

Gromov-Hausdorff limits of flat Riemannian surfaces

Dmitry Sustretov ¹

Abstract

I study Gromov-Hausdorff limits of complex curves endowed with singular flat metrics of constant diameter. I formulate a criterion that the limit is collapsed in terms of a certain piecewise affine weight function on the dual intersection complex of a semi-stable model of the degeneration introduced by Kontsevich and Soibelman. I describe the collapsed and non-collapsed limits, which are, respectively, metric graphs and finite collections of complex curves with flat metrics glued along finitely many points. I show that the collapsed limit of any positive genus can occur.

Contents

1	Introduction	1
2	Background	3
2.1	Gromov-Hausdorff distance	3
2.2	Dual intersection complexes and the weight function	4
3	Limits of flat curves	6
3.1	Models and charts	6
3.2	Asymptotic distance estimates	8
3.3	Shape of the limit	11

1 Introduction

In the paper [KS06] Kontsevich and Soibelman formulate a series of conjectures about the shape of Gromov-Hausdorff limits of certain families of complex manifolds endowed with Ricci flat metrics. These conjectures are motivated by mirror symmetry and in particular by the authors' approach to the SYZ conjecture. One considers germs of holomorphic families of compact Calabi-Yau manifolds parametrized by points of a punctured disc

¹ This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 843100 (NALIMDIF). The author was supported by the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) with ERC Grant Agreement nr. 615722 MOTMELSUM, and by Labex CEMPI (ANR-11-LABX-0007-01).

version of January 28, 2020

having maximally unipotent action of the monodromy on the middle cohomology and with a relatively ample line bundle on the total space of the family. For each element of the family one picks the Ricci flat metric with the Kähler class equal to the first Chern class of the polarizing line bundle and normalized so that the diameter is constant. Gromov-Hausdorff limits of such families are then conjectured by Kontsevich and Soibelman to carry a singular affine manifold structure with respect to which the limit metric satisfies the real Monge-Ampere equation. The real dimension of the limit manifold is half the real dimension of the elements of the family, so we speak about *collapsed* limits.

Alternatively, the limit manifold together with the singular affine structure can be recovered from the non-archimedean analytification X^{an} (in the sense of Berkovich) of the variety X over the non-archimedean field \mathbb{C}_{mer} of germs of complex functions meromorphic at 0 ([KS06, §5]). As a topological space it is a closed subset of X^{an} , the minimality locus of a certain weight function associated to a canonical form; this closed subset is called the essential skeleton of X , denoted $\text{Sk}(X)$. On a variety with trivial canonical bundle weight functions associated to different canonical forms differ by a constant and so the minimality locus does not depend on this choice. As it turns out, there exists a (generally speaking, not canonical) retraction $X^{\text{an}} \rightarrow \text{Sk}(X)$ [NX16] which is a fibration over an open dense subset of $\text{Sk}(X)$, with the fibre isomorphic to a non-archimedean torus [NXY19]; this endows $\text{Sk}(X)$ with an integral affine structure away from the discriminant locus. Kontsevich and Soibelman conjecture ([KS06, Conjecture 3, §5]) that $\text{Sk}(X)$ is isomorphic as a manifold with affine structure with singularities to the Gromov-Hausdorff limit described in the previous paragraph.

Collapsed Gromov-Hausdorff limits of Ricci-flat hyperkähler manifolds have been extensively studied in a slightly different setting: one fixes a holomorphic fibration of a single hyperkähler manifold, with a generic fibre Abelian variety, and considers a variation of the Kähler class, making it tend to the boundary of the Kähler cone [GW00], [GTZ13], [GTZ16], [TZ17]. Boucksom and Jonsson [BJ17] show that volume forms on the fibres of the degenerating family converge in a certain sense to a measure supported on $\text{Sk}(X)$. In the paper [OO18] Odaka and Oshima describe Gromov-Hausdorff limits of K3 surfaces in the set up of [KS06, Conjecture 3, §5] (though they do not discuss the relationship with $\text{Sk}(X)$). Finally, I would like to mention a recent paper [DHL19] of Ducros, Hrushovski and Loeser where they propose a framework for asymptotic integration that is motivated by the said Conjecture.

In this paper we consider the Gromov-Hausdorff limits of families of complex curves of genus ≥ 1 endowed with flat pseudo-Kähler metrics. If one compares to the set up of Kontsevich and Soibelman, I relax the assumption on the triviality of the tangent bundle and allow the metric to have conical singularities in finitely many points. We rescale the metrics with the Kähler

form $\frac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$ (where Ω is a given relative 1-form) on the fibres X_s of the family X so that $\text{diam } X_s \equiv 1$.

We show that there are two possibilities for the limit. In the collapsed case the limit is a metric graph which can be canonically represented as a certain quotient of the dual intersection complex of the special fibre of a semi-stable model of the degeneration. The quotient is defined in terms of minimality locus of the weight function associated to the form Ω , defined in [KS06] and further studied in [MN15, NX16] and [Tem16]. In the non-collapsed case, the limit is a union of flat surfaces glued along finitely many points at which the metric is singular, the gluing is determined by the minimality locus of the weight function.

The main results of this paper are Theorems 3.3.2 and 3.3.3 which describe the collapsed and non-collapse d limits. In Proposition 3.3.4 a series of degenerations of curves of genus $2k + 1$ which give rise to collapsed limits, metric graphs of any genus $k \geq 1$, are constructed. The technical heart of the paper is Section 3 where a neighbourhood of the special fibre of a model of the degeneration is covered by charts of a special form and estimates on the lengths of shortest geodiscs are derived. Section 2 provides background information about dual intersection complexes and the weight function.

Acknowledgements. I would like to thank Ehud Hrushovski for telling me about the conjectures of Kontsevich and Soibelman and suggesting the idea to use the Robinson field for the description of the Gromov-Hausdorff limit which was crucial in the initial version of this work. I would like to thank Grisha Papayanov, Jérôme Poineau, Matthew Stevenson, Ilya Tyomkin, Misha Verbitsky and Eric Walsberg for helpful discussions. I also would like to thank I.H.É.S. and Max Planck Institute, where the work on this paper was carried out, for perfect working conditions.

2 Background

2.1 Gromov-Hausdorff distance

We refer the reader to [BBI01, Chapter 7] for a comprehensive introduction into the Gromov-Hausdorff distance. Let X be a metric space with the metric d and let $A, B \subset X$ be two subsets; the *Hausdorff distance* between A and B is the infimum of real numbers $\varepsilon > 0$ such that $B_\varepsilon(X) \subset Y$ and $B_\varepsilon(Y) \subset X$, where $B_\varepsilon(A)$

$$B_\varepsilon(A) = \{ x \in X \mid \exists a \in A \ d(x, a) < \varepsilon \}$$

denotes the ε -neighbourhood of a set A . The *Gromov-Hausdorff distance* between two metric spaces (X, d) and (Y, d') is the infimum of Hausdorff distances between X and Y over all metric spaces Z and all possible isometric embeddings of $X \hookrightarrow Z, Y \hookrightarrow Z$. Note that finite metric spaces are dense in

the space of (isometry classes of) compact metric spaces with the Gromov-Hausdorff metric.

