this is well-defined, since $\Lambda^1(\tau)$ is the quotient of $A^1(\tau)$ be the subspace spanned by 1_τ.

For each element $a_\sigma \in \Lambda^p(\sigma)$ pick the unique lifting $\tilde{a}_\sigma \in A^p(\sigma)$ such that $\delta_j a_\sigma = 0$, where $j \in \underline{\sigma}$ is maximal with respect to the orientation ordering. Since $da = 0$, we have that $d\tilde{a} \in C^{q+1}(\Delta_X, \Lambda^{p-1}) \subset C^{q+1}(\Delta_X, A^p)$. Pick a face $\tau \subset \Delta_X$. Let $j < k$ be two topmost vertices of τ in the orientation ordering. We have

$$\delta_k \tilde{a}_{\partial_i \tau} = 0, \text{ for all } i \in \underline{\tau}, i \neq k, \qquad \delta_j \tilde{a}_\sigma = 0,$$

where $\sigma = \partial_k \tau$.

The elements of the form $1_\tau \wedge b \in A^p(\tau)$ are characterized by the property

$$\delta_i(1_\tau \wedge b) = \delta_j(1_\tau \wedge b),$$

for any vertices $i, j \in \underline{\tau}$, moreover, for any vertex $i \in ul\tau$,

$$\delta_i(1_\tau \wedge b) = b|_{\langle\tau\rangle}.$$

In particular,

$$\delta_k(d\tilde{a})_\tau = \sum_{i\in\tau} \operatorname{sgn}(i,\underline{\tau})\delta_k a_{\partial_i\tau} = (-1)^{|\underline{\tau}|}\delta_k\tilde{a_\sigma} = (-1)^{|\underline{\tau}|}(\delta_k - \delta_j)\tilde{a}_\sigma.$$

But since $\tilde{a}_\sigma|_{\langle e(\sigma)\rangle} = a_\sigma$, we have

$$(d\tilde{a})_\tau = (-1)^{|\underline{\tau}|}1_\tau \wedge \delta_{jk}a_\sigma|_\tau = (Na)_\tau,$$

and we conclude. $\square$

5. Cohomology of Λ^p

5.1. **Proof of Theorems A and A'.** Let α, σ be faces such that $\partial_i\alpha = \sigma$. Denote the unique vector in $\lambda^1(\alpha;\sigma)$ such that $\delta_{ik}dz_{ik,\sigma} = 1$ for some (equivalently, any) $k \in \underline{\sigma}$ as $dz_{i,\sigma}$. Clearly,

$$\sum_{i\in\underline{\sigma}} dz_{i,\partial_i\sigma} = 0.$$

Let $G^\bullet$ be the filtration on the total complex $sK^\bullet(\Delta_X,\Lambda^p))$ induced by the filtration

$$G^m K^r(\sigma,\Lambda^p) = F^{|\underline{\sigma}|+r-m}K^r(\sigma,\Lambda^p).$$

Lemma 5.1.1. *The zeroth page of the spectral sequence associated to the filtration G on $sK^\bullet(\Delta_X,\Lambda^p))$ has the following form*

$$ {}^p_G E_0^{i,j}(Y) = \bigoplus_{r\geq 0}\bigoplus_{|\sigma|=i+j-r+1}\bigoplus_{\tau\supset^{p-r-j}\sigma} {}^\prime H^{2r}(Y_\tau)\otimes\lambda^j(\sigma)$$

with the differential $d_0 : {}^p_G E_0^{i,j} \to {}^p_G E_0^{i,j+1}$ defined as follows on primitive tensors $a_\tau\otimes b_\tau \in {}^\prime H^{2r}(Y_\tau)\otimes\lambda^j(\sigma)$:

$$d_0(a_\tau\otimes b_\tau) = \sum_{l\in\underline{\tau}\setminus\underline{\sigma}} a_\tau\otimes \operatorname{sgn}(l,\underline{\tau})(dz_{l,\sigma}\wedge b_\tau)$$

where k is the maximal vertex in $\underline{\sigma}$ with respect to the orientation ordering.

Proof. The first statement follows from the fact that

$$ {}^p_G E_0^{i,j}(Y) = \operatorname{gr}^i_G sK^{i+j}(\Delta_X,\Lambda^p) = \bigoplus_{r\geq 0}\bigoplus_{r+|\underline{\sigma}|-1=i+j}\operatorname{gr}^i_G K^r(\sigma,\Lambda^p) =$$
$$= \bigoplus_{r\geq 0}\bigoplus_{|\underline{\sigma}|=i+j-r+1}\operatorname{gr}^{j+1}_F K^r(\sigma,\Lambda^p).$$

The contribution of the vertical differential $d' : K^r(\sigma,\Lambda^p) \to K^{r+1}(\sigma,\Lambda^p)$ to

$$d_0 : \bigoplus_{r\geq 0}\bigoplus_{|\underline{\sigma}|=i+j-r+1}\operatorname{gr}^{j+1}_F K^r(\sigma,\Lambda^p) \to \bigoplus_{r\geq 0}\bigoplus_{|\underline{\sigma}|=i+j-r+2}\operatorname{gr}^{j+2}_F K^r(\sigma,\Lambda^p)$$

is zero because d' preserves the filtration F.

We identify $\operatorname{gr}^{j+1}_F K^r(\sigma,\Lambda^p)$ with $\bigoplus_{\tau\supset^{p-r-j}\sigma} {}^\prime H^{2r}(Y_\tau)\otimes\lambda^j(\sigma)$ via the isomorphism $\mathrm{id}\otimes\kappa^\tau_{\sigma,k}$ where k is the vertex in $\underline{\sigma}$ that is maximal with respect to the orientation ordering. Let us compute the contribution of the horizontal differential $d'' : K^{q,r}(\Delta_X,\Lambda^p) \to K^{q+1,r}(\Delta_X,\Lambda^p)$.

Consider a sum of primitive tensors $v = \sum_{\tau\supset\sigma} a_\tau \otimes b_\tau \in K^r(\sigma,\Lambda^p)$. Take some $l \in \underline{\tau}\backslash\underline{\sigma}$ and let $\partial_l\alpha = \sigma$. Then the images of v and $d''v$ in $\mathrm{gr}_F^{j+1} K^r(\sigma,\Lambda^p)$ and $\mathrm{gr}_F^{j+2} K^r(\alpha,\Lambda^p)$ are, respectively,

$$\sum_{\tau\supset^{p-r-j}\sigma} a_\tau \otimes \kappa^\tau_{\sigma,k}(b_\tau), \qquad \mathrm{sgn}(l,\underline{\alpha}) \sum_{\tau\supset^{p-r-j-1}\alpha} a_\tau \otimes \kappa^\tau_{\alpha,k}(b_\tau),$$

with

$$\kappa^\tau_{\alpha,k}(b_\tau) = \mathrm{sgn}(l,\underline{\tau}\setminus\underline{\sigma}) dz_{lk,\sigma} \wedge \kappa^\tau_{\sigma,k}(b_\tau).$$

Since

$$\mathrm{sgn}(l,\underline{\alpha})\,\mathrm{sgn}(l,\underline{\tau}\setminus\underline{\sigma}) = \mathrm{sgn}(l,\underline{\tau})$$

by Lemma 4.3.1, the statement of the lemma follows. □

For any face τ and a number $n \le |\underline{\sigma}|$ consider the complex $S^\bullet_n(\tau)$

$$0 \to \bigoplus_{\sigma\subset^n\tau} \lambda^0(\sigma) \to \bigoplus_{\sigma\subset^{n-1}\tau} \lambda^1(\sigma) \to \ldots \to \lambda^n(\tau) \to 0$$

with the differential sending $a \in \lambda^p(\sigma)$ to $\sum_{\partial_l\alpha=\sigma} \mathrm{sgn}(l,\underline{\sigma}) dz_{l,\sigma} \wedge a$.

Lemma 5.1.2. *For any τ and any $n < |\underline{\tau}|$*

$$H^i(S^\bullet_n(\tau)) = \begin{cases} \mathbb{Q} & i = 0, \\ 0, & i > 0. \end{cases}$$

Moreover, the unique up to scalar 0-th cohomology class is represented by a cocycle $a = (a_\sigma)_{\sigma\subset^n\tau}$ with

$$a_\sigma = \prod_{i\in\underline{\sigma}} (-1)^{|\underline{\tau}|}\,\mathrm{sgn}(i,\underline{\tau})$$

Proof. The proof is by induction on n. Let's check the base of induction: clearly, $S^0_0(\tau) = \lambda^0(\tau)$ and $S^i_0(\tau) = 0$ for $i > 0$. For the induction step, let k be some vertex in $\underline{\sigma}$ and consider the exact sequence

$$0 \to S^\bullet_n(\tau,\partial_k\tau) \to S^\bullet_n(\tau) \to S^\bullet_{n-1}(\partial_k\tau) \to 0,$$

where $S^\bullet_n(\tau,\partial_k\tau)$ is the subcomplex with the terms

$$S^i_n(\tau,\partial_k\tau) = \bigoplus_{\tau\supset^{n-i}\sigma,\sigma\not\subset\partial_k\sigma} \lambda^i(\sigma).$$

Since $S^\bullet_{n-1}(\partial_k\tau)$ only has cohomology $\mathbb{Q}$ in degree 0, and vanishing cohomology in all other degrees by induction hypothesis, in order to prove that $S^\bullet_n(\tau)$ has the same property it would suffice to show that the complex $S^\bullet_n(\tau,\partial_k\tau)$ is acyclic.

