Gromov-Hausdorff limits of flat Riemannian surfaces

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Abstract

I study Gromov-Hausdorff limits of complex curves endowed with singular flat metrics of constant diameter. I formulate a criterion that the limit is collapsed in terms of a certain piecewise affine weight function on the dual intersection complex of a semi-stable model of the degeneration introduced by Kontsevich and Soibelman. I describe the collapsed and non-collapsed limits, which are, respectively, metric graphs and finite collections of complex curves with flat metrics glued along finitely many points. I show that the collapsed limit of any genus can occur.

Contents

1 Introduction

In the paper [KS06] Kontsevich and Soibelman formulate a series of conjectures about the shape Gromov-Hausdorff limits of certain families of complex manifolds endowed with Ricci flat metrics. These conjectures are motivated by mirror symmetry and in particular by the authors' approach to the SYZ conjecture. One considers germs of holomorphic families of compact Calabi-Yau manifolds parametrized by points of a punctured disc having maximally unipotent action of the monodromy on the middle cohomology and with a relatively ample line bundle on the total space of the family. For each element of the family one picks the Ricci-flat metric with the Kähler class equal to the Chern class of the polarizing line bundle and normalized so that the diameter is constant. Gromov-Hausdorff limits of such families are then conjectured by Kontsevich and Soibelman to carry a singular affine

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manifold structure with respect to which the limit metric satisfies the real Monge-Ampere equation. The real dimension of the limit manifold is half the real dimension of the elements of the family, so we speak about *collapsed* limits.

Alternatively, the limit manifold together with the singular affine structure can be recovered from the non-archimedean analytification X^{an} (in the sense of Berkovich) of the variety X over the non-archimedean field of germs of complex functionos meromorphic at 0 (Conjecture 3, §5, [KS06]). As a topological space it is a closed subset of X^{an}, the minimality locus of a certain weight function associated to a canonical form; this closed subset is called the essential skeleton $\mathrm{Sk}(X^{\mathrm{an}})$ of X^{an}. On a variety with trivial canonical bundle weight functions associated to different canonical forms differ by a constant and so the minimality locus does not depend on this choice (though the singular affine structure does). There exists a retraction $X^{\mathrm{an}} \to \mathrm{Sk}(X^{\mathrm{an}})$, which is a fibration over an open dense subset of $\mathrm{Sk}(X^{\mathrm{an}})$, with the fibre isomorphic to a non-archimedean torus; this endows $\mathrm{Sk}(X^{\mathrm{an}}$ with an integral affine structure away from the discriminant locus.

Collapsed Gromov-Hausdorff limits of Ricci-flat hyperkähler manifolds have been extensively studied in a slightly different setting: one fixes a holomorhic fibration of a single hyperkähler manifold, with a generic fibre Abelian variety, and considers a variation of the Kähler class, making it tend to the boundary of the Kähler cone [GW00], [GTZ13], [GTZ16], [TZ17]. In the recent paper [OO18] Odaka and Oshima reduce the computation of the limits of K3 surfaces in the set up described in the previous paragraph to degeneration of the Kähler class using Kähler rotation and careful study of the moduli of K3 surfaces. Let us also mention the related work of Boucksom and Jonsson [BJ16] on asymptotic behaviour of volume forms on degenerations of Calabi-Yau manifolds.

In this paper we consider the Gromov-Hausdorff limits of families complex curves of genus ≥ 1 endowed with flat pseudo-Kähler metrics, relaxing the assumption on the triviality of the tangent bundle and allowing the metric to have conical singularities. Following Kontsevich and Soibelman, we rescale the metrics with the Kähler form $\dfrac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$ (where Ω is a given relative 1-form) on the fibres X_s of the family X so that $\operatorname{diam} X_s \equiv 1$.

There are two possibilites for the limit. In the collapsed case the limit is a metric graph which can be canonically represented as a certain quotient of the dual intersection complex of the special fibre of a semi-stable model of the degeneration. The quotient is defined in terms of the weight function associated to the form Ω, defined in [KS06] and further studied in [MN15, NX16] and [Tem16]. In the non-collapsed case, the limit is a union of flat surfaces glued along finitely many singular points, the gluing is determined by the minimality locus of the weight function.

The main results of this paper are Theorems 3.13 and 3.14 which describe the collapsed and non-collapsed limits. In Proposition 3.15 a series of degenerations of curves of genus $2k+1$ which give rise to collapsed limits, metric graphs of any genus $k \geq 1$, are constructed. The technical heart of the paper is Section 3 where the total space of the degeneration near zero is covered by charts of a special form and estimates on the lengths of shortest geodiscs are derived. Section 2 provides background information about the dual intersection complexes and on the weight function.

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2 Background

2.1 Gromov-Hausdorff distance

Let X be a metric space with the metric d and let $A, B \subset X$ be two subsets; the *Hausdorff distance* between A and B is the infimum of real numbers $\varepsilon > 0$ such that $B_\varepsilon(X) \subset Y$ and $B_\varepsilon(Y) \subset X$, where $B_\varepsilon(A)$

$$B_\varepsilon(A) = \{ \ x \in X \ \mid \ \exists a \in A \ \ d(x,a) < \varepsilon \ \}$$

denotes the ε-neighbourhood of a set A. The *Gromov-Hausdorff distance* between two metric spaces (X,d) and (Y,d') is the infimum of Hausdorff distances between X and Y over all metric spaces Z and all possible isometric embeddings of $X \hookrightarrow Z$, $Y \hookrightarrow Z$. Note that finite metric spaces are dense in the space of (isometry classes of) compact metric spaces with the Gromov-Hausdorff metric.

If X is a complex curve and Ω is a holomorphic 1-form on X then $\omega = \dfrac{i}{2}(\Omega \wedge \bar{\Omega})$ defines a pseudo-Kähler form an X. Such pseudo-Riemannian surfaces are locally isometric to Euclidean plane away from the zeroes of Ω, where they have conical singularities, and have trivial holonomy. They are also called *translation surfaces* since they can be glued from polygonal domains on a plane via identification of opposite sides by translations (see, for example, [Zor06]). Such surfaces can be regarded as metric spaces with a complete inner metric via the shortest geodesic metric.