If X is a complex curve and Ω is a holomorphic 1-form on X then $\omega = \frac{i}{2}(\Omega \wedge \bar{\Omega})$ defines a positive semi-definite (1,1)-form on X , so if I the complex structure then $g(x, y) = \omega(x, Iy)$ is a pseudo-Riemannian metric. Such pseudo-Riemannian surfaces are locally isometric to the Euclidean plane away from the zeroes of Ω , where they have conical singularities, and have trivial holonomy. They are also called *translation surfaces* since they can be glued from polygonal domains on a plane via identification of opposite sides by translations (see, for example, [Zor06]). The shortest geodesic metric on such spaces is a complete inner metric.

Let X be a variety over the field \mathbb{C}_{mer} of germs of functions meromorphic at 0. To give X is the same as to give a germ of a family $\mathcal{X} \rightarrow \mathbb{D}_\varepsilon^\circ$ over the punctured disc $\mathbb{D}_\varepsilon^\circ = \{x \in \mathbb{C} \mid 0 < |x| < \varepsilon\}$ for some sufficiently small ε . Assume that the genus $g(X)$ of X is greater than or equal to 1, and let $\Omega \in H^0(X, \Omega_{X/\mathbb{C}_{\text{mer}}})$ be given. Denote $\omega_s = \frac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$ the Kähler forms on each fibre X_s for $0 < |s| < \varepsilon$ and let $\tilde{\omega}_s = \omega_s / \text{diam}(\mathcal{X}_s, \omega_s)^2$ be the rescaled Kähler form (so that $\text{diam}(\mathcal{X}_s, \tilde{\omega}_s) \equiv 1$). Drawing analogy with the conjecture of Kontsevich and Sobelman one can ask:

Question: what is the limit in the Gromov-Hausdorff metric of $(X_s, \tilde{\omega}_s)$ as $s \rightarrow 0$?

The answer will be given in the Section 3.3.

2.2 Dual intersection complexes and the weight function

Everything in this section is valid over a discretely valued field K with a value ring R ($K = \mathbb{C}_{\text{mer}}$ in the rest of the article), and the valuation $v_K : K^\times \rightarrow \mathbb{R}$. For a scheme \mathcal{X} over R we denote \mathcal{X}_0 the scheme-theoretic fibre over the closed point of R .

Recall that a *model* \mathcal{X} is a flat scheme over R such that $\mathcal{X} \otimes_R K \cong X$. It is called an *snc model* if $(\mathcal{X}_0)_{\text{red}}$ is a divisor with strict normal crossings.

Definition 2.2.1 (Dual intersection complex). *Let \mathcal{X} be an snc model of a projective curve X , and let the central fibre \mathcal{X}_0 equal $\sum_{i=1}^m N_i E_i$ where E_1, \dots, E_m are irreducible components of \mathcal{X}_0 . The dual intersection complex $\Delta_{\mathcal{X}}$ of the special fibre \mathcal{X}_0 is a metric graph that has vertices $[E_1], \dots, [E_m]$ and edges $[\sigma]$ of length $l(\sigma) = (N_i, N_j) / N_i N_j$ for each point $\sigma \in E_i \cap E_j$.*

For any edge σ we denote $\partial\sigma$ the set of its ends. For any vertex $[E_i] \in \Delta_{\mathcal{X}}$ we will denote $\text{St}([E_i])$ the *star* of $[E_i]$, the set of edges σ such that $[E_i] \in \partial\sigma$.

The dual intersection complex of any snc model embeds into the *Berkovich analytification* X^{an} of X . Let v_K be the valuation on the base field K . As a topological space the analytification X^{an} is defined to be the set of pairs

$$X^{\text{an}} := \{ (\xi, v) \mid \xi \in X, v : K(\xi) \rightarrow \mathbb{R} \text{ valuation}, v|_K = v_K \}$$

with the weakest topology that makes evaluation maps $v \mapsto v(f)$ continuous for any $f \in K[U], U \ni x$.

The construction of the embedding $\Delta(\mathcal{X}) \hookrightarrow X^{\text{an}}$ goes back to [Ber90], we recall here a more direct approach following [MN15, Proposition 3.1.4], [BFJ16, Section 3]. We identify a face σ joining two components E_i and E_j with an interval in $\mathbb{R}_{\geq 0}^2$ given by the equation $N_j x + N_i y = 1$ (note that the metric on σ has nothing to do with the Euclidean metric on this interval). A point with coordinates (α, β) is identified with a quasi-monomial valuation as follows. Let $x, y \in \mathcal{O}_{\mathcal{X}, \sigma}$ be some local parameters at an intersection point $\sigma \in E_i \cap E_j$ such that x is a local equation for E_i and y is a local equation for E_j . Define the following valuation:

$$v_\alpha : K(X)^\times \rightarrow \mathbb{R} \quad f \mapsto \min_{f_{ij} \neq 0} \alpha i + \beta j$$

where f_{ij} are the coefficients of an expansion $f = \sum_{i,j=1}^{\infty} f_{ij} x^i y^j$; by [MN15, Proposition 2.4.6] this definition does not depend on the choice of the local equations for E_i, E_j . One can show ([MN15, Proposition 3.2.2]) that if f has no zeroes or poles that contain σ then $v_\alpha(f)$ is an affine function of α .

Remark. The part of the topological space X^{an} that consists of valuations on the function field of X can be metrized, see, for example, [BPR16, 5.58]. The image of the embedding $\Delta_{\mathcal{X}} \rightarrow X^{\text{an}}$ clearly lies in this part, and one can show that the embedding is isometric with respect to this metric.

Let \mathcal{Y} be the blow-up of a point $\sigma \in E_i \cap E_j$. Then $\Delta_{\mathcal{Y}}$ is obtained from $\Delta_{\mathcal{X}}$ by the subdivision of the edge that joins $[E_i]$ and $[E_j]$ and that corresponds to the intersection of E_i and E_j that has been blown up; the new point is the divisorial valuation corresponding to the exceptional divisor of the blow-up. If \mathcal{Y} is a blow-up of a smooth point $x \in E_i$ of \mathcal{X}_s then $\Delta_{\mathcal{Y}}$ is obtained by adjoining an edge to $[E_i]$ in $\Delta_{\mathcal{X}}$, so the latter dual intersection complex can be naturally regarded as a subgraph of $\Delta_{\mathcal{Y}}$.

For a blow-up $f : \mathcal{Y} \rightarrow \mathcal{X}$ with the exceptional divisor E there exists a natural map $r : \Delta_{\mathcal{Y}} \rightarrow \Delta_{\mathcal{X}}$ which retracts the edge containing the point $[E]$ if $f(E) \notin (\mathcal{X}_0)_{\text{sing}}$, or sends it to the barycenter of the interval joining $[E_i]$ and $[E_j]$ if $f(E) \subset E_i \cap E_j$. Since any model \mathcal{Y} that dominates \mathcal{X} is obtained as a sequence of blow-ups of points in the central fibre, the map r can be defined as a composition of such maps for any dominant $\mathcal{Y} \rightarrow \mathcal{X}$. The retraction map can be defined in a less ad hoc and still explicit way in any dimension, see [BFJ16, Theorem 3.1], [MN15, Proposition 3.1.7].

