Hessian metrics with distribution coefficients on a 2-sphere

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Abstract

Let Δ be a 2-sphere endowed with an affine structure away from a finite set of points $P \subset \Delta$, and assume that the monodromy of the associated connection ∇ on $\Delta \setminus P$ around any point from P is unipotent. I show that there exists a pseudo-metric tensor with distribution coefficients on Δ that is non-degenerate on $\Delta \setminus P$ and that locally is of the form ∇df for some convex function f. In particular, if X_∞ is the canonical nearby fibre of a Type III degeneration of K3 surfaces in Kulikov form, $\Delta_X \cong S^2$ is the dual intersection complex of the central fibre and Δ_X has simple affine structure singularities, existence of such "Hessian metric" on Δ_X implies that the map $H^1(\Delta_X, \Lambda^1) \to \operatorname{gr}^2_W H^2(X_\infty)$, constructed previously in [Sus22], where W is the monodromy weight filtration on $H^2(X_\infty)$ and Λ^1 is the push-forward of the sheaf of parallel 1-forms along the open embedding $\Delta \setminus P \hookrightarrow \Delta$, is an isomorphism.

Contents

1. Introduction

An affine structure on a manifold M is a choice of a torsion-free flat connection ∇ on its tangent bundle. This structure gives a natural setting to define a notion of convexity. One can consider the *sheaves of* (p, q)*-forms* [Shi07]: the sheaves of smooth sections of the bundle $\wedge^p T^*M \otimes \wedge^q T^*M$. The sheaves of (p, q)-forms form a bi-graded complex with two differentials: d and ∇. A Hessian of a function f is naturally represented by a $(1, 1)$-form ∇df, and a smooth function f on M is convex if ∇df is positive definite as a symmetric tensor.

A notion similar to the notion of (p, q)-forms has received recent interest in relation to the formalism of *superforms* [Lag12], developed by Lagerberg based on an idea of Berndtsson [Ber06, Section 8]. This formalism was developed with an eye towards application in tropical geometry and was also later adopted to the setting of Berkovich analytic spaces, see [CL12, Gub16]. The superforms form a bigraded complex, similar to the Dolbeault complex in complex geometry, with differentials d', d''. The cohomology groups of the rows of this complex compute [JSS19] the cohomology of certain sheaves on tropical varieties defined by Itenberg, Katzarkov, Mikhalkin and Zharkov [IKMZ19]. These cohomology groups, under certain conditions, are isomorphic to even graded pieces of the cohomology of the nearby fibres of a degeneration of complex projective manifolds.

In [Sus22] I define a graded sheaf of algebras $\Lambda^\bullet$ on the dual intersection complex Δ_X of the central fibre of a degeneration $f : X \to S$ of complex manifolds over a disc, provided that the central fibre $f^{-1}(0)$ is a strictly normal crossings divisor. I show that if

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additionally the monodromy action on a nearby fibre $f^{-1}(t), t \neq 0$ is unipotent, and the central fibre is projective, then for all $p, q \geq 0$ there exist morphisms

$$H^q(\Delta_X, \Lambda^p) \to \operatorname{gr}^W_{2p} H^{p+q}(X, \mathbb{Q}).$$

In contrast to the results of Itenberg et al. these morphisms in general are not guaranteed to be injective or surjective. In [Sus22, Section 6] I have studied the cohomology of sheaves Λ^p in the case of a Type III degeneration of K3 surfaces in the Kulikov form, and have found a sufficient condition for injectivity. In this case Δ_X is a 2-sphere and the sheaf Λ^1 defines an affine structure (which was also recently studied in [Eng18, AET19]) on a complement of a finite set P points ($|P| \leq 24$). The morphisms above are injective subject to existence of a certain combinatorial datum on Δ_X (a convex PL metric on a Λ^1-torsor over Δ_X). I also show that if the monodromy of the affine structure around each point $p \in P$ is unipotent and a primitive as an element of $\mathrm{SL}_2(\mathbb{Z})$ (in appropriate local coordinates), then the morphism above is surjective.