Let X be a variety over the field $\mathbb{C}_{\mathrm{mer}}$ of germs of functions meromorphic at 0. To give X is the same as to give a germ of a family $\mathcal{X} \to \mathbb{D}_\varepsilon^\circ$ over the punctured disc $\mathbb{D}_\varepsilon^\circ = \{ \ x \in \mathbb{C} \ \mid \ |x| < \varepsilon \ \}$ for some sufficiently small ε. Assume that the genus $g(X)$ of X is greater than or equal to 1, and let

$\Omega \in H^0(\mathscr{X}, \Omega_{\mathscr{X}/\mathbb{A}^1})$ be given. Denote $\omega_s = \dfrac{i}{2}(\Omega \wedge \bar{\Omega})$ the Kähler forms on each fibre $\mathscr{X}_s$ for $|s|< \varepsilon$ and let $\tilde{\omega}_s = \omega_s/\mathrm{diam}(\mathscr{X}_s, \omega_s)^{1/2}$ be the rescaled Kähler form (so that the diameter of $\mathrm{diam}(\mathscr{X}_s, \tilde{\omega}_s) \equiv 1$). By analogy with the conjecture of Kontsevich and Sobelman one can ask:

Question: what is the limit in the Gromov-Haudorff metric of $(\mathscr{X}_s, \tilde{\omega}_s)$ as $s \to 0$?

The answer will be given in the Section 3.4.

2.2 Dual intersection complexes and the weight function

Everything in this section is valid over a discretely valued field K with a value ring R ($K = \mathbb{C}_{\mathrm{mer}}$ in the rest of the article), and the valuation $v_k : K^\times \to \mathbb{R}$. For a scheme $\mathscr{X}$ over R we denote $\mathscr{X}_0$ the fibre over the closed point of R.

Recall that a *model* $\mathscr{X}$ is a flat scheme over R such that $\mathscr{X} \otimes_R K \cong X$. It is called an *snc model* if $(\mathscr{X}_0)_{\mathrm{red}}$ is an snc divisor in $\mathscr{X}$. We call a divisor $D \in \mathrm{Div}(\mathscr{X})$ *vertical* if $\mathrm{supp}\, D \subset \mathscr{X}_0$ and *horizontal* if no irreducible components of $\mathrm{supp}\, D$ are vertical.

Definition 2.1 (Dual interesection complex). *Let $\mathscr{X}$ be an snc model of a projective curve X, $\mathscr{X}_0 = \sum N_i E_i$. The* dual intersection complex $\Delta_{\mathscr{X}}$ *of the special fibre $\mathscr{X}_0$ is a metric graph that has a vertex $[E_i]$ for each irreducible component of $\mathscr{X}_0$. The edges between vertices $[E_i]$ and $[E_j]$ are in bijective correspondence with connected components of $E_i \cap E_j$ and are of length $(N_i, N_j)/N_i N_j$.*

For any edge σ we denote $\partial\sigma$ the set of its ends. For any vertex $i \in \Delta_{\mathscr{X}}$ we will denote $\mathrm{St}(i)$ the *star of* i, the set of edges σ such that $i \in \partial\sigma$.

The dual intersection complex of any snc model embeds into the *Berkovich analytification* X^{an} of X. Let v_K be the valuation on the base field K. As a topological space the analytification is defined to be the set of pairs

$$X^{\mathrm{an}} := \{ \ (\xi, |\cdot|) \ \mid \ \xi \in X, v : K(\xi) \to \mathbb{R} \text{ valuation } , v|_K = v_K \ \}$$

with the weakest topology that makes evaluation maps $v \mapsto v(f)$, for any $f \in K[U], U \ni x$ continuous.

The construction of the embedding $\Delta(\mathscr{X}) \hookrightarrow X^{\mathrm{an}}$ goes back to [Ber98], we recall here a more direct approach following [MN15, Proposition 3.1.4], [BFJ16, Section 3]. We identify a face joining two components E_i and E_j with an interval in $\mathbb{R}^n$ that connects two points $(0, N_i)$ and $(N_j, 0)$. A point of this interval with coordinates $(\alpha, N_j(1 - \dfrac{\alpha}{N_i}))$ is identified with a quasi-monomial valuation as follows. Let x, y be the local parameters at the intersection $E_\sigma = E_i \cap E_j$. Define the valuation

$$v_\alpha : k(X)^\times \to \mathbb{R} \qquad f \mapsto \min_{f_{ij} \neq 0} \alpha i + N_i(1 - \frac{\alpha}{N_j})j$$

where f_{ij} are the coefficients of the expansion $f = \sum f_{ij} x^i y^j$. It is clear from this definition that if f has no zeroes or poles intersecting E_σ then $v_\alpha(f)$ is an affine function of α.

It is also possible to define the metric on X^{an} so that the embedding $\Delta_{\mathscr{X}} \to X^{\mathrm{an}}$ is isometric.

Fact 2.2. *Let K' be a finite extension of K, with $R' \subset K'$ its value ring. Let $X' = X \otimes K'$, then there exists a surjection $(X')^{\mathrm{an}} \to X^{\mathrm{an}}$. Then there exists a model $\mathscr{X}'$ of such that the induced map $\Delta_{\mathscr{X}'} \to \Delta_{\mathscr{X}'}$ multiplies distances by $[K' : K]$.*

Remark. Let $\mathscr{Y}$ be a blow-up of the intersection of two divisors E_i and E_j. Then $\Delta_{\mathscr{Y}}$ is obtained from $\Delta_{\mathscr{X}}$ by the subdivision of the edge that joins $[E_i]$ and $[E_j]$ and that corresponds to the intersection of E_i and E_j that has been blown up.

If $\mathscr{Y}$ is a blow-up of a smooth point $x \in E_i$ of $\mathscr{X}_s$ then $\Delta_{\mathscr{Y}}$ is obtained by adjoining an edge to $[E_i]$ in $\Delta_{\mathscr{X}}$, so the latter dual intersection complex can be naturally regarded as a subgraph of $\Delta_{\mathscr{Y}}$.

Remark. For a blow-up $f : \mathscr{Y} \to \mathscr{X}$ there exists a natural map $r : \Delta_{\mathscr{Y}} \to \Delta_{\mathscr{X}}$ which retracts the edge containing the point corresponding to the exceptional divisor E if $f(E) \notin \mathscr{X}_{\mathrm{sing}}$, and or sends it to the barycenter of the interval joining $[E_i]$ and $[E_j]$ if $f(E) \subset E_i \cap E_j$. Since any model $\mathscr{Y}$ that dominates $\mathscr{X}$ is obtained as a sequence of blow-ups of points in the central fibre, the map r can be defined as a composition of such maps for any dominant $\mathscr{Y} \to \mathscr{X}$. The retraction map can be defined in a less ad hoc and still explicit way in any dimension, see [BFJ16, Theoerem 3.1], [MN15, Proposition 3.1.7].