Let $\Omega \in H^0(X, \Omega_X^1)$. Define the weight function $\text{wt}_\Omega : \Delta_{\mathcal{X}} \rightarrow \mathbb{R} \cup \{+\infty\}$ as the function that takes the following values on the divisorial valuations

$$\text{wt}_\Omega([E_i]) = \frac{1 + \text{ord}_{E_i}(\Omega)}{N_i}$$

where $\text{ord}_{E_i}(\Omega)$ is the order of vanishing at the divisor E_i of Ω regarded as the rational section of the relative canonical bundle $\omega_{\mathcal{Y}/R}$ of a model that

has E_i as one of the components of the central fibre. By Proposition 4.2.4 [MN15] wt_Ω is well-defined (i.e. does not depend on the model \mathcal{Y}) and by Proposition 4.4.5 *loc.cit* its extension by continuity to the whole of X^{an} gives rise to a function that is piece-wise affine on the faces of $\Delta_{\mathcal{X}}$ for any snc model \mathcal{X} .

It follows from this definition that the weight function is compatible with the embeddings of dual intersection complexes: if $f : \mathcal{Y} \rightarrow \mathcal{X}$ is a dominant morphism of snc models, then

$$\text{wt}_\Omega^{\mathcal{Y}}|_{\Delta_{\mathcal{X}}} = \text{wt}_\Omega^{\mathcal{X}}$$

Furthermore, by [MN15, Proposition 3.1.6] if $r : \Delta_{\mathcal{Y}} \rightarrow \Delta_{\mathcal{X}}$ is the retraction then $\text{wt}_\Omega^{\mathcal{Y}}(v) \geq \text{wt}_\Omega^{\mathcal{X}}(r(v))$.

An alternative treatment of the weight function using the non-archimedean analytic techniques can be found in [Tem16].

3 Limits of flat curves

3.1 Models and charts

From now on $K = \mathbb{C}_{\text{mer}}$.

Definition 3.1.1 (Snc model of a pair). *A pair (\mathcal{X}, Ω') of a model \mathcal{X} and a relative 1-form Ω' such that $\Omega'|_X = \Omega$ is called a model of the pair (X, Ω) . It is called an snc model of the pair (X, Ω) if the reduction of $\text{div}(\Omega') \cup \mathcal{X}_s$ is snc.*

Lemma 3.1.2. *Let (\mathcal{X}, Ω') be a model of a pair (X, Ω) where X is a projective curve. Then there exists an snc model \mathcal{Y} and a dominant morphism $f : \mathcal{Y} \rightarrow \mathcal{X}$ such that $(\mathcal{Y}, f^*\Omega')$ is an snc model of the pair (X, Ω)*

Proof. Suffices to take \mathcal{Y} to be a log-resolution of $\mathcal{X}_0 \cup \text{div}(\Omega)$. Indeed, $\text{div}(f^*\Omega) = f^*\text{div}(\Omega)$. \square

Admitting a slight abuse of notation, given a K -variety X and its model \mathcal{X} over R we will denote by X_s (resp. \mathcal{X}_s) the fibres for s close enough to 0 of the corresponding families over a disc (resp. punctured disc). We will also denote ω_0 the $(1, 1)$ -form $\frac{i}{2}(\Omega_0 \wedge \bar{\Omega}_0)$ defined on the smooth part of \mathcal{X}_0 .

Proposition 3.1.3. *Let (\mathcal{X}, Ω') be an snc model of a pair (X, Ω) . Let σ be an edge and $\{i, j\} \in \partial\sigma$. Assume that $\text{div}(\Omega) \cup E_i \cup E_j$ is a divisor with snc support. Let $x_\sigma, y_\sigma \in \mathcal{O}_{\mathcal{X}_s, \sigma}$ be the local equations of E_i, E_j , respectively, $x_\sigma^{N_i} y_\sigma^{N_j} = t$ be the local equation of $E_i \cup E_j$ near σ , a, b are integers, $a \geq 0, b \leq 0$ such that $aN_j + bN_i = (N_i, N_j)$ and put $z_\sigma = x_\sigma^a y_\sigma^{-b}$. Then for s sufficiently close to 0*

$$\frac{i}{2}(\Omega'_s \wedge \bar{\Omega}'_s) = |s|^{2(\text{wt}_\Omega(E_j) + \frac{b}{N_j}k-1)} |z_\sigma|^{2k} |u|^{2d} \frac{|z_\sigma|}{|z_\sigma|} \wedge d \text{Arg } z_\sigma$$

where $z_\sigma = x_\sigma^a y_\sigma^{-b}$, $k = \frac{(\text{wt}_\Omega([E_i]) - \text{wt}_\Omega([E_j]))N_i N_j}{(N_i, N_j)}$ and $u \in \mathcal{O}_{\mathcal{X}_0, \sigma}^\times$.

Proof. Denote $c = \text{ord}_{E_i}(\Omega')$, $d = \text{ord}_{E_j}(\Omega')$. By a standard computation (see, for example, [MN15, 4.1.4]) the form $\frac{dy}{x_\sigma^{N_i-1} y_\sigma^{N_j}}$ generates $\omega_{\mathcal{X}/R}$. Since

$$\frac{dz_\sigma}{z_\sigma} = a \frac{dx_\sigma}{x_\sigma} - b \frac{dy_\sigma}{y_\sigma} = \left(b - \frac{aN_j}{N_i}\right) \frac{dy_\sigma}{y_\sigma} + \frac{adt}{N_i t}$$

it follows that

$$\text{div}\left(\frac{dz_\sigma}{z_\sigma}\right) = \text{div}\left(\frac{dy_\sigma}{y_\sigma}\right) = (N_i - 1)E_i + (N_j - 1)E_j$$

Then $\Omega'_s = ux_\sigma^{c-N_i+1} y_\sigma^{d-N_j+1} \frac{dz_\sigma}{z_\sigma}$ for some unit $u \in \mathcal{O}_{\mathcal{X}_0, \sigma}^\times$. Since

$$\frac{dz_\sigma}{z_\sigma} \wedge \frac{d\bar{z}_\sigma}{\bar{z}_\sigma} = -2i \frac{d|z_\sigma|}{|z_\sigma|} \wedge d \text{Arg } z_\sigma$$

it is left to show that

$$x_\sigma^{c-N_i+1} y_\sigma^{d-N_j+1} = s^{\text{wt}_\Omega([E_j]) + \frac{b}{N_j}k-1} z_\sigma^k$$

Note that by definition $\text{wt}_\Omega([E_i]) = \frac{1+c}{N_i}$, $\text{wt}_\Omega([E_j]) = \frac{1+d}{N_j}$ and $b\frac{N_j}{N_i} + ak = (\text{wt}_\Omega([E_i]) - \text{wt}_\Omega([E_j]))N_i$. Expanding and simplifying the rhs we get

$$\begin{aligned} & x_\sigma^{\text{wt}_\Omega([E_j])N_i + b\frac{N_i}{N_j}k - N_i} y_\sigma^{1+d+bk-N_j} x_\sigma^{ak} y_\sigma^{-bk} \\ &= x_\sigma^{\text{wt}_\Omega([E_j])N_i - N_i} y_\sigma^{1+d-N_j} \\ &= x_\sigma^{1+c-N_i} y_\sigma^{1+d-N_j} \end{aligned}$$