Notice that for any faces $\alpha,\beta,\eta \subseteq \sigma$ such that $\underline{\alpha}\cup\underline{\beta} = \underline{\eta}$ and $\underline{\alpha}\cap\underline{\beta} = \{k\}$, and any numbers $m_1, m_2, m_1 \le |\underline{\alpha}|, m_2 \le |\underline{\beta}|$,

$$S^\bullet_{m_1+m_2}(\eta,\partial_k\sigma) \cong S^\bullet_{m_1}(\alpha,\partial_k\alpha) \otimes S^\bullet_{m_2}(\beta,\partial_k\beta)$$

(recall that the tensor product in the right hand side is the total complex of the double complex $(S^\bullet_{m_1}(\alpha,\partial_k\alpha)\otimes S^\bullet_{m_2}(\beta,\partial_k\beta), d_\alpha, d_\beta)$). In particular, denoting $\alpha_l \subset \sigma$ the 1-faces with $\underline{\alpha_l} = \{l,k\}$, we have

$$S^\bullet_n(\sigma,\partial_k\sigma) = \bigoplus_{\tau\subset\partial_k\sigma,|\underline{\tau}|=n}\ \bigotimes_{l\in\underline{\partial_k\sigma}} S^\bullet_{1_{\underline{\tau}}(l)}(\alpha_l,l),$$

where $1_{\underline{\tau}} : \underline{\partial_k\sigma} \to \{0,1\}$ is the indicator function of $\underline{\tau}$. The complex $S^\bullet_1(\alpha_l,l)$ has the form $\mathbb{Q} \to \mathbb{Q}$ with non-trivial differential, and so is acyclic, and $S^\bullet_0(\alpha_i,\partial_k\alpha)$ has unique non-trivial term $\mathbb{Q}$ in degree 0. Therefore, since there is at least one $l \in \underline{\tau}$, the tensor product is acyclic, and so is $S_n^\bullet(\sigma, \partial_k\sigma)$.

For the last statement of the Lemma, direct computation gives

$$(da)_\sigma = \sum_{i\in\underline{\sigma}} \left(\prod_{j\in\underline{\partial_i}\sigma} (-1)^{|\underline{\tau}|} \operatorname{sgn}(j,\underline{\tau}) \right) \operatorname{sgn}(i,\underline{\tau}) dz_{i,\partial_i\sigma} =$$

$$= (-1)^{|\underline{\tau}|} \prod_{j\in\underline{\sigma}} (-1)^{|\underline{\tau}|} \operatorname{sgn}(j,\underline{\tau}) (\sum_{i\in\underline{\sigma}} dz_{i,\partial_i\sigma}) = 0.$$

So a is indeed a cocycle representing a non-trival class in $H^0(S_n^\bullet(\sigma))$. □

Lemma 5.1.3. *The first page of the spectral sequence associated to the filtration G on $sK^\bullet(\Delta_X, \Lambda^p)$ consists of a single row:*

$$({}^{\,p}_{G}E_1^{\bullet,0}(Y), d_1) = (sD_p^\bullet(Y), d).$$

Proof. By Lemma 5.1.1,

$${}^{\,p}_{G}E_0^{i,j}(Y) = \bigoplus_{r=0}^{\min\{i,p\}} \bigoplus_{|\tau|=p-2r+i+1} {}'H^{2r}(Y_\tau) \otimes S_{p-r}^j(\tau),$$

so by Lemma 5.1.2, ${}^{\,p}_{G}E_1^{i,j}(Y) = 0$ unless $j = 0$, and

$${}^{\,p}_{G}E_1^{i,0}(Y) = \bigoplus_{r=0}^{\min\{i,p\}} \bigoplus_{|\tau|=p-2r+i+1} {}'H^{2r}(Y_\sigma) = \bigoplus_{r=0}^{\min\{i,p\}} {}'H^{2r}(Y^{(p-2r+i+1)}) = sD_p^i(Y).$$

Fix p and denote for brevity $L^\bullet = sK^\bullet(\Delta_X, \Lambda^p))$. The differential $d_1 : H^{i+j}(\operatorname{gr}_G^i L^\bullet) \to H^{i+j+1}(\operatorname{gr}_G^{i+1} L^\bullet)$ is the boundary morphism on the cohomology associated to the short exact sequence

$$0 \to \operatorname{gr}_G^{i+1} L^\bullet \to G^i L^\bullet / G^{i+2} L^\bullet \to \operatorname{gr}_G^i L^\bullet \to 0$$

If v is an element of $G^{i+j}L$ that represents a cohomology class $[v] \in H^{i+j}(\operatorname{gr}_G^i L^\bullet)$ then $dv \in G^{i+j+1}L^{i+1}$ and dv represents $d_1([v])$ in $H^{i+j+1}(\operatorname{gr}_G^{i+1} L^\bullet)$.

Take

$$[v] = \sum_{r\geq 0} \sum_{|\tau|=p-2r+i+1} a_\tau \in \bigoplus_{r\geq 0} \bigoplus_{|\tau|=p-2r+i+1} {}'H^{2r}(Y_\tau) \cong {}'H^i(\operatorname{gr}_G^i L^\bullet),$$

We identify

$$H^i(\operatorname{gr}_G^i L^\bullet) = \bigoplus_{|\tau|=p-2r+i+1} {}'H^{2r}(Y_\tau) \otimes S_{p-r}^0(\tau) \text{ and } \bigoplus_{|\tau|=p-2r+i+1} {}'H^{2r}(Y_\tau)$$

via the isomorphism that sends the cocycle a from the statement of Lemma 5.1.2, to 1. Denote vol_σ^τ the unique element of $\lambda^{p-r}(\tau;\sigma)$ such that $\kappa_\sigma^\tau(\mathrm{vol}_\sigma^\tau) = 1_\sigma$. Then the cohomology class of $[v]$ can be represented by a sum

$$v = \sum_{r\geq 0} \sum_{\substack{|\tau|=p-2r+i+1 \\ \sigma \subset^{p-r} \tau}} v_{\sigma,\tau} \in G^i L, \qquad v_{\sigma,\tau} \in K^r(\sigma, \Lambda^p),$$

where

$$(v_{\sigma,\tau})_\tau = \prod_{l\in\underline{\sigma}} (-1)^{|\tau|} \operatorname{sgn}(l,\underline{\tau}) \cdot a_\tau \otimes \mathrm{vol}_\sigma^\tau.$$

Then

$$d'(v_{\sigma,\tau}) = \prod_{l\in\underline{\sigma}}(-1)^{|\tau|}\operatorname{sgn}(l,\underline{\tau})\cdot\sum_{l\in\underline{\tau}\setminus\underline{\sigma}} N_l \operatorname{Gys}^{\tau}_{\partial_l\tau}(a_\tau)\otimes\delta_{lk}\operatorname{vol}^{\tau}_{\sigma}|_{\partial_l\tau}+\sum_{l\in\underline{\sigma}} N_i \operatorname{res}_\tau \operatorname{Gys}^{\tau}_{\partial_l\tau}(a_\tau)\otimes\delta_{lk}\operatorname{vol}^{\tau}_{\sigma}.$$

Since $\operatorname{vol}^{\tau}_{\sigma} \in \lambda^{p-r}(\tau;\sigma)$ we have that $\delta_{lk}\operatorname{vol}^{\tau}_{\sigma} = 0$ for all $l \in \underline{\sigma}$, and the summands in the second sum vanish. And since for $l \in \underline{\tau}\setminus\underline{\sigma}$ we have

$$\kappa^{\partial_l\tau}_{\sigma,k}(\delta_{lk}\operatorname{vol}^{\tau}_{\sigma}) = \operatorname{sgn}(l,\underline{\tau}\setminus\underline{\sigma})\kappa^{\tau}_{\sigma,k}(\operatorname{vol}^{\tau}_{\sigma})$$

(as in the proof of Lemma 4.3.2), we have that the image of $d'(a_\tau\otimes\operatorname{vol}^{\tau}_{\sigma})$ in $\operatorname{gr}^{i+1}_G L^{i+1}$ is

$$\sum_{l\in\underline{\tau}\setminus\underline{\sigma}} N_l \operatorname{sgn}(l,\underline{\tau}\setminus\underline{\sigma})\frac{\prod_{l'\in\underline{\sigma}}(-1)^{|\underline{\tau}|}\operatorname{sgn}(l',\underline{\tau})}{\prod_{l'\in\underline{\partial_l\sigma}}(-1)^{|\underline{\tau}|-1}\operatorname{sgn}(l',\underline{\tau})}\operatorname{Gys}^{\tau}_{\partial_i\tau}(a_\tau) = \sum_{l\in\underline{\tau}\setminus\underline{\sigma}} N_l\operatorname{sgn}(l,\underline{\tau})\operatorname{Gys}^{\tau}_{\partial_i\tau}(a_\tau)$$

by Lemma 4.3.1.

Similarly, since $\operatorname{vol}^{\beta}_{\alpha}|_{\sigma} = \operatorname{vol}^{\tau}_{\sigma}$ for any pair of faces $\beta\supset\alpha$ such that $\partial_i\beta=\tau$ and $\partial_i\alpha=\sigma$ for some $i\notin\underline{\tau}$,

$$d''(v)_\beta = \operatorname{sgn}(l,\underline{\alpha})a_\beta\otimes\operatorname{vol}^{\beta}_{\alpha} = \operatorname{sgn}(l,\underline{\alpha})a_\tau|_{Y_\beta}\otimes\operatorname{vol}^{\beta}_{\alpha}.$$

Therefore, the image of $d''(v)_\beta$ in $H^{i+1}(\operatorname{gr}^{i+1}L^\bullet)$ is then

$$\operatorname{sgn}(l,\alpha)\frac{\prod_{l\in\underline{\sigma}}(-1)^{|\tau|}\operatorname{sgn}(l,\underline{\tau})}{\prod_{l\in\underline{\alpha}}(-1)^{|\beta|}\operatorname{sgn}(l,\underline{\beta})}a_\tau|_{Y_\alpha} = \operatorname{sgn}(l,\underline{\beta})a_\tau|_{Y_\alpha}$$

by Lemma 4.3.1.

We observe that $d_1 = d'+d''$ coincides with the differential of the complex $sD^\bullet_p$. ☐

Coboundary morphism N associated to the exact sequence (3.1) is analogous to the eigenwave morphism in tropical geometry, introduced by Mikhalkin and Zharkov in [MZ14]. See Proposition 3.5 [JRS17] for the comparison between a coboundary morphism of a sequence analogous to 3.1 in tropical geometry. We will now show that under the isomorphism from Lemma 5.1.3 the coboundary morphism corresponds to the morphism N on the weight spectral sequence for the limit mixed Hodge structure induced by the logarithm of the monodromy morphism.