The purpose of this paper is to show that the sufficient condition for injectivity holds using the formalism of superforms on Δ_X, the foundations for which has been laid in [Sus22, Sections 5.2-3]. The main result of this paper may also be of independent interest:

Theorem A. Let Δ be the 2-sphere endowed with affine structure away from finitely many points $P = \{ p_1, \ldots, p_n \}$. Then there exists a positive symmetric d''-closed $(1,1)$-supercurrent on Δ that is bounded below by a strictly positive $(1,1)$-form.

For the proof of this statement see Proposition 4.2.1. By Proposition 3.4.1ii the current T from Theorem A has a local potential, i.e. locally $T = d'd''f$ for some convex (not necessarily smooth) function f. If T is smooth on an open set $U \subset \Delta \setminus P$, then it follows from these consideratiosn that it defines a Hessian metric on U, which explains the title of this paper.

This theorem is an analog in affine geometry of the theorem, due to Lamari [Lam99] and Buchsdal [Buc99] (see [BHPVdV15, IV.3] for an exposition), that states that any smooth compact complex surface with an even Betti number is Kähler. The role of the latter condition in the complex case translates via the exact sequence of sheaves of *real* vector spaces

$$0 \to \mathbb{R} \to \mathcal{O}_X \xrightarrow{\mathrm{Im}} \mathcal{H}_X \to 0,$$

where $\mathcal{H}_X$ is the sheaf of pluriharmonic functions, to the fact that the morphism

$$H^1(X, \mathbb{R}) \to H^1(X, \mathcal{O}_X)$$

in the associated long exact sequence is an isomorphism, implying further, crucially for the proof, that $H^1(X, \mathcal{H})$ injects into $H^2(X, \mathbb{R})$. A similar role is played in the affine situation by the sequence

$$0 \to \mathbb{R} \to A^1 \to \Lambda^1 \to 0,$$

where A^1, Λ^1 are the pushforwards from the complement of the finite sets of affine structure singularities of the sheaves of affine functions and parallel 1-forms, respectively. When Δ is a 2-sphere, we have $H^1(\Delta, \mathbb{R}) = 0$ and

$$H^1(\Delta, A^1) \to H^1(\Delta, \Lambda^1)$$

is injective, so the cohomology of the sheaf A^1 can be identified with the cohomology clases in $H^1(\Delta, \Lambda^1)$ represented by symmetric $(1,1)$-superforms. The key step in the proof, similarly to the proofs of Buchsdahl and Lamari, is showing that the "affine Bott-Chern" cohomology $H^1(\Delta, A^1)$ is naturally isomorphic to its dual vector space.

1.1. **Structure of the paper.** The background on superforms on simplicial complexes is recalled in Section 2. Section 3 contains the main preparation steps for the proof of the Theorem A: I describe a modification $\tilde{\Delta}$ of the 2-sphere Δ endowed with an affine structure with singularities, for which an analogue of Serre duality holds, prove a criterion for a existence of a Käler supercurrent, and show that the sheaf A^1 admits a certain resolution, so that a natural injective morphism $H^1(\tilde{\Delta}, A^1) \to H^1(\tilde{\Delta}, \Lambda^1)$ exists. In Section 4 I show that $H^1(\tilde{\Delta}, A^1)$ is naturally isomorphic to its dual, and use this to prove Theorem A.

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2. Background

2.1. **Superforms on real vector spaces.** Let V be a real vector space. We denote V^0 and V^1 two copies of V and call a *superform* on V a differential form on $\bar{V} = V^0 \oplus V^1$ that is invariant under translations along vectors from V^1.