□

Corollary 3.1.4. *Let (\mathcal{X}, Ω') be an snc model of a pair (X, Ω) . Assume $\text{wt}_\Omega([E_i]) < \text{wt}_\Omega([E_j])$ for all $j \in \text{St}(i)$. Then $\Omega/s^{\text{wt}_\Omega([E_j]) + b/N_j - 1}|_{E_i}$ is regular and non-zero.*

We will need an snc model of X that satisfies the following technical assumption in order to describe the collapsed limit in Theorem 3.3.2:

Assumption A. *For all prime divisors $E_i, E_j \subset \mathcal{X}_0$ and for each $\sigma \in E_i \cap E_j$ there exists a neighbourhood U of σ such that U_t is connected for t close enough to 0.*

Lemma 3.1.5. *For any pair (X, Ω) there exists a finite extension $K' \supset K$ such that the pair $(X \otimes K', \Omega \otimes K')$ has an snc model that satisfies Assumption A.*

Proof. By semi-stable reduction [MKKSD73] there exists a finite extension $K' \supset K$ such that $X \otimes K'$ has an snc model \mathcal{X} with all the irreducible components of \mathcal{X}_0 having multiplicity 1, so the Assumption A is satisfied. Then notice that Assumption A is stable under blow-ups and apply Lemma 3.1.2. \square

Proposition 3.1.6. *Let (\mathcal{X}, Ω') be an snc model of a pair (X, Ω) that satisfies Assumption A. Then there exists a cover $U^\alpha \subset X(\mathbb{C})$ of a neighbourhood of \mathcal{X}_0 that is indexed by vertices and edges of $\Delta_{\mathcal{X}}$ that satisfies the following properties:*

(i) *for all edges σ , the set U^σ is defined in a neighbourhood of σ by the inequalities*

$$C_\sigma |s|^{a/N_i} \leq |z_\sigma| \leq D_\sigma |s|^{-b/N_j}$$

for some constants $C_\sigma, D_\sigma > 0$, U_s^σ is connected for all s close enough to 0;

(ii) *for any vertex i and any edge σ , $U^i \cap U^\sigma \neq \emptyset$ if and only if $i \in \partial\sigma$;*

Proof. Let ϵ be a number such that $x_\sigma^{N_i} y_\sigma^{N_j} = s$ are the equations of \mathcal{X}_s in the total space of the degeneration for $|s| < \epsilon$ for all edges $\sigma \in \Delta_{\mathcal{X}}$.

Observe that the inequalities from the property (i) can be rewritten as $|x_\sigma| \leq \frac{1}{C_\sigma^{N_j/(N_i, N_j)}}$ and $|y_\sigma| \leq \frac{1}{D_\sigma^{N_i/(N_i, N_j)}}$, and pick the constants C_σ, D_σ so that $U^\sigma \cap U^\tau \neq \emptyset$ for each vertex i and each $\sigma, \tau \in \text{St}(i)$. The property (i) follows from Assumption A.

The complement of the union of U^σ over all edges σ in the neighbourhood of the special fibre defined by the inequality $|s| < \epsilon$ consists of connected components W^i that are in bijective correspondence with irreducible components $E_i \subset \mathcal{X}_0$. Define $U^i = W^i \cup \bigcup_{\sigma \in \text{St}(i)} \partial U^\sigma$. Then the property (ii) is satisfied by construction. \square

Lemma 3.1.7. *Let $u \in \mathcal{O}_{\mathcal{X}, \sigma}^\times$ and let W be a set defined by the inequalities $|s|^\alpha \leq |z_\sigma| \leq |s|^\beta$, $|s| < \epsilon$ in the neighbourhood of σ . If $\alpha \leq a/N_i$, $\beta \geq -b/N_j$ then $\sup_{x \in W_s} |u| = O(1)$, and if $\alpha < a/N_i$, $\beta > -b/N_j$ then $\sup_{x \in W_s} |u| = o(1)$, as $|s| \rightarrow 0$.*

Proof. The conclusion follows immediately after observing that the set W_s is the intersection of the curve $x^{N_i} y^{N_j} = s$ and the rectangle $|x| \leq |s|^{a/N_i - \alpha}$, $|y| \leq |s|^{\beta + b/N_j}$. \square

3.2 Asymptotic distance estimates

Lemma 3.2.1. *Let (\mathcal{X}, Ω') be an snc model of the pair (X, Ω) . Consider the fibres X_s with the Kähler metric $\omega_s = \frac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$. Pick functions $z_\sigma \in \mathcal{O}_{\mathcal{X}, \sigma}$*

as in Proposition 3.1.3 and assume that

$$\Omega' = c_\sigma z_\sigma^\alpha s^\beta (1+u) dz_\sigma$$

for some $\alpha, \beta \in \mathbb{Z}$, and $u \in \mathfrak{m} \subset \mathcal{O}_{\mathcal{X}_s, \sigma}$.

Let $\{\eta_s, \eta'_s\}$ be a collection of points in U_s^σ and let $\gamma_s \subset U_s^\sigma$ be a shortest path between the points η_s and η'_s . Then the length of γ_s as $s \rightarrow 0$ is

$$|c_\sigma| \cdot |\ln|z_\sigma(\eta'_s)| - \ln|z_\sigma(\eta_s)|| \cdot |s|^\beta (1 + o(1))$$

if $\alpha = -1$ and

$$(|z_\sigma(\eta'_s)|^{\alpha+1} - |z_\sigma(\eta_s)|^{\alpha+1}) \frac{|s|^\beta}{\alpha+1} (1 + o(1))$$

otherwise.

Proof. The Riemannian metric tensor on in polar coordinates $\rho = |z_\sigma|, \theta = \text{Arg } z_\sigma$ is given by the expression

$$g_s = |c_\sigma|^2 |z_\sigma|^{2\alpha} |s|^\beta |1+u|^2 (d\rho \otimes d\rho + \rho^2 d\theta \otimes d\theta)$$

in the neighbourhood of σ . Let $\gamma'_s : [0, 1] \rightarrow U_s^\sigma$ be the path given by

$$\gamma'_s(\tau) = ((1-\tau)|z_\sigma(\eta_s)| + \tau|z_\sigma(\eta'_s)|) \exp(i \text{Arg}(\eta_s))$$

and $\gamma''_s : [0, 1] \rightarrow U_s^\sigma$ be defined by

$$\gamma''_s(\tau) = |z_\sigma(\eta'_s)| \exp(i(\tau \text{Arg } \eta_s + (1-\tau) \text{Arg } \eta'_s))$$

Clearly,

$$\frac{d}{d\tau} \gamma'_s(\tau) = (|z_\sigma(\eta'_s)| - |z_\sigma(\eta_s)|) d\rho \quad \frac{d}{d\tau} \gamma''_s(\tau) = (\text{Arg } \eta'_s - \text{Arg } \eta_s) d\theta$$