Lemma 5.1.4. *The map*

$$N: {}^{p}_{G}E^{i,0}_1(Y)\to {}^{p-1}_{G}E^{i+1,0}_1(Y)$$

induced by the morphism $N: sK^r(\Delta_X,\Lambda^p)\to sK^{r+1}(\Delta_X,\Lambda^{p-1})$ is identity.

Proof. As in the proof of Lemma 5.1.3 take

$$[v] = \sum_{r\geq 0}\sum_{|\tau|=p-2r+i+1} a_\tau \in H^i(\operatorname{gr}^i_G sK^\bullet(\Delta_X,\Lambda^p)) \cong {}^{p}_{G}E^{i,0}_1(Y)$$

represented by an element

$$v = \sum_{r\geq0}\sum_{\substack{|\tau|=p-2r+i+1\\ \sigma\subset^{p-r}\tau}} v_{\sigma,\tau},\quad v_{\sigma,\tau}\in K^r(\sigma,\Lambda^p),\quad (v_{\sigma,\tau})_\tau = \left(\prod_{l\in\underline{\sigma}}(-1)^{|\underline{\tau}|}\operatorname{sgn}(l,\underline{\tau})\right)a_\tau\otimes\operatorname{vol}^{\tau}_{\sigma},$$

Take a face $\alpha\subset\tau$, let j,k be the two topmost vertices in $|\underline{\alpha}|$, and denote $\sigma=\partial_k\alpha$, then since $\delta_{jk}\operatorname{vol}^{\tau}_{\sigma} = \operatorname{sgn}(j,\underline{\tau}\setminus\underline{\sigma})\operatorname{vol}^{\tau}_{\alpha}$, we have

$$N_\alpha(v_{\sigma,\tau}) = (-1)^{|\underline{\tau}|+|\underline{\sigma}|+1}\prod_{l\in\underline{\sigma}}(-1)^{|\underline{\tau}|}\operatorname{sgn}(l,\underline{\tau})\operatorname{sgn}(j,\underline{\tau}\setminus\underline{\sigma})\cdot a_\alpha\otimes\operatorname{vol}^{\tau}_{\alpha}.$$

The image of $N_\alpha(v_{\sigma,\tau})$ in $H^{i+1}(\mathrm{gr}_G^{i+1} sK^\bullet(\Delta_X, \Lambda^{p-1})$ is thus by Lemma 4.3.1

$$(-1)^{|\underline{\tau}|+|\underline{\sigma}|+1} \frac{\prod_{l\in\underline{\sigma}}(-1)^{|\underline{\tau}|}\operatorname{sgn}(l,\underline{\tau})}{\prod_{l\in\underline{\alpha}}(-1)^{|\underline{\tau}|}\operatorname{sgn}(l,\underline{\tau})} \operatorname{sgn}(j, \underline{\tau}\setminus\underline{\sigma}) \cdot a_\alpha = \\ = (-1)^{|\underline{\sigma}|+1}\operatorname{sgn}(j,\underline{\tau})\operatorname{sgn}(j,\underline{\tau}\setminus\underline{\sigma})a_\alpha = (-1)^{|\underline{\sigma}|+1}\operatorname{sgn}(j,\underline{\alpha})a_\alpha,$$

which is equal to just a_α, since j is maximal in $\underline{\alpha}$. □

Theorem 5.1.5. *Let $f : X \to S$ be a unipotent degeneration and assume that there exists a cohomologically Kähler class in $H^2(Y)$. Then for all $p, q \geq 0$, there exists a map*

$$H^q(\Delta, \Lambda^p) \to \mathrm{gr}_W^{2p} H^{p+q}(X_\infty).$$

Assume that $\Lambda^\bullet$ is regular at every face of Δ_X. Then the map commutes with the logarithm of monodromy morphism N on the right, and the coboundary morphism of the short exact sequence (3.1) on the left.

Proof. The proof is a conjunction of Lemmas 4.1.2, 4.3.2, 5.1.1, 5.1.2, 5.1.3, 5.1.4 and Fact 2.4.2. □

Theorem 5.1.6. *Assume that there exists a combinatorial Lefschetz class $\omega \in H^2(Y)$, then the morphism*

$$H^\bullet(\Delta, \Lambda^\bullet) \to \mathrm{gr}^W_{2\bullet} H^\bullet(X_\infty)$$

constructed in Theorem A is injective and $\dim H^{n-q}(\Delta_X, \Lambda^{n-p}) = \dim H^q(\Delta_X, \Lambda^p)$. *If $\Lambda^\bullet$ is regular at any face $\sigma \subset \Delta_X$ and the morphism N is well-defined then*

$$N^{p-q} : H^q(\Delta_X, \Lambda^p) \to H^p(\Delta_X, \Lambda^q)$$

is an isomorphism.

Proof. By Fact 2.4.2 $(K^{\bullet,\bullet} \otimes \mathbb{R}, N, L_\omega)$ is a Hodge-Lefschetz module, since $\omega|_{Y_\sigma}$ is a Lefschetz class for all strata Y_σ. Since ω is also combinatorial, the operator L_ω preserves the subcomplex ${}'K^{\bullet,\bullet}$, so it is a Hodge-Lefschetz submodule of $(K^{\bullet,\bullet}, N, L_\omega)$. By Fact 2.4.4, the inclusion $\operatorname{Ker}\square \cap {}'K^{\bullet,\bullet} \hookrightarrow \operatorname{Ker}\square$ induces the inclusion of cohomology of complexes $s\,{}'D^\bullet_{2p} \hookrightarrow sD^\bullet_{2p}$ for all p. The last statement of the theorem follows from the definition of combinatorial Lefschetz classes and Fact 2.4.4. □

Proposition 5.1.7. *If $X \to S$ is a degeneration of curves such that the central fibre has at least one double point, then any cohomologically Kähler class is combinatorial Lefschetz.*

Proof. Immediate, since ${}'H^2(Y^{(1)}) = H^2(Y^{(1)})$. □

If $X' \to S^*$ is a degeneration of polarized Abelian varieties over a punctured disc with multiplicative reduction, then by [Kün98], there exists a smooth projective model $f : X \to S$ with a central fibre such that its irreducible components are toric varieties.

Proposition 5.1.8. *If $X \to S$ is a Künemann-Mumford degeneration with toric central fibre then*

$$H^i(X_\infty) \cong \bigoplus_{p+q=i} H^q(\Delta_X, \Lambda^p)$$

and

$$N^{q-p} : H^q(\Delta_X, \Lambda^p) \to H^q(\Delta_X, \Lambda^p)$$

is an isomorphism.

Proof. The irreducible components of the central fibre are toric and intersect along the irreducible components of the toric boundary, therefore, ${}'H^i(Y^{(j)}) \cong H^i(Y^{(j)})$ and ${}'D^\bullet(Y) \cong D^\bullet(Y)$, which implies the first statement. Any cohomologically Kähler class satisfies the conditions of Theorem 5.1.6, which implies the second statement. □

The morphism constructed in Theorem A can be more concretely described with the help of the notion of PL metrized virtual line bundle (intruduced in [KT02b], see also [Yu15]).

Definition 5.1.9. A *virtual line bundle* is a torsor under the sheaf Λ^1. If L is a virtual line bundle, a *piece-wise linear metrization* (or *PL metrization*) of ω is a section of $h \in H^0(L \otimes \bar{\Lambda}^1)$. The *curvature of h at σ* is the following element of $H^2(Y_\sigma)$

$$c_1(L,h)_\sigma = \sum_{i \in \overline{\mathrm{St}}(\sigma)} \tilde{h}_\sigma(i) N_i c_1(\mathcal{O}(Y_i))|_{Y_\sigma}$$

where $\tilde{h}_\sigma$ is a lifting of h to a section an element of $\bar{A}^1(\sigma)$. We denote $c_1(L,h) = \sum_\sigma c_1(L,h)_\sigma \in \bigoplus_\sigma H^2(Y_\sigma)$.

Clearly, if (L,h) and (L',h') are two PL metrized virtual line bundles then

$$c_1(L \otimes L', h+h') = c_1(L,h) + c_1(L,h').$$

Proposition 5.1.10. *Let $a \in C^1(\Delta, \Lambda^1)$ and let L be the corresponding virtual line bundle.*

If $Na = 0 \in H^2(\Delta_X, \Lambda^0)$ then L admits a PL metrization and for any PL metrization h the class $\mathrm{sp}(c_1(L,h)) \in H^2(X_\infty)$ is equal to the image of a in $\mathrm{gr}_2^W H^2(X_\infty)$.

Proof. Consider the exact sequence

$$\dots H^1(\Delta_X, A^1) \to H^1(\Delta_X, \Lambda^1) \xrightarrow{N} H^2(\Delta_X, \Lambda^0) \to \dots$$

Since $a \in \operatorname{Ker} N$, it can be lifted to a cocycle $\tilde{a} \in C^1(\Delta_X, A^1) \hookrightarrow C^1(\Delta_X, \bar{A}^1)$. Since the sheaf $\bar{A}^1$ is flabby,

$$0 \to C^0(\Delta_X, \bar{A}^1) \to C^1(\Delta_X, \bar{A}^1) \to C^2(\Delta_X, \bar{A}^1) \to \dots$$

is exact in degrees > 0. In particular, there exists a cochain $\tilde{b} \in C^0(\Delta_X, \bar{A}^1)$ such that $d\tilde{b} = \tilde{a}$. The diagram of simplicial cochain complexes arising from the morphism $\bar{A}^1 \to \bar{\Lambda}^1$

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^0(\Delta_X, \bar{A}^1) & \longrightarrow & C^1(\Delta_X, \bar{A}^1) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^0(\Delta_X, \bar{\Lambda}^1) & \longrightarrow & C^1(\Delta_X, \bar{\Lambda}^1) & \longrightarrow & \dots \end{array}$$

is commutative, and therefore there exists a cochain $b \in C^0(\Delta_X, \bar{\Lambda}^1)$ such that $db = a$. Now observe that $(C^\bullet(\Delta_X, \bar{\Lambda}^1), d) \cong (K^{\bullet,0}(\Delta_X, \Lambda^1), d'')$ and that the data of a cochain b defines a PL metric h on the virtual line bundle L corresponding to a, with $c_1(L,h) = d''b$. The cocyle $(0, -d''b) \in K^{1,0}(\Delta_X, \Lambda^1) \oplus K^{0,1}(\Delta_X, \Lambda^1)$ is then cohomologous to the image of a in $sK^1(\Delta_X, \Lambda^1)$. It follows that the image of a in $\mathrm{gr}_2^W H^2(X_\infty)$ is $(0, -c_1(L,h)) = \mathrm{sp}(-c_1(L,h))$.