Let $\Omega \in H^0(X, \Omega^1_X)$. Define the weight function $\mathrm{wt}_\Omega : \Delta_{\mathscr{X}} \to \mathbb{R} \cup \{+\infty\}$ as the function that takes the following values on the divisorial valuations

$$\mathrm{wt}_\Omega([E_i]) = \frac{1 + \mathrm{ord}_{E_i}(\Omega)}{N_i}$$

where $\mathrm{ord}_{E_i}(\Omega)$ is the order of vanishing at the divisor E_i of Ω regarded as the rational section of the relative canonical bundle $\Omega_{\mathscr{Y}/R}$. By Proposition 4.2.4 [MN15] wt_Ω is well-defined (i.e. does not depend on the model $\mathscr{X}$) and by Proposition 4.4.5 *loc.cit* its extension by continuity to the whole of X^{an} gives rise to a function that is piece-wise affine on the faces of $\Delta(\mathscr{X})$ for any snc model $\mathscr{X}$.

It follows from this definition that the weight function is compatible with the embeddings of dual intersection complexes: if $f : \mathscr{Y} \to \mathscr{X}$ is a dominant morphism of snc models, then

$$\mathrm{wt}^{\mathscr{Y}}_\Omega |_{\Delta(\mathscr{X}_s)} = \mathrm{wt}^{\mathscr{X}}_\Omega$$

Furthermore, it is shown in [MN15] that $\mathrm{wt}_\Omega^{\mathscr{Y}}(v) \geq \mathrm{wt}_\Omega^{\mathscr{Y}}(r(v))$ where $r : \Delta(\mathscr{Y}) \to \Delta(\mathscr{X})$ is the retraction.

An alternative treatment of the weight function using the non-archimedean analytic techniques can be found in [Tem16].

3 Limits of flat curves

3.1 Patching

If X and Y are metric spaces and $R \subset X \times Y$ is a relation one defines the *distortion* of R to be

$$\operatorname{dis} R = \sup_{(x,y),(x',y') \in R} |d_X(x,y) - d_Y(x',y')|$$

One easily observes that if both projections of R on X and Y are surjective then $d_{GH}(X,Y) < \operatorname{dis} R/2$, and conversely, for any metric spaces X, Y such that $d_{GH}(X,Y) < \epsilon$ the relation $R = \{(x,y) \in X \times Y \mid d(x,y)\}$, where d is the metric on $X \sqcup Y$ that realizes the bound, satisfies $\operatorname{dis} R < 2\epsilon$ ([BBI01, Theorem 7.3.25]).

Recall that a metric is called an *inner metric* if the distance beween two points is defined as an infimum of a length functional on some class of admissible paths (see Section 2 of [BBI01] for the detailed definition).

Proposition 3.1. *Let (X_t, d_t) be a collection of metric spaces indexed by the points of a punctured disc $\mathbb{D}^\circ = \{ \ x \in \mathbb{C} \ \mid \ 0 < |x| < \epsilon \ \}$, and let $X_t = \cup_{i=1}^n U_t^i$ be a finite covering of X_t by path-connected sets, for each t. Let $\bar{U}^i$ be a collection of metric spaces endowed with inner metrics which we will by abuse of notation denote as d independently of the index i. Let $\sim$ be an equivalence relation on $\sqcup \bar{U}^i$, with $q : \ \sqcup \bar{U}^i \to \bar{X} = \sqcup \bar{U}^i/\sim$ the projection map. Assume that there exists a number N such that the function $\bar{d} : \bar{X} \times \bar{X} \to \mathbb{R}$*

$$\bar{d}([x],[y]) = \inf \sum_{i=1}^N d(p_i, q_i) \tag{1}$$

where the infimum is taken over all finite sequences $p_1, \ldots, p_N, q_1, \ldots, q_N \in \sqcup_{i=1}^n \bar{U}^k$ such that $x \sim p_1, p_{i+1} \sim q_i, q_N \sim y$ defines a metric on $\bar{X}$. Let $R_t^i \subset \bar{U}^i \times U^i$ be a family of relations. Assume:

1. *for all t sufficiently close to 0 the set of connected components of $U_t^i \cap U_t^j$ is finite and in bijective correspndence with the set of connected components of $\bar{U}^i \cap \bar{U}^j$;*

2. *define*

$$R_t^{ij} = \{(x,y) \in (\bar{U}^i \cup \bar{U}^j) \times (U^i \cup U^j) \mid \exists x' \sim x, x' \in \bar{U}^i \sqcup \bar{U}^j \ (x', y) \in R^i \cup R^j\}$$

for all i, j, $\operatorname{dis} R_t^{ij} \to 0$ as $t \to 0$ (so in particular for all i, $\operatorname{dis} R_t^{ij} \to 0$ as $t \to 0$).

Then for the relation

$$R_t = \{(x, y) \in \bar{X} \times X_t \mid \exists i \, \exists x' \in \bar{U}^i, q(x') = x, (x', y) \in R^i\}$$

$\operatorname{dis} R_t \to 0$ *as* $t \to 0$.

Proof. For a sequence of sets $\bar{U}^{i_1}, \ldots, \bar{U}^{i_L}$ such that $\bar{U}^{i_k} \cap \bar{U}^{i_{k+1}} \neq \emptyset$ consider all shortest paths $\bar{\gamma}$ that start in $\bar{U}^{i_1}$ and end in $\bar{U}^{i_L}$ passing through U^{i_k} in the order of indexing. Denote $\bar{U}^{i_k, i_{k+1}}$ the connected components of $\bar{U}^{i_k} \cap U^{i_{k+1}}$ through which γ passes. We subdivide γ by picking some points $\bar{x}_k \in \bar{U}^{i_k, i_{k+1}}$ and denote $\bar{\gamma}^k$ the length of the segment of γ between $\bar{x}_k$ and $\bar{x}_{k+1}$. Since the metric on $\bar{X}$ is an inner metric, γ^k is the length of a shortest such path.

By Assumption 1 whenever $x^1 \in U_t^{i_1}, x^L \in U^{i_L}$ there exists a path connecting x^1 and x^L passing through U^{i_k} in the order of indexing. Assume that $(\bar{x}^1, x^1) \in R_t^{i_1}, (\bar{x}^L, x^L) \in R_t^{i_L}$ and that γ is the shortest such path. Adopting the notation similar to the previous paragraph, the length of γ is $\sum \gamma^k$.

Let δ be such that $\operatorname{dis} R_t^{i_k, i_{k+1}} < \epsilon$ when $|t| < \delta$. Then for γ picked as above $\sum \gamma^k - \sum \bar{\gamma}^k < 2L\epsilon$ when $|t| < \delta$, because at the very least one can pick γ to pass through points $x^k \in U_t^{i_k, i_{k+1}}$ such that $(\bar{x}^k, x^k) \in R^{i_k, i_{k+1}}$.