Assume for definiteness that $|z_\sigma(\eta'_s)| > |z_\sigma(\eta_s)|$. Then

$$L(\gamma'_s) = \int_0^1 \sqrt{g_s(\dot{\gamma}'_s(\tau), \dot{\gamma}'_s(\tau))} d\tau = \int_{|z_\sigma(\eta_s)|}^{|z_\sigma(\eta'_s)|} |c_\sigma| \rho^\alpha |s|^\beta |1+u(\rho e^{i \text{Arg } \eta_s}, s)| d\rho$$

Denote $I_{\epsilon, s} = [|s|^{a/N_i + \epsilon}, |s|^{-b/N_j - \epsilon}]$ and let $H_s = [|z_\sigma(\eta_s)|, |z_\sigma(\eta'_s)|]$; denote $A_{\epsilon, s}, B_{\epsilon, s}$ the endpoints of the interval $I_{\epsilon, s} \cap H_s$. The latter integral can be represented, for $\epsilon > 0$, as the sum of two integrals

$$\begin{aligned} & \int_{H_s} |c_\sigma| \rho^\alpha |s|^\beta |1+u(\rho e^{i \text{Arg } \eta_s}, s)| d\rho = |c_\sigma| |s|^\beta \left(\int_{H_s \setminus I_{\epsilon, s}} \rho^\alpha |1+u(\rho e^{i \text{Arg } \eta_s}, s)| d\rho + \right. \\ & \left. + \int_{I_{\epsilon, s}} \rho^\alpha |1+u(\rho e^{i \text{Arg } \eta_s}, s)| d\rho \right) \end{aligned}$$

Let us first consider the case $\alpha = -1$. By Lemma 3.1.7

$$\begin{aligned} L(\gamma'_s) &= |c_\sigma| |s|^\beta \sup_{\epsilon > 0} (\ln B_{\epsilon,s}/A_{\epsilon,s})(1 + o(1)) + \\ &\quad + C \max\{A_{\epsilon,s} - |z_\epsilon(\eta_s)|, 0\} + C \max\{|z_\epsilon(\eta'_s)| - B_{\epsilon,s}, 0\} \\ &= |c_\sigma| |s|^\beta \ln |z_\sigma(\eta'_s)|/|z_\sigma(\eta_s)|(1 + o(1)) \end{aligned}$$

Similarly, one derives for $\alpha \neq -1$,

$$L(\gamma'_s) = \frac{|s|^\beta}{\alpha + 1} (|z_\sigma(\eta'_s)|^{\alpha+1} - |z_\sigma(\eta_s)|^{\alpha+1})(1 + o(1))$$

Clearly, $L(\gamma_s) \leq L(\gamma'_s) + L(\gamma''_s)$ and $L(\gamma''_s) = O(|s|^\beta |z_\sigma(\eta'_s)|)$.

On the other hand, denoting the polar coordinates of γ_s by $\gamma_{s,\rho}, \gamma_{s,\theta}$ we have

$$\begin{aligned} L(\gamma_s) &= \int_0^1 \sqrt{g_s(\dot{\gamma}(\tau), \dot{\gamma}(\tau))} d\tau \\ &= \int_0^1 \sqrt{|c_\sigma|^2 (\gamma_{s,\rho})^{2\alpha} |s|^{2\beta} |1 + u(\gamma_s)|^2 ((\dot{\gamma}_{s,\rho})^2 + \gamma_{s,\rho}^2 (\dot{\gamma}_{s,\theta})^2)} d\tau \\ &\geq \int_0^1 |c_\sigma| (\gamma_{s,\rho})^\alpha |s|^\beta |1 + u(\gamma_s)| (\dot{\gamma}_{s,\rho}) d\tau \end{aligned}$$

The last expression has the same asymptotics as $L(\gamma'_s)$ and we conclude. \square

Lemma 3.2.2. *Let (\mathcal{X}, Ω') be an snc model of a pair (X, Ω) and let E_i be an irreducible component of \mathcal{X}_0 . Let $a_s, b_s \in \mathcal{X}_s, |s| < \varepsilon$ be collections of points such that $\lim_{s \rightarrow 0} a_s = a_0, \lim_{s \rightarrow 0} b_s = b_0$ for some $a_0, b_0 \in E_i \subset \mathcal{X}_s$. Assume that $\text{wt}_\Omega([E_i]) = 1, \text{wt}_\Omega([E_j]) > 1$ for all $[E_j] \in \text{St}([E_i])$ and that $a_0, b_0 \notin (\mathcal{X}_0)_{\text{sing}}$. Let $\gamma_s \subset U^i$ be a shortest path for the metric $\omega_s = \frac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$ on X_s and that connects a_s and b_s . Then*

$$\lim_{s \rightarrow 0} l(\gamma_s) = l(\gamma_0)$$

and the limit is finite.

Proof. As was observed before in the proof of Lemma 3.1.3 the form $\Omega'_0|_{E_i}$ can be written down in a Zariski neighbourhood of $\sigma \in E_i \cap E_j$ as

$$\Omega'_0|_{E_i} = u x_\sigma^{(\text{wt}_\Omega([E_i])-1)N_i} y_\sigma^{(\text{wt}_\Omega([E_j])-1)N_j} \frac{dy_\sigma}{y_\sigma}$$

where $u \in \mathcal{O}_{\mathcal{X},\sigma}^\times$ and x_σ, y_σ are local equations of E_i, E_j respectively. We may assume that E_i is covered by such neighbourhoods, passing to a model where finitely many points of E_i are blown up if it is not the case; notice that in this case the assumptions about the weight remain true. Since $(\text{wt}_\Omega([E_j]) - 1)N_j > 0$ and integral, $\Omega'_0|_{E_i}$ is regular and non-zero. The statement then follows from the continuity of the form $\frac{i}{2}\Omega' \wedge \bar{\Omega}'$. \square

Lemma 3.2.3. *Let (\mathcal{X}, Ω') be an snc model of a pair (X, Ω) that satisfies Assumption A. Let $k = \min_{x \in \Delta_{\mathcal{X}}} \text{wt}_{\Omega}(x)$ and let $\Sigma \subset \Delta(\mathcal{X}_s)$ be the set of faces of $\Delta_{\mathcal{X}}$ where $\text{wt}_{\Omega} \equiv k$. Let $\Omega'' = \Omega'/s^{k-1}$ and let $\omega''_s = \frac{i}{2}(\Omega''_s \wedge \bar{\Omega}''_s)$. The asymptotics of the diameter of X_s as $s \rightarrow 0$ with respect to the Kähler metric $\omega_s = \frac{i}{2}(\Omega_s \wedge \bar{\Omega}_s)$ is*

$$\text{diam } X_s = (\text{diam}(X_0, \omega''_0) |s|^{k-1} (1 + o(1)))$$

if $\dim \Sigma = 0$, and

$$\text{diam } X_s = c \ln |s| |s|^{k-1} (1 + o(1))$$

if $\dim \Sigma = 1$, for some constant c .