If h' is another metrization of L then there exists a cochain b' such that $d''b = a$ and $d'b' = c_1(L,h')$, a is cohomologous to $(0, -c_1(L,h')) = \mathrm{sp}(-c_1(L,h))$. □

5.2. **Superforms on dual intersection complexes.** Let $O \subset e(\mathrm{St}(\sigma))$ be an open set that intersects $e(\sigma)$ non-trivially. Call two germs α and β of (p,q)-superforms defined in a neighbourhood of O in $T(\sigma)$ equivalent if

$$\alpha|_{e(\tau)} = \beta|_{e(\tau)}$$

for any face $\tau \supset \sigma$. For any open set $U \subset \mathrm{St}(\sigma)$ that intersects σ non-trivially define the set of superforms on U to be the set of equivalence classes of germs of superforms in a neighbourhood of $e(U) \subset T(\sigma)$. It is easy to check that differentials d' and d'' map equivalence classes of germs of superforms to equivalence classes, same is true about the map $N : A^{p,q} \to A^{p-1,q+1}$.

If $\tau \supset \sigma$ then the inclusion $\mathrm{St}(\tau) \subset \mathrm{St}(\sigma)$ induces a natural map $r : T(\tau) \to T(\sigma)$. Clearly, $e(\mathrm{St}(\tau))$ is mapped to $e(\mathrm{St}(\sigma))$ under this map. We call the pullback of a superform α defined on an open set $U \subset \mathrm{St}(\sigma)$ along this map its *restriction* to $r(U) \cap \mathrm{St}(\tau)$.

Definition 5.2.1 (Superforms on Δ_X)**.** For an open set $U \subset \Delta_X$, let Σ_U be the collection of faces σ such that $U \cap \mathring{\sigma} \neq \emptyset$. A *smooth function f* on an open subset $U \subset \Delta_X$ is a collection $(f_\sigma)_{\sigma \in \Sigma_U}$, where α_σ is a germ of a smooth function on a neighbourhood of $U \cap \mathring{\sigma}$ in $\mathrm{St}(\sigma)$, such that whenever $\sigma \subset \tau$, the restriction of f_σ to $U \cap \mathrm{St}(\tau)$ is f_τ. We denote the sheaf of smooth functions on Δ_X as $\mathscr{A}^{0,0}_X$.

The sheaves of (p,q)-superforms on Δ_X are defined to be

$$\mathscr{A}^{p,q}_X := \mathscr{A}^{0,0}_X \otimes_{\mathbb{R}} \Lambda^p \otimes_{\mathbb{R}} \Lambda^q$$

Lemma 5.2.2. *The definition of superforms on Δ_X is invariant under subdivisions induced by blow-ups of the strata of X_0.*

Proof. The definition of sheaves Λ^p is invariant under the subdivisions by Proposition 3.3.2.

Let σ be a face of Δ_X and let Y_σ be the stratum being blown up, giving rise to a new degeneration $X' \to X$. For any face $\tau \subset \Delta_{X'}$ such that $\tau \subset \sigma$ we have by Proposition 3.3.2 $H^0(\mathrm{St}(\tau), A^1) \cong H^0(\mathrm{St}(\sigma), A^1)$ and therefore $T(\tau) \cong T(\sigma)$. The invariance of the definition of the □

If f is a local section of $\mathscr{A}^{0,0}$ on a subset $U \subset \mathrm{St}(\sigma)$ then the coefficients of the forms $d'f$ and $d''f$ are germs of smooth functions in the neighbourhood of $e(U) \subset e(\mathrm{St}(\sigma)) \subset T(\sigma)$ and therefore define local sections of sheaves $\mathscr{A}^{1,0}$ and $\mathscr{A}^{0,1}$. We define differentials d', d'' on all sheaves $\mathscr{A}^{p,q}$ using formulas (2.1). Similar approach applies to maps J and N as well.

Proposition 5.2.3.

i) for all $p \geq 0$ $\mathrm{Ker}\{d'' : \mathscr{A}^{p,0}_X \to \mathscr{A}^{p,1}_X\} \cong \Lambda^p_X \otimes \mathbb{R}$*;*

ii) $\mathrm{Im}\{d'' : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}\} = \mathrm{Ker}\{d'' : \mathscr{A}^{p,q+1} \to \mathscr{A}^{p,q+2}\}$ *for all* $p \geq 0$*;*

Proof. The first statement is straightforward as soon as one observes that d''-closed $(p,0)$-superforms on $\mathrm{St}(\sigma)$ for some face σ can be identified with translation-invariant p-forms on $T(\sigma)$, which in turn can be identified with the sections $H^0(\mathrm{St}(\sigma), \Lambda^p)$.

Further, it follows from Definition 5.2.1 that the second statement reduce to the corresponding statement about superforms on a vector space. Therefore it is true by [Lag12b, Lemma 1.10] or [Jel16, Theorem 2.16]. □

5.3. **The monodromy morphism on superforms.** We will call *local coordinates* at σ any set of affine functions $x_1, \dots, x_n$ on $T(\sigma)$ such that $dx_1, \dots, dx_n$ form a base of a cotangent space of $T(\sigma)$ at any (equivalently, every) point of $T(\sigma)$. Let σ be a face of Δ_X, assume that $\Lambda^\bullet$ is regular at σ and let $x_1, \dots, x_m$ be local coordinates at σ. Then

$$x_0 = 1 - \sum_{i=1}^{m} x_i$$

is an affine function on $T(\sigma)$ which is a restriction of a linear function on $H^0(\mathrm{St}(\sigma), A^1)^*$ and $x_0, \dots, x_m$ is a basis of $H^0(\mathrm{St}(\sigma), A^1)$.

Lemma 5.3.1. *For any $p > 0$ the space $H^0(\overline{\mathrm{St}}(\sigma), A^p \otimes \mathbb{R})$ is isomorphic to the span of sections in $H^0(\mathrm{St}(\sigma), \mathscr{A}^{p-1,0})$ of the form*

$$\sum_{i \in I} \mathrm{sgn}(i, I) x_i d' x_{I \setminus \{i\}}, \qquad x_0 d' x_J,$$

for all multi-indices $I, J, |I| = p, |J| = p - 1$. Moreover, if $a \in H^0(\mathrm{St}(\sigma), A^p)$, $\pi : A^p \otimes \mathbb{R} \to \Lambda^p \otimes \mathbb{R}$ is the natural projection and a is represented by a superform η then $\pi(a)$ is represented by the superform $d'\eta$.

Proof. Recall that the sections in $H^0(\mathrm{St}(\sigma), \Lambda^p \otimes \mathbb{R})$ correspond to translation-invariant differential p-forms on $T(\sigma)$, sections in $H^0(\mathrm{St}(\sigma), A^p \otimes \mathbb{R})$ correspond to translation-invariant p-forms on $H^0(\mathrm{St}(\sigma), A^1 \otimes \mathbb{R})^*$, and the morphism π is induced by the restriction of forms to $T(\sigma)$.

If $(p-1, 0)$-superform η on $T(\sigma)$ is given by a formula from the statement of the Lemma, for some coefficients $a_{I,i}, b_i$, then it can be lifted to a unique $(p-1, 0)$-form $\tilde{\eta}$. Further, $d'\tilde{\eta}$ is a d''-closed $(p, 0)$-superform, which is the same as a translation-invariant p-form, and so gives rise to a section of $A^p \otimes \mathbb{R}$ over $\mathrm{St}(\sigma)$. Clearly this map establishes a bijective correspondence between two natural bases of the subspace $H^0(\mathrm{St}(\sigma), A^1)$ and the space of translation-invariant p-forms on $H^0(\mathrm{St}(\sigma), A^1)^*$, and so is an isomorphism. $\square$

Proposition 5.3.2. *Assume that $\Lambda^\bullet$ is regular at any face $\sigma \subset \Delta_X$. Then for any $p \geq 0$, there exists a distinguished triangle in the derived category of sheaves on Δ_X*

$$\Lambda^p \to A^{p+1} \to \Lambda^{p+1} \xrightarrow{N} \Lambda^p[1]$$

where the last morphism is given by the morphism $N : \mathscr{A}^{p+1,q} \to \mathscr{A}^{p,q+1}$.

Proof. It suffices to show that the 0-th cohomology of the cone complex $\mathrm{Cone}(N)$ is isomorphic to A^p, its cohomology in all other degrees vanishes since this is the case for the source and the destination of the morphism of complexes N.

Let $x_1, \dots, x_m$ be local coordinates at a face σ. If $\beta \in H^0(\mathrm{St}(\sigma), \mathscr{A}^{p,0}_X)$ is a d''-closed superform then it is of the form $\sum_{|I|=p} b_I d' x_I$ for some constants b_I, and $N\beta$ is of the form

$$\sum_{|I|=p-1} \sum_{j=1}^{m} (-1)^p \, \mathrm{sgn}(j, I \cup \{j\}) b_{I \cup \{j\}} d' x_I \wedge d'' x_j.$$

If $\alpha = \sum_{|I|=p-1} f_I d' x_I$ then $d''\alpha = N\beta$ implies

$$\frac{\partial f_I}{\partial x_j} = \mathrm{sgn}(j, I \cup \{j\}) b_{|I| \cup \{j\}}$$

for all I such that $|I| = p-1$ and all $j, 1 \leq j \leq n$. Therefore,

$$f_I = \begin{cases} \sum_{j=1}^m \operatorname{sgn}(j, I \cup \{j\}) b_{I \cup \{j\}} x_j + a_I, & j \notin I, \\ 0, & j \in I. \end{cases},$$

for some constant a_I. To conclude observe that the space of forms $\alpha \in \mathscr{A}^{p-1,0}$ such that there exists a form $\beta \in \mathscr{A}^{p,0}$ such that $d''\alpha = N\beta$ coincides with the space of froms from the statement of Lemma 5.3.1. $\square$

Corollary 5.3.3. *For any $p, q \geq 0$, N is the coboundary morphism in the long exact sequence*

$$0 \to H^q(\Lambda^p) \to H^q(A^{p+1}) \to H^q(\Lambda^p) \xrightarrow{N} H^{q+1}(\Lambda^p) \to$$
$$\to H^{q+1}(A^{p+1}) \to H^{q=1}(\Lambda^{p+1}) \xrightarrow{N} H^{q+2}(\Lambda^p) \to H^{q+2}(A^{p+1}) \to \dots$$

associated to the short exact sequenece (3.1).