Assume now that $\sum \bar{\gamma}^k - \sum \gamma^k > 2L\epsilon$ for some t such that $|t| < \delta$. But then one can pick a path $\bar{\eta}$ in $\bar{X}$ passing through the points $\bar{y}^k$ such that $(\bar{y}^k, x^k) \in R^{i_k, i_{k+1}}$ and the length of $\bar{\eta}$ can be estimated as

$$\sum_{k=1}^{L} \bar{\eta}^k < \sum_{k=1}^{L} (\gamma^k + 2\epsilon) < \sum_{k=1}^{L} \bar{\gamma}^k$$

since $\operatorname{dis} R^{i_k, i_{k+1}} < \epsilon$, which contradicts the fact that γ is the shortest path connecting $\bar{x}_1$ and $\bar{x}_L$ and passing through $\bar{U}^{i_k}$ in the order of indexing.

Since a distance between two points is the minimum of lengths of finitely many paths of the form considered above, and since in each case $L \leq N$, we conclude that

$$|\bar{d}(\bar{x}_1, \bar{x}_L) - d_t(x_1, x_L)| < 2N\epsilon$$

for t sufficiently close to 0, and therefore $\operatorname{dis} R_t \to 0$. □

Proposition 3.2. *In the setting of Proposition 3.1 assume that $\operatorname{dis} R_t^i \to 0$ as $t \to 0$ and that the diameters of connected components of $U_t^i \cap U_t^j$ tend to zero as $t \to 0$. Then $\operatorname{dis} R_t^{ij} \to 0$.*

Proof. Adopt the notation of the previous proposition. Let $\bar{\gamma}$ be a path connecting two points $\bar{x}_1 \in U^{i_1}$ and $x_2 \in U^{i_2}$ and passing through a point

$\bar{x}_{12}$ in a connected component $\bar{U}^{i_1,i_2} \subset U^{i_1} \cap U^{i_2}$. Let $x_1, x_2 \in X_t$ be such that $(\bar{x}_1, x_1) \in R_t^{i_1}$, $(\bar{x}_2, x_2) \in R_t^{i_1}$ and let γ be a shortest path connecting x_1 and x_2 and passing through $U_t^{i_1,i_2}$. By the same reasoning as in the previous proposition it suffices to show that $d_t(x_1, x_2)$ and $\bar{d}(\bar{x}_1, \bar{x}_2)$ differ by at most ϵ for t small enough. Pick $\bar{x}_{12}$ such that $(\bar{x}_{12}, x_{12}) \in R_t^{i_1,i_2}$. Then by triangle inequality

$$|d(x_1, x_{12}) + d(x_{12}, x_2) - d(x_1, y) + d(y, x_2)| \leq 2 \operatorname{diam} U_t^{i_1,i_2}$$

and therefore

$$|d_t(x_1, x_2) - \bar{d}(\bar{x}_1, \bar{x}_2)| < 2 \operatorname{diam} U_t^{i_1,i_2} + 2 \operatorname{dis} R_t^{i_1,i_2}$$

Since both the diameter of $U_t^{i_1,i_2}$ and $\operatorname{dis} R^{i_1,i_2}$ tend to 0, the conclusion follows. □

3.2 Models and covers

From now on $K = \mathbb{C}_{\mathrm{mer}}$.

Definition 3.3 (Snc model)**.** *A scheme $\mathscr{X}$ flat over R such that $\mathscr{X} \otimes_R K \cong X$ is called a* model of X. *If the reduction of the special fibre $\mathscr{X}_s$ is an snc divisor then such $\mathscr{X}$ is called an* snc model *of X.*

A pair $(\mathscr{X}, \Omega')$ of a model $\mathscr{X}$ and a relative 1-form Ω' such that $\Omega'|_X = \Omega$ is called a model of the pair (X, Ω). *It is called an* snc model of the pair (X, Ω) *the reduction of* $\operatorname{div}(\Omega') \cup \mathscr{X}_s$ *is snc.*

By a slight abuse of notation, for a K-varity X and its model $\mathscr{X}$ over R we will denote by X_s (resp. $\mathscr{X}_s$) the fibres for s close enough to 0 of the corresponding families over a disc (resp. punctured disc). We will also denote ω_0 the $(1,1)$-form $\dfrac{i}{2}(\Omega_0 \wedge \bar{\Omega}_0)$ defined on the smooth part of $\mathscr{X}_0$.

Lemma 3.4. *Let $(\mathscr{X}, \Omega')$ be an model of a pair (X, Ω) where X is a projective curve. Then there exists an snc model $\mathscr{Y}$ and a dominant morphsm $f : \mathscr{Y} \to \mathscr{X}$ such that $(\mathscr{Y}, f^*\Omega')$ is an snc model of the pair (X, Ω)*

Proof. Suffices to take $\mathscr{Y}$ to be a log-resolution of $\mathscr{X}_0 \cup \operatorname{div}(\Omega)$. Indeed, $\operatorname{div}(f^*\Omega) = f^* \operatorname{div}(\Omega)$. □

Let $(\mathscr{X}, \Omega')$ be an snc model. For the next two Propositions we will fix the following data: for each 1-face $\sigma \in \Delta(\mathscr{X}_s)$ such that $\partial\sigma = \{i, j\}$ let $x_\sigma, y_\sigma \in \mathcal{O}_{\mathscr{X}_s, E_\sigma}$ be the local equations of E_i, E_j, respectively, $x_\sigma^{N_i} y_\sigma^{N_j} = t$ be the local equation of $E_i \cup E_j$ near E_σ, a, b are integers such that $aN_j + bN_i = (N_i, N_j)$ and $z_\sigma = x_\sigma^{-a} y_\sigma^b$. With this set up, $\operatorname{div}(dz_\sigma/z_\sigma) = -E_i - E_j$.

Proposition 3.5. *Let $(\mathscr{X}, \Omega')$ be an snc model of a pair (X, Ω). Let σ be a 1-face and $\{i, j\} \in \partial\sigma$. Assume that $\operatorname{div}(\Omega)_{hor}$ and $E_i \cup E_j$ intersect transversely. Then on each fibre $\mathscr{X}_s$*

$$\frac{i}{2}(\Omega'_s \wedge \bar{\Omega}'_s) = |s|^{2(\mathrm{wt}_\Omega(E_i)+\frac{a}{N_i}k)}|z_\sigma|^{2k}|u|^2\frac{d|z_\sigma|}{|z_\sigma|} \wedge d\operatorname{Arg} z$$

where $z_\sigma = x_\sigma^a y_\sigma^{-b}, k = \dfrac{(\mathrm{wt}_\Omega(E_j) - \mathrm{wt}_\Omega(E_i))N_iN_j}{(N_i, N_j)}$ and $u \in \mathcal{O}^\times_{\mathscr{X}_s, E_\sigma}$.