Proof. Consider the cover constructed in Proposition 3.1.6.

Since $\text{wt}_{\Omega'/t^m} = \text{wt}_{\Omega'} - m$ for any integer m , by Lemma 3.1.3

$$\text{diam } U_s^i = (\text{diam}(U_0^i, \omega''_0) |s|^{\text{wt}_{\Omega}([E_i])-1} (1 + o(1)))$$

for any vertex $[E_i] \in \Delta_{\mathcal{X}}$.

By the same consideration, after applying Lemmas 3.2.1 and 3.2.2 we get

$$\text{diam } U_s^\sigma = d_\sigma \ln |s| |s|^{\text{wt}_{\Omega}([E_i])-1} (1 + o(1))$$

when $\text{wt}_{\Omega}([E_i]) = \text{wt}_{\Omega}([E_j])$ for $[E_i], [E_j] \in \partial\sigma$,

$$\text{diam } U_s^\sigma = d_\sigma |s|^{\text{wt}_{\Omega}([E_i])-1} (1 + o(1))$$

when $\text{wt}_{\Omega}([E_i]) < \text{wt}_{\Omega}([E_j])$, for some positive real constants d_σ .

It follows that $\text{diam}((U^\beta)_s) = o(\text{diam}(U^\alpha)_s)$ for any $\alpha \in \Sigma$ and $\beta \notin \Sigma$. Now notice that $\text{diam } U^\sigma$ has the same asymptotics as $|s| \rightarrow 0$ for all edges $\sigma \in \Sigma$ if $\dim \Sigma = 1$, and that $\text{diam } U^i$ have the same asymptotics for all $i \in \Sigma$ if $\dim \Sigma = 0$. Therefore, $\text{diam}(X_s, \omega''_s)$ in the first case tends to the diameter of (X_0, ω''_0) , and $\text{diam}(X_s, \omega''_s)$ tends to some positive real number (the diameter of the graph $\Delta_{\mathcal{X}}$ with lengths of edges adjusted) in the second case. The conclusion of the Lemma follows. \square

To describe the Gromov-Hausdorff limit of a family of curves $(X_s, \tilde{\omega}_s)$ we will distinguish two cases: *collapsed* limit, when the diameter of (X_s, ω_s) is of order $(\ln |t| |t|^k)(1 + o(1))$ for some k , and *non-collapsed* limit otherwise.

3.3 Shape of the limit

If X and Y are two (pseudo)metric spaces and $R \subset X \times Y$ is a relation one defines the *distortion* of R to be

$$\text{dis } R = \sup_{(x,y),(x',y') \in R} |d_X(x,y) - d_Y(x',y')|$$

One easily observes that if both projections of R on X and Y are surjective then $d_{GH}(X, Y) \leq \text{dis } R/2$, and conversely, for any metric spaces X, Y such that $d_{GH}(X, Y) \leq \epsilon$ the relation $R_\epsilon = \{(x, y) \in X \times Y \mid d(x, y) < \epsilon\}$, where d is the metric on $X \sqcup Y$ that realizes the bound, satisfies $\text{dis } R_\epsilon < 2\epsilon$ ([BBI01, Theorem 7.3.25]).

Recall that a metric is called an *inner metric* if the distance between two points is defined as an infimum of a length functional on some class of admissible paths (see Section 2 of [BBI01] for the detailed definition).

Lemma 3.3.1. *Let (X, d) and (Y, d') be two pseudometric spaces, let $\cup_{i=1}^n U_i = X$ and $\cup_{j=1}^n V_j = Y$ be two coverings by path-connected sets, and let $R \subset X \times Y$ be a relation. Assume that*

- (i) *the 1-nerves of $\{U_i\}$ and $\{V_j\}$ are isomorphic, that is, for all $i \neq j$ the connected components U_σ of $U_i \cap U_j$ are in bijective correspondence with connected components V_σ of $V_i \cap V_j$;*
- (ii) *for all i, j (including $i = j$), for all connected components $U_\sigma \subset U_i \cap U_j$ the relation $R \cap (U_\sigma \times V_\sigma)$ projects surjectively on U_σ, V_σ ;*

Then there exists a number N , depending only on the nerve of $\{U_i\}$ and $\{V_i\}$ such that

$$\text{dis } R \leq \max_{i_0, \dots, i_N} \sum_{k=1}^N \text{dis } R \cap (U_{i_k} \times V_{j_k})$$

where the maximum is taken over such sequences $\{i_k\}$ that $U_{i_k} \cap U_{i_{k+1}} \neq \emptyset$ for all k .

Proof. Let N be a number such that any path in X (or, equivalently, Y) passes consecutively through a sequence of elements of the cover U_{i_0}, \dots, U_{i_L} , $L \leq N$, so that any finite subsequence starting and ending with the same element occurs at most once.

Let $x_0, x_L \in X, x_0 \in U_{i_0}, x_L \in U_{i_L}$ and subdivide the shortest path between x_0 and x_L by adding points x_k so that $x_k, x_{k+1} \in U_{i_k}$ for some sequence $\{i_k\}$. Pick y_0, \dots, y_L so that $y_k, y_{k+1} \in V_{i_k}, (x_i, y_i) \in R$. Then

$$\begin{aligned} d'(y_0, y_L) &\leq \sum_{k=0}^L d'(y_k, y_{k+1}) \\ &\leq \sum_{i=0}^L d(x_k, x_{k+1}) + \text{dis } R \cap (U_{i_k} \times V_{i_k}) \\ &\leq d(x_0, x_L) + \sum_{k=0}^L \text{dis } R \cap (U_{i_k} \times V_{i_k}) \end{aligned}$$

By the symmetric argument we obtain that

$$d(x_0, x_L) \leq d'(y_0, y_L) + \sum_{k=0}^L \text{dis } R \cap (U_{i_k} \times V_{i_k})$$

Since $L \leq N$, we can conclude. \square

Theorem 3.3.2 (Collapsed limit). *Let (\mathcal{X}, Ω') be an snc model of (X, Ω) that satisfies Assumption A. Let $k = \min_{x \in \Delta_{\mathcal{X}}} \text{wt}_{\Omega}(x)$ and let $\Sigma \subset \Delta(\mathcal{X}_s)$ be the union of vertices and edges of $\Delta_{\mathcal{X}}$ where $\text{wt}_{\Omega} \equiv k$. Assume that $\dim \Sigma = 1$. Let $x \sim y$ for $x, y \in \Delta_{\mathcal{X}}$ if and only if there exists a path $\gamma : [0, 1] \rightarrow \Delta_{\mathcal{X}}$ joining x and y such that $|\gamma^{-1}(\Sigma)| < \infty$. The Gromov-Hausdorff limit of $(X_s, \tilde{\omega}_s)$ as $s \rightarrow 0$ is isometric to $\Delta_{\mathcal{X}}/\sim$ endowed with the metric that stretches each edge σ by the factor $|c_{\sigma}|$ and renormalized so that $\text{diam}(\Delta_{\mathcal{X}}/\sim) = 1$.*