6. Λ^p on Kulikov degenerations of K3 surfaces

6.1. **Singular affine structure and sheaf Λ^1.** A *Kulikov degeneration* $f : X \to S$ is a unipotent snc degeneration such that the central fibre is reduced and $K_X = 0$. By a theorem of Kulikov, Persson and Pinkham [Kul77, PP81] any unipotent snc degeneration $f' : X' \to S$ of K3 surfaces can be made a Kulikov degeneration $f : X \to S$ after a bimerorphic modification $X' \dashrightarrow X$ over S. Depending on the unipotency rank of the monodromy, Δ_X is homeomorphic to a point, to an interval, or to a 2-sphere. In the latter case, the case of maximally unipotent monodromy, or *Type III degeneration*, Y_i are rational surfaces, double curves are smooth rational curves, and since Δ_X is a manifold, the double curves on a given component Y_i form a cycle.

An *anticanonical pair* is the data of a smooth projective surface V and a divisor $D \in |-K_V|$ that is a sum of rational curves D_i forming a cycle and intersecting normally. By adjunction formula one easily sees that $(Y_i, \sum Y_{ij})$ are anticanonical pairs. To an anti-canonical pair one can associate a polygon, called its *pseudo-fan*, endowed with a singular affine structure ([GHK15, Section 1.2], [Eng18, Section 3], [AET19, Section 8]).

An *affine structure* on a manifold V is a flat torsion-free connection on its tangent bundle. An *integral affine structure* is the data of an affine structure together with a flat $\mathbb{Z}$-local system $T^{\mathbb{Z}}V \subset TV$ which spans TV. For any point $p \in V$ it is possible to find a local coordinate system $x_1, \dots, x_n$ on a neighbourhood U of p, so that the action of ∇ on differential forms coincides with the de Rham differential. If additionally $T^{\mathbb{Z}}V$ is chosen, one can choose such coordinate system in a way that $\partial/\partial x_1, \dots, \partial/\partial x_n$ generate $T^{\mathbb{Z}}V$. Given two such local charts, a transition function between them belongs to $\mathrm{SL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$. Providing an atlas of charts with such transition functions is equivalent to defining an integral affine structure.

By a *singular affine structure* on a surface we understand an affine structure on a complement of finitely many points. A point is called a singularity of a given singular affine structure if the affine structure cannot be extended to this point.

Definition 6.1.1 (Singular affine structure on a pseudo-fan)**.** A pseudo-fan of an anti-canonical pair (V, D) is a triangulated surface with boundary endowed with a singular integral affine structure on the interior as follows. The triangles are identified with triangles in $\mathbb{R}^2$, spanned by vectors corresponding to the irreducible components of D, all vectors originating in a fixed point o. The gluing is defined by identifying pairs of adjacent triangles, possibly with repetitions (in case D is irrecudible) with pairs of triangles of lattice volume 1, spanned by three vectors g_i, g_j, g_k (corresponding to a chain of irreducible components D_i, D_j, D_k) such that

$$g_i + g_k = d_j \cdot g_j,$$

where $d_j = -D_j^2$ if D_k is a smooth rational curve and $d_j = -D_j^2 + 2$ if D_j is a nodal rational curve.

Consider a Kulikov degeneration $f : X \to S$. If $o \in \Delta_X$, the closed star $\overline{\mathrm{St}}(o)$ can be identified with the pseudo-fan of the anti-canonical pair $(Y_o, \sum_i Y_{oi})$. By [Eng18, Proposition 3.10], the singular integral affine structures on stars of all vertices of Δ_X glue together to singular affine structure on the whole of Δ_X.

We are now going to compare this singular affine structure with the one defined by the sheaf Λ^1 (since sheaves Λ^1 were defined over $\mathbb{Q}$, we will not concern ourselves with integrality in this discussion). In view of Definition 3.3.3 we can pass to a subdivision of Δ_X by blowing up some strata, so that we can ensure that there are no double curves that intersect in more than one point. Since such blow-ups introduce components of multiplicity > 1, we will have to modify the definition of the affine structure on a pseudo-fan to take the multiplicities into account. Note that we do not have to consider the degenerate case when the divisor in an anticanonical pair is a nodal curve, since such pairs do not occur as irreducible components of the central fibre of Kulikov models.

Definition 6.1.2 (Singular affine structure on Δ_X)**.** Let $f : X \to S$ be a degeneration of K3 surfaces obtained from a Kulikov model by a sequence of blow-ups of strata. For any irreducible component Y_o of the central fibre the singular affine structure on $\mathrm{St}(o)$ is defined by specifying affine structure on stars of all 1-dimensional faces containing o as follows. Let $\sigma, \eta \subset X$ be a pair of adjacent triangles, $\underline{\sigma} = \{ o, i, j \}, \underline{\eta} = \{ o, j, k \}$, define affine structure on $\sigma \cup \eta$ by identifying σ and η with a pair of triangles in $\mathbb{R}^2$, of lattice volume $1/N_\sigma, 1/N_\eta$ respectively, spanned by three vectors g_i, g_j, g_k such that

$$N_i g_i + N_k g_k = d_{oj} \cdot N_j g_j,$$

where $d_{oj} = -Y_{oj}^2$ on Y_o.

Lemma 6.1.3. *For any pair of adjacent triangles σ, η as above*

$$N_o d_{jo} + N_j d_{oj} = N_i + N_k.$$

Proof. Follows immediately from writing out the restriction of $\sum N_i Y_i$, which is a principal divisor, to Y_{oj}. □

Proposition 6.1.4. *Definition 6.1.2 is consistent with respect to subdivisions of Δ_X induced by blow-ups of strata.*

Proof. Let g_i, g_j, g_k be vectors corresponding to double curves Y_{oi}, Y_{oj}, Y_{ok}. We need to consider the following modifications that will affect the subdivision of Δ_X:

i) blow up of any of Y_{oi}, Y_{oj}, Y_{ok};
ii) blow up of any of Y_{ij}, Y_{jk};
iii) blow up of any of the triple points Y_{oij}, Y_{ojk}.

In all cases, denote Y_e the exceptional divisor.

Case (i). Let Y_{ok} be the center of the blow-up for definiteness. Then $N_e = N_o + N_k, g_e = N_k/N_e g_k$ spans a triangle of volume $N_\eta \cdot N_k/N_e = 1/(N_o N_j N_e)$ with g_j and

$$N_j d_{oj} g_j = N_i g_i + N_k g_k = N_i g_i + N_e g_e,$$

implying that the singular affine structure on the union of σ with the triangle oje is the restriction of the singular affine structure on the union of σ and η. The case when Y_{oi} is the center of the blow-up is treated similarly.

Let Y_{oj} be the center of the blow-up. Then $N_e = N_o + N_j, g_e = N_j/N_e g_j$ spans triangles of volume $1/(N_o N_i N_e)$ and $1/(N_o N_k N_e)$ with g_i and g_k, respectvely. We have $d_{oe} = d_{oj}$ and

$$N_e d_{oe} g_e = N_j d_{oj} g_j = N_i g_i + N_k g_k.$$

Let us now check that the affine structure near e is the restriction of the affine structure on $\sigma \cup \eta$ before subdivision. The exceptional divisor Y_e is a projective bundle over Y_{oj}, the projectivisation of $\mathcal{O}_{Y_{oj}}(d_{oj}) \oplus \mathcal{O}_{Y_{oj}}(d_{jo}))$, and Y_{ei}, Y_{ek} are its fibres, so $d_{ei} = d_{ek} = 0$. The curves Y_{oe}, Y_{ej} are its sections and we have $d_{eo} = -d_{ej} = d_{jo} - d_{oj}$.

We have the following vectors originating in e:

$$h_i = g_i - g_e, h_j = g_j - g_e, h_k = g_k - g_e, h_o = -g_e$$

Clearly,

$$N_o h_o + N_j h_j = -N_o g_e + N_j g_j - N_j g_e = N_j g_j - N_e g_e = 0 = N_i \cdot 0 \cdot h_i = N_k \cdot 0 \cdot h_k,$$

so the affine structure on the subdivided σ as per Definition 6.1.2 coincides with the flat affine structure on σ before subdivision, and similarly for η.

To check that the subdivision does not affect the affine structure on the adjacent halves of triangles σ, τ, apply Lemma 6.1.3 and the identity for $N_e d_{oe} g_e$ already established, so we get

$$\begin{aligned} N_i h_i + N_k h_k = N_i g_i + N_k g_k - (N_i + N_k) g_e = N_e d_{oe} g_e - (N_o d_{jo} + N_j d_{oj}) g_e = \\ = (N_o d_{oj} + N_j d_{oj} - N_o d_{jo} - N_j d_{oj}) g_e = -N_o d_{eo} g_e = N_o d_{eo} h_o, \end{aligned}$$

and analogously,

$$\begin{aligned} N_i h_i + N_k h_k = N_i g_i + N_k g_k - (N_i + N_k) g_e = N_j d_{oj} g_j - (N_e d_{je} + N_j d_{ej}) g_e = \\ = N_j d_{oj} g_j - N_j d_{jo} g_j - N_j d_{ej} g_e = N_j d_{ej}(g_j - g_e) = N_j d_{ej} h_j. \end{aligned}$$

Case (ii). Let Y_{jk} be the center of the blow-up for definiteness, and let Y_e be the exceptional divisor. The strict transform Y'_o of Y_o is a blow up of Y_o in the triple point Y_{ojk} with the exceptional curve Y_{oe}. Clearly, $d_{oe} = 1$, and $N_e = N_j + N_k$. Denote d'_{oj}, d'_{ok} the negative self-intersection numbers of the strict transforms of the double curves Y_{oj}, Y_{ok}. Clearly, $d'_{oj} = d_{oj} + 1, d'_{ok} = d_{ok} + 1$. Then putting $g_e = (N_j g_j + N_k g_k)/N_e$, we get

$$N_j g_j + N_k g_k = N_e g_e = N_e d_{oe} g_e,$$

and

$$N_i g_i + N_e g_e = N_i g_i + N_j g_j + N_k g_k = N_j(d_{oj} + 1) g_j = N_j d'_{oj} g_j,$$

and observe that the subdivision does not affect the affine structure.