The space of superforms admits a bidegree decompostion which is similar to the bidegree decomposition of forms in complex geometry. Let $\pi : V^0 \oplus V^1 \to V^0$ be the natural projeciton. We call pullbacks $\pi^*\alpha$ of p-forms on V^0 *superforms of bidegree* $(p, 0)$ on V. There exists a natural decomposition $T^*\bar{V} = T^*V^0 \oplus T^*V^1$ and a morphism $J : T^*\bar{V} \to T^*\bar{V}$ that swaps the two dirrect summands of the cotangent space. This morphism induces a self-map on the superforms. Denoting $\mathscr{A}^{p,0}$ the sheaf of $(p, 0)$-superforms, we denote

$$\mathscr{A}^{0,q} = J \cdot \mathscr{A}^{p,0} \qquad \mathscr{A}^{p,q} := \mathscr{A}^{p,0} \otimes_{\mathbb{R}} \mathscr{A}^{0,q}\,.$$

The de Rham differential applied to a (p, q)-superform yields a $(p+1, q)$-superform. We will denote de Rham differential d'. Similarly to the differential d^c of complex geometry, one introduces the second differential

$$d'' = J \circ d' \circ J : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}$$

Pick a collection of affine functions $x_1, \ldots, x_n$ on V. We will denote

$$d'x_I = d'x_{i_1} \wedge \ldots \wedge d'x_{i_p}, \qquad d''x_J = d''x_{j_1} \wedge \ldots \wedge d''x_{j_q}$$

for multi-indices $I = \{ i_1, \ldots, i_p \}, J = \{ j_1, \ldots, j_q \}, i_1 < \ldots < i_p, j_1 < \ldots < j_q$. One easily checks that d', d'', J are given by the following formulas

$$\begin{aligned} d'\alpha &= \textstyle\sum_{|I|=p,|J|=q} \dfrac{\partial f_{IJ}}{\partial x_i} d'x_i i \wedge d'x_I \wedge d''x_J \\ d''\alpha &= (-1)^p \textstyle\sum_{|I|=p,|J|=q} \dfrac{\partial f_{IJ}}{\partial x_i} d'x_i i \wedge d'x_I \wedge d''x_j \wedge d''x_J \\ J\alpha &= (-1)^{pq} \textstyle\sum_{|I|=p,|J|=q} f_{IJ} d'x_J \wedge d''x_I \end{aligned}$$

Denote $\mathrm{ev} : TV \otimes T^*V \to \mathbb{R}$ the natural pairing between tangent vectors and covectors. The coevaluation morphism $\mathrm{coev} : \mathbb{R} \to TV \otimes T^*V$ is the unique morphism that makes the compositinos of the following maps

$$\begin{aligned} &T^*V \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} T^*V \otimes (TV \otimes T^*V) \cong (T^*V \otimes TV) \otimes T^*V \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} T^*V \\ &TV \xrightarrow{\mathrm{coev} \otimes \mathrm{id}} (TV \otimes T^*V) \otimes TV \cong TV \otimes (T^*V \otimes TV) \xrightarrow{\mathrm{id} \otimes \mathrm{ev}} TV \end{aligned}$$

identities.

Definition 2.1.1 (Monodromy morphism)**.** Let V be a real vector space. For any open set $U \subset V$ define the morphism $N : \mathscr{A}^{p,q}(U) \to \mathscr{A}^{p-1,q+1}(U)$ to be the composition of the morphisms

$$\begin{aligned} C^\infty(U)\otimes\wedge^p T^*V\otimes\wedge^q T^*V &\to C^\infty(U)\otimes\wedge^p T^*V\otimes(TV\otimes T^*V)\otimes\wedge^q T^*V\\ &\to C^\infty(U)\otimes(\wedge^p T^*V\otimes TV)\otimes(T^*V\otimes\wedge^q T^*V)\\ &\to C^\infty(U)\otimes\wedge^{p-1} T^*V\otimes\wedge^{q+1} T^*V \end{aligned}$$

where the first map is given by the coevaluation map.