Proof. We adopt the notation for local coordinates from Proposition 3.1.6. By Lemma 3.2.3, $\text{diam } X_s = c \ln|s| |s|^{k-1} (1 + o(1))$ for some constant c . We identify each edge σ with an interval $[-\frac{b|c_{\sigma}|}{N_j c}, \frac{a|c_{\sigma}|}{N_i c}]$. Define the map $f_s : X_s \rightarrow \Delta_{\mathcal{X}}$ as follows:

$$f_s(x, s) = \begin{cases} \frac{|c_{\sigma}|}{c} \Xi_{\sigma} \left(\frac{\ln|z_{\sigma}|}{\ln|s|} \right) \in [\sigma], & \text{if } x \in U^{\sigma} \\ [E_i] & \text{if } x \in U^i \end{cases}$$

where $\Xi_{\sigma} : [-b/N_j - \ln D_{\sigma}, a/N_i - \ln C_{\sigma}] \rightarrow [-b/N_j, a/N_i]$ is the linear bijection. Let R_s be the graph of f_s . It follows from Assumption A and the fact that $f_s|_{U_s^{\sigma}}$ is surjective onto $[\sigma]$ that R_s satisfies the conditions of Lemma 3.3.1.

Consider the metric $\tilde{\omega} = \frac{\omega_s}{\text{diam } X_s^2}$ on X . It follows from Lemma 3.2.1 that $\text{dis } R_s \cap (U^{\sigma} \times [\sigma]) \rightarrow 0$ as $s \rightarrow 0$ for $\sigma \in \Sigma$. Since by Lemma 3.2.3 $\text{diam } U_s^{\sigma} = o(\text{diam } X_s)$ for any $[\sigma] \not\subseteq \Sigma$, $\text{dis } R_s \cap (U^{\sigma} \times [\sigma]) \rightarrow 0$ for such $[\sigma]$. Since by Lemma 3.2.2 $\text{diam } U_s^i \rightarrow 0$, $R_s \cap (U^i \times [E_i]) \rightarrow 0$ as $s \rightarrow 0$.

Therefore, by Lemma 3.3.1, $\text{dis } R_s \rightarrow 0$ as $s \rightarrow 0$. If $q : \Delta_{\mathcal{X}} \rightarrow \Delta_{\mathcal{X}}/\sim$ is the projection on the quotient then clearly $\text{dis } q = 0$, since Δ/\sim is the metric space associated to the pseudometric space Δ . It follows that if \bar{R}_s is the graph of $q \circ f_s$ then $\text{dis } \bar{R}_s \rightarrow 0$ as $s \rightarrow 0$. It follows that X_s converges in the sense of Gromov-Hausdorff to $\Delta_{\mathcal{X}}/\sim$. \square

Remark. The metric graph $\Delta_{\mathcal{X}}/\sim$ does not depend on the choice of a model \mathcal{X} . Indeed, observe that if \mathcal{Y} is a model that dominates \mathcal{X} then $\Delta_{\mathcal{Y}}$ contracts onto $\Delta_{\mathcal{X}}$ and it follows from Proposition 4.3.4 [MN15] that $\Sigma_{\mathcal{Y}} = \Sigma_{\mathcal{X}}$, then use the fact that any two models are related by a series of blow-ups and blow-downs of points in the special fibre.

As was observed in [BN16, Lemma 3.4.5], if the genus of E_i is 0 then $[E_i] \in \Delta_{\mathcal{X}}$ belongs to Σ if and only if some adjacent edge belongs to Σ . In the case of non-collapsed limit the components E_i of the central fibre such that $[E_i] \in \Sigma$ can thus be regarded as surfaces endowed with a flat metric, since by Lemma 3.2.2 the restriction of Ω'/t^{k-1} to each such E_i is a regular 1-form.

Theorem 3.3.3 (Non-collapsed limit). *Let (\mathcal{X}, Ω') be an snc model of (X, Ω) as in the previous theorem and assume that $\dim \Sigma = 0$. Let \sim be the smallest equivalence relation on X_0 containing the relation defined as follows:*

1. $\sigma \sim \tau$ for $\sigma \subset E_i, \tau \subset E_j$ if $[E_i], [E_j] \in \Sigma$ and there exists a path $\gamma : [0, 1] \rightarrow \Delta_{\mathcal{X}}$ such that $\gamma(0) = [E_i], \gamma(1) = [E_j]$, initial segment of γ passes through $[\sigma]$, and final segment of γ passes through $[\tau]$,
2. $x \sim \sigma$ if $i \notin \Sigma, x \in E_i, \partial[\sigma] \cap \Sigma \neq \emptyset$, and there exists a path $\gamma : [0, 1] \rightarrow \Delta_{\mathcal{X}}$ such that $\gamma(0) = [E_i]$, and $[\sigma] \subset \gamma([0, 1])$.

Then the limit of $(X_s, \tilde{\omega}_s)$ as $s \rightarrow 0$ is $X_0/\sim = (\cup_{[E_i] \in \Sigma} E_i)/\sim$ with the metric renormalized so that the diameter of the space is 1.

Proof. The central fibre \mathcal{X}_0 is a deformation retract of the total space X , (see [Cle77] for an explicit construction of a retraction). Let $r : X \rightarrow \mathcal{X}_0$ be a retraction, and denote r_s its restriction to X_s . Let R_s be the graph of r_s .

Then by Lemmas 3.2.3 and 3.2.2, $\text{dis } R_s \rightarrow 0$ as $s \rightarrow 0$ if we consider X_0 as a pseudo-metric space with the metric given by the form Ω'/t^{k-1} . The relation \sim described in the statement of the Theorem identifies points at distance 0 between each other. It then follows that X_0/\sim is the Gromov-Hausdorff limit of X_s as $s \rightarrow 0$. \square

We will now study for the purpose of illustration of Theorem 3.3.2 the possible shapes of the Gromov-Hausdorff limits it describes.

We will use the description of the graph Laplacian of the weight function due to Baker and Nicaise [BN16], which we quickly recall. By a *weighted graph* we understand a metrized graph Γ with the set of vertices $V(\Gamma)$ and with infinite edges allowed, and a pair of functions $N, g : V(\Gamma) \rightarrow \mathbb{Z}$. Given an snc model (\mathcal{X}, Ω) of (X, Ω) , one associates a weighted graph as follows: take $\Delta_{\mathcal{X}}$ and attach infinite edges at the vertices which correspond to components having non-trivial intersection with $\text{div}(\Omega)$. A *divisor* on Γ is a formal combination of the vertices of Γ . Let $f : \Gamma \rightarrow \mathbb{R}$ be a function that is affine on every edge of Γ , then the *Laplacian of f* is the divisor $\Delta(f) = \sum_{i \in V(\Gamma)} a_i v_i$ where a_i is the sum of outward slopes of f at v_i . The *canonical divisor of Γ* is the divisor

$$K_{\Gamma} = \sum_v N_v (\text{val}(v) + 2g(v) - 2)v$$

where $\text{val}(v)$ is the valency of the vertex v . By [BN16, Theorem 3.2.3] each infinite edge running from a vertex v towards a zero $x \in X$ of a differential form Ω has an outgoing slope $N_v(1 + \deg_x(\Omega))$ and $\Delta(\text{wt}_\Omega) = K_\Gamma$.