Case (iii). Let Y_{ojk} be the center of the blow-up for definiteness. Denote as before Y_e the exceptional divisor and d'_{oj}, d'_{ok} the negative self-intersection numbers of the strict transforms of the corresponding double curves. Clearly, $N_e = N_o + N_j + N_k$, $d'_{oj} = d_{oj} + 1$, and Y_{eo}, Y_{ej}, Y_{ek} is a triangle of lines on $Y_e \cong \mathbb{P}^2$.

Let $g_e = (N_j g_j + N_k g_k)/N_e$. Then similarly to the previous case,

$$N_j g_j + N_k g_k = N_e g_e = N_e d_{oe} g_e,$$

and

$$N_i g_i + N_e g_e = N_i g_i + N_j g_j + N_k g_k = N_j d'_{oj} g_j.$$

Let $h_o = -g_e, h_j = g_j - g_e, h_k = g_k - g_e$, then the fact that the affine structure on the subdivided triangle ojk is the same as the affine structure on it before subdivision folows from the equality

$$N_o h_o + N_j h_j + N_k h_k = 0$$

and the fact that double curves on Y_e have self-intersection 1. □

Consider an anticanonical pair (V, D), then a blow up at a point $p \in D$ gives rise to an anticanonical pair (V', D') where D' is the log pull-back of D. Denote D'_i the strict transforms of irreducible components D_i of D and let D_e be the exceptional divisor; denote d_k, d'_k the self-intersection numbers of the components on V, V', respectively. If p is a smooth point of D_j then $D' = \sum D'_i$ and $d'_i = d_i, i \neq j$ and $d'_j = d_j - 1$ and (V', D') is called an *internal blow-up*. If p is a node, then the exceptional disivor D_e is a summand of D', $d'_e = 1$ and $d'_i = d_i + 1$ for two components D_i that contain p. In this case (V', D') is called a *corner blow-up*.

By Proposition 2.12[Fri15], for any anti-canonical pair (V, D) there exists a sequence of corner blow-ups resulting in a pair (V', D') and a toric variety $\bar{V}$ with toric boundary $\bar{D}$ such that (V', D') is obtained from $(\bar{V}, \bar{D})$ by a sequence of internal blow-ups. Notice that both Definition 6.1.1 and Definition 6.1.2 define a singular affine structure on $\mathbb{R}^2$, which can be regarded as the cone over the boundary of a pseudo-fan of a toric model or of $\overline{\mathrm{St}}(i)$.

Proposition 6.1.5. *If $f : X \to S$ is a Kulikov degeneration of K3 surfaces, then the sheaf Λ^1 on Δ_X coincides with the sheaf of affine functions with rational coefficients with respect to the singular affine structure given by Definition 6.1.2.*

Proof. If α is a face of dimension 2 then by definition of the sheaf Λ^1, $\Lambda^1(\sigma)$ coincides with the space of affine functions on the interior of σ.

Consider two trianges σ and η which share an edge α. Let $\underline{\sigma} = \{\, o, i, j \,\}, \underline{\beta} = \{\, o, j, k \,\}$. Then

$$\Lambda^1(\sigma) = \{\ f \in \bar{A}^1(\alpha) \ \mid\ f(o) = 0 \text{ and } N_i f(i) Y_i + N_j f(j) Y_j + N_k f(k) Y_k \sim_{Y_\alpha} 0 \ \}$$

The condition on f can be rewriten as

$$N_i f(i) Y_i.Y_j + N_k f(k) Y_k.Y_j = N_i f(i) + N_k f(k) = N_j(-Y_j^2) f(j)$$

which coincides with the condition imposed on f to be affine in a neighbourhood of α with respect to the affine structure defined in Definition 6.1.2.

Let o be a vertex in Δ_X. We perform enough blow-ups of strata so that Δ_X is a simplicial complex and so that the anti-canonical pair $(Y_o, \sum Y_{oi})$ can be obtained from a toric anti-canonical pair $(\bar{Y}_o, \sum \bar{Y}_{oi})$ by a sequence of internal blow-ups. We have

$$\Lambda^1(o) = \{\ f \in \bar{A}^1(o) \ \mid\ f(o) = 0 \text{ and } \sum N_i f(i)\, \mathcal{O}(Y_i)|_{Y_o} \sim_{Y_o} 0 \ \},$$

and we need to check that $\Lambda^1(o)$ consists of affine functions with rational coefficients (with respect to the affine structure from Definition 6.1.2) that vanish at o.

If $(Y_o, \sum Y_{oi})$ is toric, this statement is true by [Eng18, Proposition 3.9] and Corollary 3.4.2. The star of i is identified with a polygon in $\mathbb{R}^2$ with its standard integral affine structure, spanned by vectors $e_j, j \in \overline{\mathrm{St}}^0(i)$ and the elements of $\bar{\Lambda}^1(i)$ are identified with the functions that are piece-wise linear on the triangles that belong to $\overline{\mathrm{St}}(i)$. Such functions are completely determined by their values in points $j \in \overline{\mathrm{St}}^0(j)$.

Let us analize how an internal blow-up affects the affine structure on the pseudo-fan of a toric canonical pair. A sequence of blow-ups of smooth points Y_{ij} introduces a shearing transformation (see [Eng18, Section 3]) with the effect that functions in a neighbourhood of i that are affine with respect to the resulting singular affine structure must be constant along the vector e_j. If one performs internal blow-up on two components Y_{ij}, Y_{ik} such that e_j, e_k are not collinear, then there are no non-constant affine functions.

The general case follows from the following Claim.

Claim. Let $h : (Y', D') \to (Y, D)$ be an internal blow-up of a smooth point $p \in D_i$, let I be the set of irreducible components of D (and D') and let $\{ N_i \}_{i \in I}$ be a set of integers. Define the following sets

$$\mathrm{Aff}_{Y'} = \{ f : I \to \mathbb{Q} \mid \sum N_i f(i) D'_i \sim_{Y'} 0 \}, \mathrm{Aff}_Y = \{ f : I \to \mathbb{Q} \mid \sum N_i f(i) D_i \sim_Y 0 \},$$

which we will also interpret as sets of functions on $\mathbb{R}^2$ that are linear on the cones spanned by $e_1, \ldots, e_j$. Then

$$f \in \mathrm{Aff}_{Y'} \text{ if and only if } f \in \mathrm{Aff}_Y, f(i) = 0.$$

Indeed, since

$$\sum N_i f(i) D'_i = h^*(\sum N_i f(i) D_i)) - N_i E$$

and E is linearly independent from the image of $\mathrm{Pic}(Y)$ in $\mathrm{Pic}(Y)$, the above expression can vanish if and only if both summands vanish.

□

We will now show that the conditions of Theorem 5.1.6 are satisfied at least for some Type III Kulikov models.

Proposition 6.1.6. *Let Y be an snc surface such that its irreducible components are rational. Then a combinatorial class $\omega \in H^2(Y)$ is combinatorial Lefschetz if and only if $(\omega|_{Y_i})^2 > 0$ and $\omega|_{Y_\sigma} \neq 0$ for all irreducible components Y_i and double curves Y_σ.*

Proof. Necessity is immediate.

The class ω satisfying the conditions in the statement of the proposition clearly restricts to a Lefschetz class on double curves, so we only need to check whether its restrictions to Y_i are Lefschetz. A rational surface has cohomology classes of type $(0,0), (1,1), (2,2)$ only, therefore any class with non-zero square satisfies the Lefschetz property. The Hodge-Riemann bilinear relations hold immediately on $H^{0,0}(Y_i)$ and $H^{2,2}(Y_i)$ by the positivity of ω^2, and they hold on $H^{1,1}$ since the intersection form is negative definite on $\mathrm{Ker}\, L_\omega \subset H^2(Y_i)$ by Hodge index theorem. □

By the triple point formula [Kul77, 2.1], for all double curves in a Type III Kulikov model we have

$$(Y_{ij})^2|_{Y_i} + (Y_{ij})^2|_{Y_i} = -2.$$

Proposition 6.1.7. *Let $f : X \to S$ be a Type III Kulikov model such that all double curves have square -1. Then there exists a combinatorial Lefschetz class on $Y = f^{-1}(0)$.*

Proof. Let $\omega_i = \sum_{j \in \overline{\mathrm{St}}^0(i)} c_1(\mathcal{O}_{Y_i}(Y_{ij}))$. Then $\omega_i|_{Y_{ij}} = \omega_j|_{Y_{ij}}$ since

$$\omega_i . Y_{ij} = 2 + Y_{ij}^2 = \omega_j . Y_{ij},$$

so there exists a class $\omega \in H^2(Y)$ such that $\omega|_{Y_i} = \omega_i$. Clearly, this class is combinatorial and

$$\omega_i^2 = \sum_{\sigma : i \in \sigma} 2 + \sum_{j \in \overline{\mathrm{St}}^0(i)} -1 > 0 \text{ and } \omega_i|_{Y_\sigma} \neq 0,$$

so it is combinatorial Lefschetz by Proposition 6.1.6. □

6.2. **Positive** $(1,1)$**-superforms.** One can observe that the data of a $(1,1)$-superform $\omega \in \mathscr{A}^{1,1}(O), O \subset V$ such that $N\omega = 0$ is equivalent to that of a (pseudo-)metric tensor. We call ω positive if the symmetric 2-form it defines on a tangent space T_xV is *positive* for all points $x \in O$. A section $\omega \in H^0(\Delta_X, \mathscr{A}^{1,1})$ is called *positive* if its restriction to each $\overline{\mathrm{St}}(\sigma)$ gives rise to a positive (1,1)-superform on $\overline{\mathrm{St}}(\sigma) \subset T\Delta_X\sigma$.