We also define $\bar N = J\circ N\circ J : \mathscr{A}_V^{p,q}\to\mathscr{A}_V^{p+1,q-1}$.

The definition of the morphism N is due to Yifeng Liu [Liu17].

For any ordered set $I=\{\,i_1,\dots,i_k\,\}, i_1<\dots<i_k$ and any $i=i_l\in I$ we will denote

$$\operatorname{sgn}(i,I)=\frac{i_1\wedge\dots\wedge i_k}{i_l\wedge i_1\wedge\dots\wedge\widehat{i_l}\wedge\dots i_k}=(-1)^{l-1}.$$

The morphisms N and $\bar N$ can be written explicitly as follows:

$$\begin{aligned} N(\sum f_{IJ}d'x_I\wedge d''x_J) &= \sum_{i=1}(-1)^{p-1}\operatorname{sgn}(i,I)d'x_{I\setminus\{i\}}\wedge d''x_i\wedge d''x_J\\ \bar N(\sum f_{IJ}d'x_I\wedge d''x_J) &= \sum_{j\in J}\operatorname{sgn}(j,J)d'x_I\wedge d'x_j\wedge d''x_{J\setminus\{j\}} \end{aligned}$$

Lemma 2.1.2.

i) $\bar N = J\circ N\circ J$;

ii) $[d'',N]=0, \qquad [\bar N,d']=0.$

Proof. Define

$$\tilde J\left(\sum_{|I|=p,|J|=q} f_{IJ}d'x_{i_1}\wedge\dots\wedge d'x_{i_p}\wedge d''x_{j_1}\wedge\dots\wedge d''x_{j_q}\right)= \\ =\sum_{|I|=p,|J|=q} f_{IJ}d'x_{i_p}\wedge\dots\wedge d'x_{i_1}\wedge d''x_{j_q}\wedge\dots\wedge d''x_{j_1}.$$

Then clearly $\bar N=\tilde J\circ N\circ\tilde J$. Now it is easy to see that

$$J=(-1)^{pq}(-1)^{p(p-1)/2}(-1)^{q(q-1)/2}\tilde J=(-1)^{(p+q)(p+q-1)/2}J$$

and so the first statement follows.

The first part of the second statement is [Liu17, Lemma 2.2], and the second one follows from the first one by i). ☐

2.2. **Superforms on an affine simplicial space.** Although Δ is a manifold, the singularities of the affine structure make the sheaves of parallel forms ill-behaved, as if Δ was singular. Notably, the sheaf of parallel 2-forms on Δ is a constant sheaf with 1-dimensional fibres, but near any point $x\in P$ there is only a 1-dimensional space of parallel 1-forms. It follows that parallel 2-forms cannot always be represented as wedges of parallel 1-forms even locally. In the terminology of [Sus22], she graded sheaf of parallel forms is not regular near the singularities of affine structure.

In order to fix this we will have to leave the category of manifolds endowed with an affine structure with singularities. Given a simplicial complex and a certain sheaf of functions on it that can serve as “affine coordinates”, one can define the sheaves of superforms. We will call such spaces *affine simplicial spaces*. In particular, Δ is one, but instead of working with Δ we will work with a polyhedral affine space $\tilde\Delta$ which will have the desirable properties.

Definition 2.2.1 (Affine simplicial spaces)**.** We call a pair (Δ, A^1) of a topological space and a sheaf of continuous functions on it an *affine simplicial space* if Δ admits a structure of a simplicial complex and A^1 is a subsheaf of the sheaf of continuous piece-wise affine functions on Δ that are linear on each face.

Let Δ be a surface and let ∇ be a flat connection that defines an affine structure on Δ. One can define the sheaves of superforms on Δ by putting

$$\mathscr{A}^{p,q} = \mathscr{A}^p \otimes_{\mathbb{R}} \mathscr{A}^q,$$

where $\mathscr{A}^p, \mathscr{A}^q$ are sheaves of real p- and q-forms, respectively. The differentials d', d'' are given by the de Rham differential d and the connection ∇, respectively.