Proof. Denote $c = \operatorname{ord}_{E_i}(dz_\sigma), d = \operatorname{ord}_{E_j}(dz_\sigma)$.

Consider the following $\mathbb{Q}$-divisor

$$D = (\mathrm{wt}_\Omega(E_i) + \frac{a}{N_i}k)\operatorname{div}(t) + k\operatorname{div}(z_\sigma) + \operatorname{div}(dz_\sigma/z_\sigma)$$

It would suffice to show that $D = cE_i + dE_j$.

Expanding, we get

$$D = (1 + c + ak)E_i + (N_j\,\mathrm{wt}_\Omega(E_i) + ak\frac{N_j}{N_i})E_j - akE_i + bkE_j - E_i - E_j$$

Further expanding the defintion of $\mathrm{wt}_\Omega(E_i)$ and combining similar terms one obtains

$$D = cE_i + ((1 + c + ak)\frac{N_j}{N_i}E_j + bkE_j - E_j$$

and since

$$N_j/N_ia + b = (N_i, N_j)/N_i \text{ and } N_j(\mathrm{wt}_\Omega(E_j) - \mathrm{wt}_\Omega(E_i)) = k(N_i, N_j)/N_i,$$

one gets after the simplification that $D = cE_i + (N_j\,\mathrm{wt}_\Omega(E_j) - 1)E_j = cE_i + dE_j$. □

Proposition 3.6. *Let $(\mathscr{X}, \Omega')$ be an snc model. Then there exists a cover $U^\alpha \subset X(\mathbb{C})$ indexed by 0- and 1-faces of $\Delta(\mathscr{X}_s)$ of a tubular neigbourhood of $\mathscr{X}_s$ that satisfies the folowing properties:*

1. *for all 1-faces σ, the set U^σ is defined in a neighbourhood of E_σ by the inequalities*
$$C_\sigma|t|^{a/N_i} \leq |z_\sigma| \leq D_\sigma|t|^{-b/N_j}$$
for some constants $C_\sigma, D_\sigma > 0$;

2. *for any 0-face i and 1-face σ, $U^i \cap U^\sigma \neq \emptyset$ if and only if $i \in \partial\sigma$;*

Proof. Let ϵ be the smallest number such that $x_\sigma^a y_\sigma^{-b} = t$ are the equations of $\mathscr{X}_t$ for $|t| < \epsilon$ for all 1-faces $\sigma \in \Delta_{\mathscr{X}}$.

Observe that the inequalities from the property 1 can be rewritten as $|x_\sigma| \leq \dfrac{1}{C_\sigma^{N_i}}$ and $|y_\sigma| \leq \dfrac{1}{D_\sigma^{N_j}}$, and pick the constants C_σ, D_σ so that $U^\sigma \cap U^\tau \neq \emptyset$ for each 0-face i and each $\sigma, \tau \in \operatorname{St}(i)$.

Then the complement of $\cup U^\sigma$ in the neighbourhood of the special fibre defined by the inequality $|t|< \epsilon$ consists of connected components W^i that are in bijective correspondence with irreducible components of the special fibre. Define $U^i = W^i \cup \bigcup_{\sigma\in \mathrm{St}(i)} \partial U^\sigma$. Then the property 2 is satisfied by construction. $\square$

Lemma 3.7. *Let $u \in \mathfrak{m} \subset \mathcal{O}_{\mathscr{X},E_\sigma}$ and let W be a germ of a neighbourhood of E_σ definde by inequalities $|t|^\alpha \leq |z_\sigma| \leq |t|^\beta$ where $\alpha > a/N_i$, $\beta < b/N_j$. Then $\sup_{x\in W_s}|u| \to 0$ as $|s| \to 0$.*

Proof. The conclusion follows immediately after observing that the set W_s is the intersection of the curve $x^{N_i}y^{N_j} = s$ and the rectangle $|x|\leq |t|^\alpha$, $|y|\leq t^\beta$. $\square$

Corollary 3.8. *Let $(\mathscr{X},\Omega')$ be an snc model of a pair (X,Ω). Assume $\mathrm{wt}_\Omega(i) < \mathrm{wt}_\Omega(l)$ for all $j \in \mathrm{St}(i)$. Then $\Omega/t^{\mathrm{wt}_\Omega(i)-a/N_i-1}|_{E_i}$ is regular and non-zero.*

3.3 Asymptotic distance estimates

Lemma 3.9. *Let $(\mathscr{X},\Omega')$ be an snc model of the pair (X,Ω). Consider the fibres X_s with the Kälher metric $\omega_s = \frac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$.*

Pick functions $z_\sigma \in \mathcal{O}_{\mathscr{X},E_\sigma}$ as in Proposition 3.5 and assume that

$$\Omega' = c_\sigma z_\sigma^\alpha t^\beta (1+u) dz_\sigma$$

for some $\alpha,\beta \in \mathbb{Z}$, and $u \in \mathfrak{m} \subset \mathcal{O}_{\mathscr{X}_s,E_\sigma}$.

Let $\{\eta_s,\eta'_s\}$ be a collection of points in $(U^\sigma)_s$ and let $\gamma_s \subset U^\sigma$ be a shortest path between the points η_s and η'_s. Then the length of γ_s as $s \to 0$ is

$$|c_\sigma|\cdot|\log|z_\sigma(\eta'_s)| - \log|z_\sigma(\eta_s)|\cdot|s|^\beta(1+o(1))$$

if $\alpha = -1$ and

$$(|z_\sigma(\eta'_s)|^{\alpha+\beta+1} - |z_\sigma(\eta_s)|^{\alpha+\beta+1})(1+o(1))$$

otherwise.

Proof. The Riemannian metric tensor associated to ω_s has the form

$$g_\Omega = |c|^2|z_\sigma|^{2\alpha}|s|^{2\beta}|1+u|^2(d|z_\sigma|\otimes d|z_\sigma| + |z_\sigma|\, d\,\mathrm{Arg}\, z_\sigma \otimes d\,\mathrm{Arg}\, z_\sigma)$$

in the polar coordinates.