Proposition 6.2.1. *For any $p \geq 0$ there exists a quasi-isomorphism from the complex $(H^0(\Delta_X, \mathscr{A}_X^{p,\bullet}), d'')$ to the complex of singular chains $C^\bullet(\Delta_X, \Lambda^p \otimes \mathbb{R})$.*

Proof. Let ω be a p,q-superform defined in a neighbourhood of a face $\sigma \subset \Delta_X$. By definition, ω is in a neighbourhood of σ by a germ $\tilde{\omega}_\sigma$ of a p,q-form in a neighbourhood of $e(\overline{\mathrm{St}}(\sigma)) \subset T_\sigma\Delta_X$. The latter can also be regarded as a Λ^p-valued q-form, which we will denote $\bar{\omega}_\sigma$. Define

$$I_p(\omega) = \sum_{|\underline{\sigma}|=p+1} \int_\sigma \bar{\omega}_\sigma$$

This map defines a morphism of complexes by the Stokes theorem for superforms:

$$I_p(d''\omega) = \sum_{|\underline{\tau}|=p+2} \int_\tau d\bar{\omega}_\sigma = \sum_{|\underline{\tau}|=p+2} \int_{\partial\tau} d\bar{\omega}_\sigma = \sum_{|\underline{\tau}|=p+2} \sum_{\sigma:\underline{\sigma}=\underline{\tau}\setminus\{i\}} \mathrm{sgn}(u,\underline{\tau}) \int_\sigma \omega_\sigma$$

Since this map defines a quasi-isomorphism between two resolutions of the sheaf Λ^p, it induces an isomorphism $H^q(\mathscr{A}^{p,\bullet}(\Delta_X), d'')) \to H^q(C^\bullet(\Delta_X, \Lambda_X^p \otimes \mathbb{R}))$ for any $q \geq 0$. $\square$

Recall that a function $f : U \to \mathbb{R}$ on a convex domain $U \subset \mathbb{R}^n$ is convex is it is continuous and all sublevel sets $U_c := \{\ x \in U \ \mid \ f(x) \leq c\ \}$ are convex. It is *strictly convex* if U_c are strictly convex, that is, if every line segment that lies in U_c is contained in the interior of U_c except maybe its endpoints. Let $\Sigma \subset \mathbb{R}^n$ be a polyhedral compelx, then a PL function on Σ is called *strictly conex* if it is convex and additionally its restriction to a neighbourhood of any polyhedron in Σ is not linear.

This definition makes on Δ_X when X is a Kulikov degeneration. In this case we call a function on $\mathrm{St}(i)$ strictly convex if its restrictions to $\mathrm{St}(\sigma)$ are convex for all σ containing i — where $\mathrm{St}(\sigma)$ is identified with a pair of triangles in $\mathbb{R}^2$ using the affine structure on Δ_X.

Proposition 6.2.2. *Let $X \to S$ be a Kulikov degeneration. If ω is a symmetric d''-closed superform on Δ_X, let (a_σ) be a cocycle that corresponds to ω under the quasi-isomorphism from Proposition 6.2.1 and let L be the corresponding virtual line bundle. Assume that away from the singularities of affine structure ω is positive definite, then there exists a PL convex metrization of L.*

Proof. A PL metrization of L is a collection of sections $h_i \in H^0(\mathrm{St}(i), \bar{\Lambda}^1)$ such that $h_i - h_j = a_\sigma$ for all one-dimensional faces $\sigma \subset \Delta_X, \underline{\sigma} = i, j$. We will regard h_i as PL functions vanishing at 0 and linear on faces contaning i.

For each vertex i take a function $f_i \in \mathscr{A}^{0,0}(\mathrm{St}(i))$ such that $d'd''f_i = \alpha|_{\mathrm{St}(i)}$ and

$$\frac{\partial f_i}{\partial x_1}(i) = \frac{\partial f_i}{\partial x_2}(i) = 0,$$

where x_1, x_2 is some coordinate system at i. To define h_i, suffices to define them on vertices that belong to $\overline{\mathrm{St}}(i)$:

$$h_i(j) = f_i(j).$$

Since $d'd''f_i = d'd''f_j = 0$, the functions f_i and f_j have the same Hessian matrix on $\overline{\mathrm{St}}(i) \cap \overline{\mathrm{St}}(j)$ and we have that

$$f_i - f_j = \frac{\partial f_i}{\partial x_1} x_1 + \frac{\partial f_i}{\partial x_2} x_2 + c$$

for some constant $c \in \mathbb{R}$. But the linear part of the difference is precisely the value of the cocycle a_{ij}. □

Corollary 6.2.3. *Let T be a positive supercurrent on Δ_X. Then there exists a virtual line bundle L on Δ_X and a PL convex metrization h.*

Proof. By [Lag11, Theorem] there exist non necessarily smooth functions f_i such that $d'd''f_i = T$ on $\overline{\mathrm{St}}(i)$ in the sense of currents. Pick some such functions and define, as in the proof of Proposition 6.2.2,

$$h_i(j) = f_i(j)$$

It is then clear from the definition that

$$a_{ij} = [h_j - h_i]$$

forms a cocyle in $H^1(\Delta_X, \Lambda^1)$ that gives rise to a virtual line bundle L with PL metrisation (h_i). □

Proposition 6.2.4. *Let $f : X \to S$ be a Kulikov degeneration, let L be a virtual line bundle on Δ_X and h be a PL metric on L. If h is strictly convex then $c_1(L,h)_i \in H^2(Y_i)$ is combinatorial Lefschetz.*

Proof. Let $h_i \in H^0(\overline{\mathrm{St}}(i), \bar{\Lambda}^1)$ be a trivialization of h in $\overline{\mathrm{St}}(i)$. As before, we will identify h_i with piece-wise linear functions such that $h_i(i) = 0$. We need to show that the class

$$c_1(L,h)_i = \sum_{j \in \overline{\mathrm{St}}^0(i)} N_i h(i) c_1(\mathcal{O}_{Y_j}(Y_{ij})) \in H^2(Y_i)$$

has a positive square.

$$c_1(L,h)_i^2 = \sum_{o \in \overline{\mathrm{St}}^0(i)} N_j^2 h(j)^2 Y_{ij}^2 + 2 \sum_{\substack{\exists \sigma \subset \overline{\mathrm{St}}(i) \\ \underline{\sigma} = \{i,j,k\}}} N_j N_k h(j) h(k).$$

Since $c_1(L,h)_i$ does not change when a linear function is added to h_i, we may assume that h_i strictly positive on all vertices of $\overline{\mathrm{St}}^0(i)$. Since h_i is strictly convex, from the definition of affine structure on Δ_X for any triple of adjacent vertices $j,k,l \in \overline{\mathrm{St}}^0(i)$ we have

$$N_j h(j) + N_k(Y_{ik}^2) \cdot h(k) + N_l h(l) > 0,$$

where Y_{ik}^2 is the self-intersection of the curve Y_{ik} on Y_i, and hence

$$N_k N_j h(k) h(j) + N_k^2 (Y_{ik}^2) \cdot (h(k))^2 + N_k N_l h(k) h(l) > 0,$$

summing up these expressions for all triples j,k,l of adjacent vertices in $\overline{\mathrm{St}}^0(i)$ we obtain the expression for $c_1(L,h)_i$, and therefore $c_1(L,h)_i > 0$. □

6.3. **Simple affine structure singularities.** Let $X \to S$ be a Kulikov degeneration and let Y_i be an irreducible component of the central fibre. Take an anticanonical pair

$$(Y_i, D), D = \sum_{\exists \sigma : \underline{\sigma} = \{ i,j \}} Y_j \cap Y_i,$$

and assume that it is obtained from a toric anticanonical pair $(\bar{Y}_i, \bar{D})$ by a single blow-up of a smooth point $p \in Y_{ij} \subset \bar{D}$. Applying Definition 6.1.2 one observes that the monodromy matrix of the affine structure around i is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in the basis consisting of vectors ij, ik. This is the simplest affine structure singularity.

Now assume that all non-toric irreducible components Y_i become toric after blowing down on every one of them of a single exceptional curve C_i that intersects the double locus of Y in one point. Let X' be the blow up in X of all such curves C_i on non-toric irreducible components Y_i of the central fibre Y. The morphisms

$$H^q(\Delta_{X'}, \Lambda^p_{X'}) \cong \operatorname{gr}^W_{2p} H^{p+q}(X'_\infty)$$

are isomorphisms since ${}'H^{2\bullet}(Y'_\sigma) = H^{2\bullet}(Y'_\sigma)$ for any stratum Y'_σ of Y. We will show in this section that the cohomology of $\Lambda^0, \Lambda^1, \Lambda^2$ did not change when passing to $\Delta_{X'}$. In the paper [Sus22] I show that Δ_X admits a Kähler superform and therefore by Propositions 6.2.2 and 6.2.4 the morphisms

$$H^q(\Delta_X, \Lambda^1_X) \cong \operatorname{gr}^W_{2p} H^{p+q}(X_\infty)$$

are isomorphisms too. This is parallel to the results of Ruddat [Rud10] that show that affine cohomology recovers full nearby fibre cohomology in case of toric degenerations with simple singularities.