If (Δ, A^1) is an affine simplicial space then one can define superforms on Δ following the procedure proposed in [Sus22, Section 5.3]. First, we define forms on simplicial complexes (or more generally, polyhedral complexes) embedded into a vector space.

Definition 2.2.2 (Superforms on polyhedral complexes)**.** Let Σ be a polyherdal complex embedded into a real vector space V. Define the following equivalence relation on the superforms defines on a neighbourhood of Σ in V: two forms η and η' are eqivalent if for any face $\sigma \subset \Sigma$, the restrictions $\eta|_\sigma$ and $\eta'|_\sigma$ coincide. We call such an equivalence class of superformsa a *superform on* Σ.

For any point $x \in \Delta$ there exists a neighbourhood U of x and a map

$$e : U \to (A^1_x)^* \cong H^0(U, A^1)^* \qquad x \mapsto [f \mapsto f(x)]$$

which factors through the subspace

$$T(x) := \{\ F \in H^0(U), A^1)^* \ \mid\ F(c) = c \,\forall c \in \mathbb{R}\ \}.$$

The latter is clearly a torsor under the vector subspace

$$H^0(U, \Lambda^1)^* \cong \{\ F \in H^0(U, A^1)^* \ \mid\ F(c) = 0\ \} \subset H^0(U, A^1)^*.$$

For example, if Δ is a 2-sphere and A^1 is $j_* \operatorname{Aff}$, where Aff is the sheaf of affine functions on the complement of finitely many points $P \subset \Delta$, then if $x \notin P$ then $e : U \hookrightarrow e(U) \subset (x)$ is an open embedding, but if the monodromy of the sheaf of affine functions on $\Delta \setminus P$ is non-trivial around $x \in P$ then $e : U \to e(U) \subset T(x)$ has positive-dimensional fibres.

Now let (Δ, A^1) be an affine simplicial space. For any face $\sigma \subset \Delta$, there exists a neighborhood $O \supset \sigma$ such that $e(O)$ is an open of a polyhedral complex in $T(x)$, which is the same for any $x \in \sigma$, on which superforms are defined as in Definition 2.2.2.

If $\tau \supset \sigma$ are two faces of Δ then the inclusion $\operatorname{St}(\tau) \subset \operatorname{St}(\sigma)$ induces a natural map $T(y) \to T(x)$ for any $y \in \sigma, x \in \tau$. Clearly, $e(\operatorname{St}(\tau))$ is mapped to $e(\operatorname{St}(\sigma))$ under this map. We call the pullback of a superform η defined on an open set of $\operatorname{St}(\sigma)$ along this map its *restriction* to $U \cap \operatorname{St}(\tau)$.

Definition 2.2.3 (Superforms on an affine simplicial space)**.** Let $U \subset \Delta$ be an open subset of an affine simplicial space, and denote Σ_U the collection of faces σ such that $U \cap \mathring{\sigma} \neq \emptyset$. A *superform* η on an open subset $U \subset \Delta_X$ is a collection $(\eta_\sigma)_{\sigma \in \Sigma_U}$, where η_σ is a germ of a superform on a neighbourhood of $U \cap \mathring{\sigma}$ in $\operatorname{St}(\sigma)$, such that whenever $\sigma \subset \tau$, the restriction of η_σ to $U \cap \operatorname{St}(\tau)$ is η_τ.

The differentials d' and d'', as well as maps $J, N, \bar{N}$ give rise to the corresponding maps of sheaves of superforms on Δ.