Let $\gamma'_s : [0,1] \to U^\sigma_s$ be the path given by

$$\gamma'_s(\tau) = |z_\sigma(\eta_s)|\exp(\tau + (1-\tau)\log\frac{|z_\sigma(\eta'_s)|}{|z_\sigma(\eta_s)|} + i\,\mathrm{Arg}(\eta_s))$$

and $\gamma''_s : [0,1] \to U^\sigma_s$ be defined by

$$\gamma''_s(\tau) = |z_\sigma(\eta'_s)| \exp(i(\tau \operatorname{Arg} \eta_s + (1-\tau) \operatorname{Arg} \eta'_s))$$

Assume that $|u| < C$ on U^σ and assume for definiteness that $|z_\sigma(\eta'_s)| > |z_\sigma(\eta_s)|$. Denote $I_{\epsilon,s} = [|s|^{a/N_i+\epsilon}, |s|^{b/N_j-\epsilon}]$ and let $H_s = [|z_\sigma(\eta_s)|, |z_\sigma(\eta'_s)|]$; denote $A_{\epsilon,s}, B_\epsilon$ the endpoints of the interval $I_{\epsilon,s} \cap H_s$. Then

$$L(\gamma'_s) = \int_0^1 \sqrt{g_\Omega(\dot{\gamma}'(\tau), \dot{\gamma}'(\tau)} d\tau = \int_{|z_\sigma(\eta_s)|}^{|z_\sigma(\eta'_s)|} |c_\sigma| \rho^\alpha |s|^\beta |1 + u(\rho e^{i \operatorname{Arg} \eta_s}, s)| d\rho$$

The latter integral can be represented, for $\epsilon > 0$, as the sum of two integrals

$$\int_{H_s} |c_\sigma| \rho^\alpha |s|^\beta |1 + v(\rho e^{i \operatorname{Arg} \eta_s}, s)| d\rho = |c_\sigma| |s|^\beta \left(\int_{H_s \setminus I_{\epsilon,s}} \rho^\alpha |1 + u(\rho e^{i \operatorname{Arg} \eta_s}, s)| d\rho + \right.$$
$$\left. + \int_{I_{\epsilon,s}} \rho^\alpha |1 + u(\rho e^{i \operatorname{Arg} \eta_s}, s)| d\rho \right)$$

Let us first consider the case $\alpha = -1$. By assumption, $|u(\rho, s)| \leq C$, for $(\rho, s) \in U^\sigma_s$, and by Lemma 3.7 $|u(\rho, s)| = o(1)$ as $(\rho, s) \in U^\sigma_s, \rho \in I_{\epsilon,s}$ and $s \to 0$, therefore,

$$\begin{aligned} L(\gamma'_s) &= |c_\sigma| |s|^\beta \inf_{\epsilon > 0} (\ln B_{\epsilon,s}/A_{\epsilon,s})(1 + o(1)) + \\ &+ C \max\{A_{\epsilon,s} - |z_\epsilon(\eta_s)|, 0\} + C \max\{|z_\epsilon(\eta'_s)| - B_{\epsilon,s}, 0\}) \\ &= |c_\sigma| |s|^\beta \ln |z_\sigma(\eta'_s)|/|z_\sigma(\eta_s)| (1 + o(1)) \end{aligned}$$

Similarly, one derives for $\alpha \neq -1$,

$$L(\gamma'_s) = \Theta(|s|^{\beta+1} |z_\sigma^{\alpha+1}(\eta'_s)| - |z_\sigma^{\alpha+1}(\eta_s)|)$$

On the other hand,

$$L(\gamma''_s) = O(|s|^\beta |z_\sigma(\eta'_s)|)$$

Clearly, $L(\gamma_s) \leq L(\gamma'_s) + L(\gamma''_s)$.

Finally,

$$L(\gamma_s) = \int_0^1 \sqrt{g(\dot{\gamma}(\tau), \dot{\gamma}(\tau)} d\tau \geq \int_0^1 |c_\sigma| |z_\sigma(\gamma_s(\tau))|^\alpha |s|^\beta |1 + u(\gamma_s(\tau))| d\tau$$

the latter expression can be shown, as before, to be equal to

$$|c_\sigma| |s|^\beta \ln |z_\sigma(\eta'_s)|/|z_\sigma(\eta_s)| (1 + o(1))$$

if $\alpha = -1$ and

$$\Theta(|s|^{\beta+1} |z_\sigma^{\alpha+1}(\eta'_s)| - |z_\sigma^{\alpha+1}(\eta_s)|)$$

otherwise. □

Lemma 3.10. *Let $(\mathscr{X}, \Omega')$ be an snc model of a pair (X, Ω) and let E_i be an irreducible component of $\mathscr{X}_s$. Let a_s, b_s be collections of points such that $\lim_{s\to 0} a_s = a_0, \lim_{s\to 0} b_s = b_0$ for some $a_0, b_0 \in E_i \subset \mathscr{X}_s$. Assume that $\mathrm{wt}_\Omega(E_i) = 1$ and that $a_0, b_0 \notin E_\sigma$ for any $\sigma \subset \Delta_{\mathscr{X}}$ such that $i \in \partial\sigma$. Consider the metric $\omega_s = \frac{i}{2}(\Omega_s \wedge \bar{\Omega})_s$ on X_s and let $\gamma_s \subset U^i$ be a shortest path that connects a_s and b_s. Then*

$$\lim_{s\to 0} l(\gamma_s) = l(\gamma_0)$$

and the limit is finite. If $\mathrm{wt}_\Omega(j) > 1$ for all $j \in \mathrm{St}(i)$ then this statement holds regardless of whether a_0, b_0 belong to any E_σ.

Proof. The first statement follows from the continuity of the form $\frac{i}{2}\Omega \wedge \bar{\Omega}$.

By Lemma 3.5 if $1 = \mathrm{wt}_\Omega(i) < \mathrm{wt}_\Omega(j)$ for all $j \in \mathrm{St}(i)$ then ω_0 does not vanish on E_i and Ω_0 does not have poles on E_i, therefore, the metric ω_0 is complete on E_i. The path γ_0, therefore, has a finite length. □

Lemma 3.11. *Let $k = \min_{x\in\Delta_{\mathscr{X}}} \mathrm{wt}_\Omega(x)$. Let $\Sigma \subset \Delta(\mathscr{X}_s)$ be the set of faces of $\Delta_{\mathscr{X}}$ where $\mathrm{wt}_\Omega \equiv k$, and let $\Omega'' = \Omega'/t^k$. The asymptotics of the diameter of X_s as $s \to 0$ with respect to the Kähler metric $\omega_s = \frac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$ is*

$$\mathrm{diam}\, X_s = c|s|^k(1 + o(1)) \text{ where } c = \sum_{i\in\Sigma} \mathrm{diam}(E_i, \frac{i}{2}\Omega''_0 \wedge \bar{\Omega}''_0)$$

if $\dim \Sigma = 0$, and

$$\mathrm{diam}\, X_s = c \log|s||s|^k(1 + o(1)) \text{ where } c = \sum_{\sigma\in\Sigma} |c_\sigma| l(\sigma)$$

if $\dim \Sigma = 1$.

Proof. Consider the cover constructed in Proposition 3.6.