Proposition 6.3.1. *The sheaf $\Lambda^\bullet_{X'}$ is regular at any face $\sigma \subset \Delta_{X'}$ and there exist a surjective morphism*

$$H^1(\Delta_X, \Lambda^1_X) \to H^1(\Delta_{X'}, \Lambda^1_{X'}).$$

and an injective morphism

$$H^0(\Delta_X, \Lambda^1_X) \hookrightarrow H^0(\Delta_{X'}, \Lambda^1_{X'}).$$

For $p \in \{ 0, 2 \}$ and any q there exist isomorphisms

$$H^q(\Delta_X, \Lambda^p_X) \cong H^q(\Delta_{X'}, \Lambda^p_{X'}).$$

Proof. Since the modification $X' \to X$ is performed in a similar manner in a neighbourhood of each non-toric component Y, so below we will concentrate on the situation and fix notation for the strata near one such component, Y_i.

Let Y_j be the irreducible component of Y that contains the double curve Y_{ij} that C_i intersects, and let Y_k and Y_l be the irreducible components of Y that intersect Y_{ij} in triple points.

We may assume that Y_j is toric and that the double curves on it are irreducible components of the toric boundary divisor, otherwise, reduce to this situation by doing a base change with respect to a finite cover $S \to S$ of sufficiently high degrre and resolving the arising singularities.

Denote Y' the central fibre of X' and Y'_i the strict transforms of the strata of Y, and denote Y'_e the exceptional divisor, whose multiplicity is $N_e = N_i + N_j$. Note that $Y'_i \cong Y_i$ and that Y'_j is a blow-up of Y_j in the triple point $C \cap Y_{ij}$. The normal bundle of C in X is a direct sum of the normal bundle of C in Y_i, which is $\mathcal{O}_C(-1)$, and the restriction of the normal bundle of Y_i in X to C which is equal to

$$\mathcal{O}_{Y_i}(-\sum_{o\in\overline{\mathrm{St}}^0(i)\setminus\{i\}} N_oY_o)|_C = \mathcal{O}_C(-1).$$

It follows that $Y'_e \cong \mathbb{P}^1\times\mathbb{P}^1$.

Since the stars of vertices k and l and the corresopnding stata do not change when passing from Δ_X to $\Delta_{X'}$, the sheaves $\Lambda^i, i=1,2$ also do not change near these vertices, and we only need to compute the sections of these sheaves near faces i,j,e,ij,j,ie,je.

We have

$$\mathcal{O}_{X'}(Y'_e)|_{Y'_{ie}} = \mathcal{O}_{Y'_{ie}}(-1),\quad \mathcal{O}_{X'}(Y'_i)|_{Y'_{ie}} = \mathcal{O}_{Y_{ie}}(-1),\quad \mathcal{O}_{X'}(Y'_j)|_{Y'_{ie}} = \mathcal{O}_{Y'_{ie}}(1),$$
$$\mathcal{O}_{X'}(Y'_e)|_{Y'_{je}} = 0,\quad \mathcal{O}_{X'}(Y'_i)|_{Y'_{je}} = \mathcal{O}_{Y'_{je}}(1),\quad \mathcal{O}_{X'}(Y'_j)|_{Y'_{je}} = \mathcal{O}_{Y'_{ie}}(-1).$$

From the considerations above one gets immediately that

$$\Lambda^1_{X'}(e) = 0, \qquad \Lambda^1(ie) = \Lambda^1(je) = \{\ f:\{\,i,j,e\,\}\to\mathbb{Q}\ \mid\ f(i)=f(j)\ \}/\,\mathrm{const}\,.$$

Any element of $\bar\Lambda^2(ie)$ or $\bar\Lambda^2(je)$ is proportional to $i\wedge j$ but

$$c^2_{ie}(i\wedge j) = c^1_{ie}(j)\otimes i \qquad c^2_{je}(i\wedge j) = c^1_{ie}(i)\otimes j$$

and both expressions are non-zero. Therefore,

$$\Lambda^2(e) = \Lambda^2(ie) = \Lambda^2(je) = 0.$$

First of all, we have

$$\mathcal{O}_{X'}(Y'_k)|_{Y'_{ij}} = \mathcal{O}_{X'}(Y_k)|_{Y_{ij}} = \mathcal{O}_{Y_{ij}}(1), \qquad \mathcal{O}_{X'}(Y'_l)|_{Y'_{ij}} = \mathcal{O}_X(Y_l)|_{Y_{ij}} = \mathcal{O}_{Y_{ij}}(1).$$

Since Y'_j is a blow-up of Y_j, $Y'_i = Y_i$ and $Y'_{ij} = Y_{ij}$, we have

$$\mathcal{O}_{X'}(Y'_i)|_{Y_{ij}} = \mathcal{O}_{Y'_{ij}}((Y'_{ij})|^2_{Y'_i}) = \mathcal{O}_{Y_{ij}}((Y_{ij})|^2_{Y_i}) = \mathcal{O}_X(Y_i)|_{Y_{ij}},$$
$$\mathcal{O}_{X'}(Y'_j)|_{Y_{ij}} = \mathcal{O}_{Y'_{ij}}((Y'_{ij})|^2_{Y'_j}) = \mathcal{O}_{Y_{ij}}((Y_{ij})|^2_{Y_j}-1) = \mathcal{O}_X(Y_j)|_{Y_{ij}}\otimes\mathcal{O}_{Y_{ij}}(-1).$$

If we identify elements of $\Lambda^1_X(ij)$, resp. $\Lambda^1_{X'}(ij)$, with maps f, resp. f', on finite sets $\{\,i,j,k,l\,\}$, resp. $\{\,i,j,k,l,e\,\}$, up to constant maps then we see that there is a natural inclusion

$$\iota:\Lambda^1_X(ij)\hookrightarrow\Lambda^1_{X'}(ij),\quad [f]\mapsto[f'],\ \text{where } f'(e)=f(i)+f(j),\ f'|_{\{\,i,j,k,l\,\}} = f|_{\{\,i,j,k,l\,\}}.$$

The sections of $\Lambda^1(ij)$ are of the form $f'+g$ where $f\in\Lambda^1_X(ij)$ and g is a function supported on $\{\,i,j,e\,\}$ such that

$$(g(e)\,\mathcal{O}_{X'}(Y'_e) + g(i)\,\mathcal{O}_{X'}(Y'_i) + g(j)\,\mathcal{O}_{X'}(Y'_j))|_{Y'_{ij}} = 0$$

To compute $\Lambda^2(ij)$ note that any element of $\bar\Lambda^2(ij)$ is represented by a tensor of the form $i\wedge a$ where a is a linear combination of e,k,l and that

$$c^2_{ij}(i\wedge a) = c^1_{ij}(a)\otimes i.$$

It follows that

$$\Lambda^2_{X'}(ij) = \{\ i\wedge a\ \mid\ a|_{ij}=0, a\in\Lambda^1_{X'}(ij)\ \}.$$

In particular, $\dim\Lambda^2_{X'}(ij) = 1$.

From the computations of $\Lambda^2_{X'}(ij)$ we conclude that Λ^2_X is a constant sheaf in the neighbourhood of i and j.

Summing up, the sections of the sheaf $\Lambda^1_{X'}$ near vertices i and j are sums of pull-backs of sections on $\Delta_X\subset\Delta_{X'}$ that are affine with respect to a certain non-singular affine structure under a certain natural projection map $\Delta_{X'} \to \Delta_X$ that collapses the triangle ije onto the its edge ij, and sections supported outside Δ_X. The sheaf $\Lambda^2_{X'}$ is a push-forward of the constant sheaf along the open embedding of the complement of the boundary of the space $\Delta_{X'}$ into $\Delta_{X'}$. It is also clear from the computations that $\Lambda^\bullet$ is regular at every face of $\Delta_{X'}$.

Since Δ_X is homotopy equivalent to $\Delta_{X'}$, there exist isomorophisms $H^\bullet(\Delta_X, \Lambda^p_X) \cong H^\bullet(\Delta_{X'}, \Lambda^p_{X'})$ for $p = 0, 2$. Is is left to treat the Λ^1 case.

Consider an inclusion $C^\bullet(\Delta_X, \Lambda^1) \hookrightarrow C^\bullet(\Delta_{X'}, \Lambda^1_{X'})$ of complexes

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{u\in\Delta_X} \Lambda^1(u) \arrow[r, "d"] \arrow[d, "\iota"] & \bigoplus\limits_{\substack{|\sigma|=2\\ \sigma\subset\Delta_X}} \Lambda^1(\sigma) \arrow[r, "d"] \arrow[d, "\iota"] & \bigoplus\limits_{\substack{|\tau|=3\\ \tau\subset\Delta_X}} \Lambda^1(\tau) \arrow[r] \arrow[d, "\iota"] & 0 \\
0 \arrow[r] & \bigoplus\limits_{v\in\Delta_{X'}} \Lambda^1(v) \arrow[r, "d"] & \bigoplus\limits_{\substack{|\alpha|=2\\ \alpha\subset\Delta_{X'}}} \Lambda^1(\alpha) \arrow[r, "d"] & \bigoplus\limits_{\substack{|\eta|=3\\ \eta\subset\Delta_{X'}}} \Lambda^1(\eta) \arrow[r] & 0
\end{tikzcd}$$

Since there are no coboundaries in degree 0 the first map is an inclusion.

For the last statement we only need to analyze the cocycles with coefficients in Λ^1_X and $\Lambda^1_{X'}$ near the glued in triangles, since away from them the sheaves are isomorphic. We keep the notation for vertices and strata from the computations above.

Let $(a_\alpha) \in \oplus_{|\alpha|=2}\Lambda^1_X(\tau)$ be a cocycle. Since a cocycle must satisfy in particular

$$d(a_\alpha)|_{\{i,j,e\}} = 0$$

but

$$\Lambda^1(ij)|_{\{i,j,e\}} \cap \Lambda^1(ie) = \Lambda^1(ij)|_{\{i,j,e\}} \cap \Lambda^1(je) = 0,$$

we have then $a_{ij}|_{\{i,j,e\}} = 0$. In particular, $(a_\alpha) \in \operatorname{Im}\iota$.

□

Note that since $H^0(\Delta_{X'}, \Lambda^1_{X'}) \cong \operatorname{gr}^W_2 H^1(X'_\infty) = 0$, $\iota : H^0(\Delta_X, \Lambda^1_X) \to H^0(\Delta_{X'}, \Lambda^1_{X'})$ is an isomorphism.

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