Proposition 2.2.4.

i) $\operatorname{Ker}\{d'd'' : \mathscr{A}^{0,0}_X \to \mathscr{A}^{0,1}_X\} \cong A^1 \otimes \mathbb{R}$*;*

ii) $N(d'd''f) = 0$ *for all* $f \in \mathscr{A}^{0,0}$*;*

iii) for all $p \geq 0$ $\mathrm{Ker}\{d'' : \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1}\} \cong \Lambda_X^p \otimes \mathbb{R}$*;*

iv) $\mathrm{Im}\{d'' : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}\} = \mathrm{Ker}\{d'' : \mathscr{A}^{p,q+1} \to \mathscr{A}^{p,q+2}\}$ *for all* $p \geq 0$*;*

Proof. The first statement follows from the fact that $\mathrm{Ker}\, d'd''$ consists of linear combinations of affine functions. Indeed,

$$d'd''f = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} d'x_i \wedge d''x_j$$

and the vanishing of all second derivatives of f is equivalent to f being affine.

Further,

$$Nd'd''\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) d''x_i \wedge d''x_j = 0.$$

The last two statements are proved in [Sus22, Seciton 5.2]. □

In particular, $(\mathscr{A}_\Delta^{p,q}, d'')$ is a resolution of Λ^p and

$$H^q(\Delta, \Lambda_\Delta^p) = \frac{\{\ \alpha \in \mathscr{A}^{p,q}(\Delta)\ \mid\ d''\alpha = 0\ \}}{d''\, \mathscr{A}^{p,q-1}(\Delta)}$$

We will further denote $H^{p,q}(\Delta) = H^q(\Delta, \Lambda^p)$.

Definition 2.2.5. If (Δ, A^1) and $(\tilde{\Delta}, \tilde{A}^1)$ are two affine simplcial spaces then a continuous map $f : \Delta \to \tilde{\Delta}$ is called a *morphism of affine simplicial spaces* if $f^*\tilde{A}^1 \cong A^1$.

2.3. **Positive supercurrents.** If $\dim V = n$ then an (n,n)-superform on V is given by the expression

$$f \cdot d'x_1 \wedge d''x_1 \wedge \ldots \wedge d'x_n \wedge d''x_n,$$

where $x_1, \ldots, x_n$ is a basis of V, ordered accordingly to the chosen orientation. An (n,n)-superform is called positive if $f \geq 0$. Call a (p,p)-superform η *symmetric* if $\eta = J\eta$.

Definition 2.3.1 (Positive superforms on a real vector space)**.** A symmetric (p,p)-superform η is *weakly positive* if

$$\eta \wedge \alpha_1 \wedge J\alpha_1 \wedge \ldots \wedge \alpha_{n-p} \wedge J\alpha_{n-p}$$

for any collection of forms $\alpha_1, \ldots, \alpha_{n-p} \in \mathscr{A}^{1,0}(V)$.

A symmetric (p,p)-superform η is *(strongly) positive* if

$$\eta = \sum_j f_j \cdot \alpha_{1j} \wedge J\alpha_{1j} \wedge \ldots \wedge \alpha_{pj} \wedge J\alpha_{pj},$$

where $f_j \geq 0$ and $\alpha_{ij} \in \mathscr{A}^{1,0}(V)$.

These definitions naturally extend to superforms on affine simplicial spaces.

Definition 2.3.2 (Positive superforms on affine simplicial spaces)**.** Let Δ be an affine simplicial space. A symmetric superform $\eta \in \mathscr{A}^{p,p}(\Delta)$ is *weakly, resp. strongly positive*, if for any $x \in \Delta$, the germ η_x is the pullback $e^*\tilde{\eta}_x$ of a germ of a weakly, resp. strongly positive form on a polyhedral complex in $T(x)$.

Definition 2.3.3 (Supercurrents)**.** Let $U \subset \Delta$ be an open set. Let $\mathscr{D}^{p,q}(U)$ be the space of (p,q)-superforms on U with compact support. We denote $\mathscr{D}'^{p,q}$ the space of continuous functions on $\mathscr{D}^{p,q}$ and call its elements *currents* on U. The supercurrents form a sheaf $\mathscr{D}'^{p,q}$ on Δ.