By Lemmas 3.5, 3.9 and 3.10

$$\mathrm{diam}(U^i)_s = \Theta(|s|^{\mathrm{wt}_\Omega(E_i)-1})$$

when $i \in \Delta_{\mathscr{X}}$ is a vertex and

$$\mathrm{diam}(U^\sigma)_s = \Theta(\log|s|\,|s|^{\mathrm{wt}_\Omega(E_i)-1})$$

when $\mathrm{wt}_\Omega(E_i) = \mathrm{wt}_\Omega(E_j)$ for $i, j \in \partial\sigma$,

$$\mathrm{diam}(U^\sigma)_s = \Theta(|s|^{\mathrm{wt}_\Omega(E_i)-1})$$

when $\mathrm{wt}_\Omega(E_i) < \mathrm{wt}_\Omega(E_j)$.

It follows that $\mathrm{diam}((U^\beta)_s) = o(\mathrm{diam}(U^\alpha)_s)$ for any $\alpha \in \Sigma$ and $\beta \notin \Sigma$. Now notice that $\mathrm{diam}\, U^\sigma$ has the same asymptotics in $|s|$ for all edges

$\sigma \in \Sigma$ if $\dim \Sigma = 1$, and that $\operatorname{diam} U^i$ have the same asymptotics in $|s|$ for $i \in \Sigma$ if $\dim \Sigma = 0$. Therefore, $\operatorname{diam}(X_s, \omega_s/(\log|s|\,|s|^k)^2)$ in the first case, $\operatorname{diam}(X_s, \omega_s/|s|^{2k})$ in the second case, tends to the diameter of X_0. Appealing to Lemmas 3.9 and 3.10 again, we compute this diameter and get the desired expressions for the asymptotics of $\operatorname{diam} X_s$. □

To describe the Gromov-Hausdorff limit of a family of curves $(X_s, \tilde{\omega}_s)$ we will distinguish two cases: *collapsed* limit, when the diameter of (X_s, ω_s) is of order $\Theta(\log|t|\,|t|^k)$ for some k, and *non-collapsed* limit otherwise.

3.4 Shape of the limit

Let $(\mathscr{X}, \Omega')$ be an snc model for (X, Ω) and adopt the notations for local coordinates from Section 3.2. Let $k = \min_{x \in \Delta_{\mathscr{X}}} \operatorname{wt}_\Omega(x)$. Let $\Sigma \subset \Delta(\mathscr{X}_s)$ be the union of vertices and edges of $\Delta_{\mathscr{X}}$ where $\operatorname{wt}_\Omega \equiv k$.

Lemma 3.12. *In the collapsed case (*$\dim \Sigma = 1$*), by Lemma 3.11,* $\operatorname{diam} X_s = c \log|s||s|^k(1 + o(1))$ *for some constant c. Consider the metric* $\dfrac{\omega_s}{\operatorname{diam} X_s^{1/2}}$. *Then for all edges $\sigma \in \Sigma \subset \Delta_{\mathscr{X}}$ such that $\partial\sigma = \{i, j\}$ there exists a relation $R_s^\sigma \subset U_s^\sigma \times [0, |c_\sigma|/cN_iN_j]$ such that* $\operatorname{dis} R_s^\alpha \to 0$.

Proof. Define the relation R^σ in the local coordinates z_σ, s as follows:

$$R_s^\sigma = \operatorname{graph}\left[(z_\sigma, s) \mapsto |c_\sigma|/c \frac{\log|z_\sigma|}{\log|s|} - \frac{a}{N_i}\right]$$

It follows from Lemma 3.9 that $\operatorname{dis} R_s^\sigma \to 0$ as $s \to 0$. □

Theorem 3.13 (Collapsed limit)**.** *Assume that degeneration (X, Ω) gives rise to a collapsed limit and let $(\mathscr{X}, \Omega')$ be an snc model as above. Let $x \sim y$ for $x, y \in \Delta_{\mathscr{X}}$ if and only if there exists a path $\gamma : [0, 1] \to \Delta_{\mathscr{X}}$ joining x and y such that $|\gamma^{-1}(\Sigma)| < \infty$. The Gromov-Hausdorff limit of $(X_s, \tilde{\omega}_s)$ is $\Delta_{\mathscr{X}}/{\sim}$ endowed with the metric that stretches each edge σ be the factor $|c_\sigma|$ and renormalized so that* $\operatorname{diam}(\Delta_{\mathscr{X}}/{\sim}) = 1$.

Proof. For any face $\sigma \in \Delta_{\mathscr{X}}$ let R^σ be the image of relation defined in Lemma 3.12 in $\Delta_{\mathscr{X}}/{\sim} \times U^\sigma$. By the definition of the equivalence relation $\sim$, the image of Σ in $\Delta_{\mathscr{X}}/{\sim}$ is in bijection with Σ, but for any $\sigma \notin \Sigma$ its image in $\Delta_{\mathscr{X}}$ is a point. Therefore, $\operatorname{dis} R_s^\sigma \to 0$ for edges $\sigma \in \Sigma$. Since by Lemma 3.11 $\operatorname{diam} U_s^\sigma = o(\operatorname{diam} X_s)$ for any $\sigma \notin \Sigma$, $\operatorname{dis} R_s^\sigma \to 0$ for $\sigma \notin \Sigma$.

Define $R^i = \{i\} \times U^i$ for any vertex $i \in \Delta_{\mathscr{X}}$. Then $\operatorname{dis} R_s^i = \operatorname{diam} U_s^i$, and by Proposition 3.10 $\operatorname{diam} U_s^i \to 0$. The connected components of $U^i \cap U^\sigma$ for $i \in \partial\sigma$ are annuli given by equations $|z_\sigma| = C_\sigma |s|^{a/N_i}$ and $|z_\sigma = D_\sigma |s|^{-b/N_j}$ and their diameters also tend to 0. By Proposition 3.2 $\operatorname{dis} R^{i\sigma} \to 0$ as $s \to 0$ if $i \in \partial\sigma$. Therefore, Proposition 3.1 applies and we conclude. □

Remark. The metric graph $\Delta_{\mathscr{X}}/\sim$ does not depend on the choice of a model $\mathscr{X}$. Indeed, if $\mathscr{Y}$ is a model that dominates $\mathscr{X}$ then $\Delta_{\mathscr{Y}}$ contracts onto $\Delta_{\mathscr{X}}$ and it follows from Proposition 4.3.4 [MN15] that $\Sigma_{\mathscr{Y}} = \Sigma_{\mathscr{X}}$, now use the fact that any two models are related by a series of blow-ups and blow-downs of subvarieties in the special fibre.