Proposition 3.3.4. *For any $k \geq 1$ the wedge sum of k circles can occur as a limit (in the sense of Theorem 3.3.2) of a family of curves of genus $2k+1$. This family admits an snc model such that all the components of the central fibre are rational.*

Proof. For the case $k = 1$ take an elliptic curve over \mathbb{C}_{mer} of bad reduction and any regular form Ω .

For $k > 1$, we will construct a weighted graph Γ^+ with a subgraph Γ of non-infinite edges of Γ^+ , and a weight function $w : \Gamma^+ \rightarrow \mathbb{R}$ affine on each edge, we will then apply [MUW17, Theorem 6.3] to get a pair (X, Ω) that gives rise to the Gromov-Hausdorff limit of the desired genus.

Let Γ be a chain of $g = 2k - 1$ cycles, C_1, \dots, C_{2k-1} joined consecutively, each cycle C_i consisting of two edges connecting vertices c_i and c_{i+1} . Attach two infinite edges (corresponding to the zeroes of the differential form) to both edges of each $C_{2i}, 1 \leq i \leq k$, subdividing them in the points of attachment $a_{2i-1}, a_{2i}, b_{2i-1}, b_{2i}, 1 \leq i \leq k$; call the resulting graph Γ^+ . Define the function $w : V(\Gamma^+) \rightarrow \mathbb{Z}$ as follows:

$$w(c_i) = 0 \quad w(a_i) = w(b_i) = 1$$

Extend it in the affine fashion to the edges of Γ^+ , with the outgoing slope 2 on the infinite edges. Put $N = 0, g = 0$ on all vertices, and let all edges be of length 1. One checks that $\Delta(\text{wt}_\Omega) = K_{\Gamma^+}$. Clearly, the minimality locus of w is the union of odd cycles C_{2i-1} , and the quotient by the equivalence relation described in Theorem 3.3.2 is of genus k .

In order to apply [MUW17, Theorem 6.3] we need to find a piecewise affine function $f : \Gamma \rightarrow \mathbb{R}$ affine on the edges such that the tropical divisor $\sum a_i + b_i$ can be presented as $K_\Gamma + \Delta(f)$. We claim that such f can be taken to be $-w|_\Gamma$. We can then check that the extended level graph $\Gamma^+(f)$ does not have inconvenient vertices (Definition 6.2, *loc.cit.*) because for all vertices one of the outgoing slopes is 0. Therefore the only condition of Theorem 6.3, *loc.cit.* that is left to check is about the edges, and it is fulfilled, since all horizontal edges belong to a horizontal simple cycle. By Theorem 6.3, *loc.cit.* there exists a variety X over \mathbb{C}_{mer} and a 1-form Ω on X and an snc model (\mathcal{X}, Ω') of the pair (X, Ω) such that $\Delta_{\mathcal{X}}$ is isomorphic to Γ and $\text{wt}_\Omega = w$ on Γ . \square

Let us conclude with two questions.

Question 1: can Proposition 3.3.4 be proved (perhaps with some additional conditions) for a fixed genus of the elements of the family \mathcal{X} ? for a given fixed \mathcal{X} , constructing appropriate Ω ?

Question 2: can one characterise non-collapsed limits starting from the description of Theorem 3.3.3?

References

- [BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001.
- [Ber90] Vladimir Berkovich. *Spectral theory and analytic geometry over non-archimedean fields*. Number 33 in *Mathematical Surveys and Monographs*. American Mathematical Soc., 1990.
- [BFJ16] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Singular semipositive metrics in non-archimedean geometry. *Journal of Algebraic Geometry*, 25(1):77–139, 2016.
- [BJ17] Sébastien Boucksom and Mattias Jonsson. Tropical and non-archimedean limits of degenerating families of volume forms. *Journal de l'École polytechnique*, 4:87–139, 2017.
- [BN16] Matthew Baker and Johannes Nicaise. Weight functions on Berkovich curves. *Algebra & Number Theory*, 10(10):2053–2079, 2016.
- [BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff. Non-archimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom.*, 3(1):63–105, 2016.
- [Cle77] C. H. Clemens. Degeneration of Kähler manifolds. *Duke Mathematical Journal*, 44(2):215–290, 1977.
- [DHL19] Antoine Ducros, Ehud Hrushovski, and François Loeser. Non-archimedean integrals as limits of complex integrals. *arXiv:1912.09162*, 2019.
- [GTZ13] Mark Gross, Valentino Tosatti, and Yuguang Zhang. Collapsing of abelian fibered Calabi–Yau manifolds. *Duke Mathematical Journal*, 162(3):517–551, 2013.
- [GTZ16] Mark Gross, Valentino Tosatti, and Yuguang Zhang. Gromov–Hausdorff collapsing of Calabi–Yau manifolds. *Communications in Analysis and Geometry*, 24(1):93–113, 2016.
- [GW00] Mark Gross and Pelham MH Wilson. Large complex structure limits of K3 surfaces. *Journal of Differential Geometry*, 55(3):475–546, 2000.

- [KS06] Maxim Kontsevich and Yan Soibelman. Affine structures and non-archimedean analytic spaces. *Progress in mathematics*, 244:321, 2006.
- [MKKSD73] David Mumford, George Kempf, Finn Knudsen, and Bernard Saint-Donat. Toroidal embeddings i. *Springer Lecture Notes*, 339, 1973.
- [MN15] Mircea Mustața and Johannes Nicaise. Weight functions on non-archimedean analytic spaces and the kontsevichsoibelman skeleton. *Algebraic Geometry*, 2(3):365–404, 2015.
- [MUW17] Martin Moeller, Martin Ulirsch, and Annette Werner. Realizability of tropical canonical divisors. *arXiv:1710.06401*, 2017.
- [NX16] Johannes Nicaise and Chenyang Xu. The essential skeleton of a degeneration of algebraic varieties. *American Journal of Mathematics*, 138(6):1645–1667, 2016.
- [NXY19] Johannes Nicaise, Chenyang Xu, and Tony Yue Yu. The non-archimedean SYZ fibration. *Compositio Mathematica*, 155(5):953–972, 2019.
- [OO18] Yuji Odaka and Yoshiki Oshima. Collapsing K3 surfaces, tropical geometry and moduli compactifications of Satake, Morgan–Shalen type. *arXiv:1810.07685*, 2018.
- [Tem16] Michael Temkin. Metrization of differential pluriforms on Berkovich analytic spaces. In *Nonarchimedean and Tropical Geometry*, pages 195–285. Springer, 2016.
- [TZ17] Valentino Tosatti and Yuguang Zhang. Collapsing hyperkähler manifolds. *arXiv:1705.03299*, 2017.
- [Zor06] Anton Zorich. Flat surfaces. *Frontiers in number theory, physics, and geometry I*, pages 439–585, 2006.

DEPARTMENT OF MATHEMATICS
 KU LEUVEN
 CELESTIJNENLAAN 200B
 B-3001 LEUVEN (HEVERLEE)
 BELGIUM
 dmitry.sustretov@kuleuven.be