As was observed in [BN16], Lemma 3.4.5, a vertex $i \in \Delta_{\mathscr{X}}$ corresponding to a component E_i which is a curve of genus 0 belongs to Σ if and only if some adjacent edge belongs to Σ. In the case of non-collasped limit the set of components E_i of the central fibre such that $i \in \Sigma$ can thus be regarded as surfaces endowed with a flat metric, since the restriction of Ω'/t^k to each such E_i is a regular 1-form.

Theorem 3.14 (Non-collapsed limit)**.** *Assume that the limit of X_s is of non-collapsed type (*$\dim \Sigma = 0$*), and let $\Omega'' = \Omega/t^k$. Let $X_\Sigma = \cup_{i\in\Sigma} E_i \subset X_0$ and let $\sim$ be the equivalence relation on X_0 defined as follows:*

1. *$E_\sigma \sim E_\tau$ for $E_\sigma \subset E_i$, $E_\tau \subset E_j$ if $i, j \in \Sigma$ and there exists a path $\gamma : [0,1] \to \Delta_{\mathscr{X}}$ such that $\gamma(0) = i, \gamma(1) = j$, initial segment of γ passes through σ, and final segment of γ passes through τ,*

2. *$x \sim E_\sigma$ if $i \notin \Sigma$, $\partial\sigma \cap \Sigma \neq \emptyset$, $x \in E_i$ and there exists a path $\gamma : [0,1] \to \Delta_{\mathscr{X}}$ such that $\gamma(0) = i$, and final segment of γ passes through σ.*

3. *is identity otherwise.*

Then the limit of $(X_s, \tilde{\omega}_s)$ as $s \to 0$ is $\cup(E_i, \Omega''_0)/\sim$ with the metric renormalized so that the diameter of the space is 1.

Proof. For a vertex $i \in \Sigma$ the U_0^i is in bijection with $U_0^i/\sim$, and for $i \notin \Sigma$, $U_0^i/\sim$ is a point. For an edge σ such that $\partial\sigma \cap \Sigma = i$ the set $U_0^\sigma/\sim$ is the projection of U_0^σ on E_i, and otherwise $U^\sigma/\sim$ is a point.

Define the relations R^i to be $\{i\} \times U^i$ for $i \notin \Sigma$ and similarly for σ such that $\partial\sigma \cap \Sigma = \emptyset$ let R^σ be $\sigma/\sim \times U^\sigma$.

If $i \in \Sigma$, topologically

$$U^i \cong E_i \setminus (\cup_{\sigma\in \mathrm{St}(i)} U^\sigma U^i \cong E_i \setminus (\cup_{\sigma\in \mathrm{St}(i)} U_0^\sigma) \times_0) \times \mathscr{D}$$

where $\mathscr{D}$ is a disc. Let R^σ be the graph of the projection on the first factor.

If $\partial\sigma \cap \Sigma = i$ and the germ of E_i is cut out by the equation $x_\sigma = 0$ near E_σ, define the relation $R_s^\sigma \subset (U_0^\sigma \cap E_i) \times U_s^\sigma$ to be

$$R^i = \text{graph} \quad (x, y) \mapsto \begin{cases} p_i(x,y) & |x_\sigma| \leq |y_\sigma| \\ E_\sigma & |x_\sigma| > |y_\sigma| \end{cases}$$

where $p_i : U^\sigma \to U_0^\sigma \cap E_i$ is a projection that coincides with the projection used in the definition on R^i on $U^\sigma \cap U_i$. By Lemma 3.10, $\operatorname{dis} R_s^i \to 0$ as $s \to 0$.

By definition of R^i and R^σ, $\operatorname{dis} R^{i\sigma} \to 0$. Proposition 3.1, therefore, applies and X_s converges to $\cup E_i/\sim$. $\square$

We will now study for the purpose of illustration of Theorem 3.13 the possible shapes of the Gromov-Hausdorff limits it describes.

We will use the description of the graph laplacian of the weight function due to Baker and Nicaise [BN16], which we quickly recall. By a *weighted graph* we understand a metrized graph Γ with the set of vertices $V(\Gamma)$ and with infinite edges allowed, and a pair of functions $N, g : V(\Gamma) \to \mathbb{Z}$. Given an snc model $(\mathscr{X}, \Omega)$ of (X, Ω), one associates a weighted graph as follows: take $\Delta_{\mathscr{X}}$ and attach infinite edges at the vertices which correspond to components having non-trivial intersection with $\mathrm{div}(\Omega)$. A *divisor* on Γ is a formal combination of the vertices of Γ. Let $f : \Gamma \to \mathbb{R}$ be a function that is affine on every edge of Γ, then the *Laplacian of f* is the divisor $\Delta(f) = \sum_{i \in V(\Gamma)} a_i v_i$ where a_i is the sum of outward slopes of f at v_i. The *canonical divisor of* Γ is the divisor

$$K_\Gamma = \sum_v N_v(\mathrm{val}(v) + 2g(v) - 2)v$$

where $\mathrm{val}(v)$ is the valency of the vertex v.

Proposition 3.15. *For any $k > 1$ the wedge sum of k circles can occur as a limit (in the sense of Theorem 3.13) of a family of curves of genus $2k+1$ which is of maximal degeneration (i.e. admits a semi-stable model such that all the components of the central fibre are rational).*

Proof. We need to construct the special fibre $\mathscr{X}_0$ and the form Ω_0, then the existence of the family $\mathscr{X}$ of curves with the special fibre $\mathscr{X}_0$ follows by [DM19]. We will do so by constructing the dual intersection graph of $\mathscr{X}_0$ and designating which components of $\mathscr{X}_0$ the zeroes of Ω_0 should specialize to. Once this is done, it is enough to specify the slope of the weight function on each edge to determine it up to an additive constant.

Let Γ' be a chain of $g = 2k+1$ cycles, $C_1, \dots, C_{2k+1}$, each consisting of two edges. Define the weight function to be constant on C_{2i-1} for $1 \leq i \leq k$, and attach two infinite edges (corresponding to zeroes of Ω_0) to both edges of each C_{2i}, subdividing them in the points of attachment, call the resulting graph Γ. The even cycles C_{2i} in Γ then consist of six edges, $e_{3i}, e_{3i+1}, e_{3_i+2}$ and $d_{3i}, d_{3i+1}, d_{3_i+2}$ that form two chains starting and ending in one point. Define the slope of the weight function to be 1, 0, -1, respectively on these triples of edges. One checks that with these slopes $\Delta(\mathrm{wt}_\Omega) = K_\Gamma$. $\square$

Let us conclude with two open questions.

Question 1: can Proposition 3.15 be proved (perhaps with some additional conditions) for a fixed genus of the elements of the family $\mathscr{X}$? for a given fixed $\mathscr{X}$, constructing appropriate Ω?

Question 2: can one characterise non-collapsed limits starting from the description of Theorem 3.14?

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