Definition 2.3.4 (Positive supercurrents)**.** Let $U \subset \Delta$ be an open set and let $T \in \mathscr{D}'^{p,q}$ be a supercurrent. We call T *weakly, resp. strongly, positive* if $T\eta \geq 0$ for any strongly, resp. weakly postive form $\eta \in \mathscr{D}^{p,q}(U)$.

Fact 2.3.5. *Let U be an open subset of an affine simplicial space Δ. Then the space $\mathscr{A}^{p,q}(U)$ is reflexive, that is, the topological dual of $\mathscr{D}'^{p,q}(U)$ is naturally isomorphic to $\mathscr{A}^{p,q}(U)$.*

Lemma 2.3.6. *Let (Δ, A^1) be a compact affine simplicial space such that A^1 is constructible with respect to a simplicial complex strucutre that contains finitely many faces. Then there exists a positive symmetric $(1,1)$-superform on Δ.*

Proof. Consider the finite covering of Δ by open stars of its vertices $\mathrm{St}(i)$ and select a positive form $\psi_i \in \mathscr{A}^{1,1}(\mathrm{St}(i))$. These forms then can be glued together to obtain a global $(1,1)$-form ψ using a partition of unity subordinate to the cover $\{\, \mathrm{St}(i) \,\}$. □

3. Affine Bott-Chern cohomology

From now on we denote Δ a 2-sphere endowed with an affine structure away from a finite set $P \subset \Delta$, and we assume that the monodromy of the affine structure around any point P is unipotent. We denote $A^1 = j_* \mathrm{Aff}$, where Aff the sheaf of affine functions on $\Delta \setminus P$ and $j : \Delta \setminus P \to \Delta$ is the open embedding.

3.1. **Modification $\tilde{\Delta}$ and Serre duality.** The sheaves of superforms that were just defined can be used to define groups analogous to Dolbeault cohomology groups:

$$H^{p,q}(\Delta) = \frac{\{\, \eta \in \mathscr{A}^{p,q}(\Delta) \;\mid\; d''\eta = 0 \,\}}{d''\, \mathscr{A}^{p,q-1}(\Delta)}$$

The proof of the main theorem that will be explained further in the paper relies on certain properties of these cohomology groups. In particular, one expects a "Serre duality":

$$H^{p,q}(\Delta) \cong (H^{2-p,2-q}(\Delta))^*.$$

However, we have $\dim H^{0,2} = 1$, but $\dim H^{0,2} = 0$: indeed, $\dim T(p) = 1$ for any affine structure singularity $p \in P$, and therefore there are no non-zero germs of $(2,0)$-superforms at any $p \in P$, and hence no global d''-closed $(2,0)$-forms.

In order to remedy this situation, consider the following modification $\tilde{\Delta}$ of Δ which is obtained by gluing simplices to Δ, a simplex for each $p \in P$. The idea of this construction stems from [Sus22, Section 6.3]: it is the modification to the dual intersection complex of the central fibre Y of a Kulikov degeneration that corresponds to blowing up of a (-1)-curve $C \subset Y_i$ that defines the simple affine structure singularity at the vertex i. Here we effectively retell this construction without a reference to a particular triangulation of Δ and without the requirement that Δ arises as a dual intersection complex.

We will explain the gluing near one point, the procedure is identical near each point. Recall that by assumption the monodromy matrix of the connection ∇ associated to the affine structure on $\Delta \setminus P$ is unipotent, and therefore the affine structure in a punctured neighbourhood of i is obtained from a non-singular affine structure on a neighbourhood of i using a single "shear", as described in, for example, [KS06, Section 6.4] or [AET19, Definition 8.3]. Consider the following model of the germ of such affine structure singularity in $\mathbb{R}^2$ that has monodromy matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in the standard basis. Let

$$W^+ = \{\, (x,y) \in \mathbb{R}^2 \;\mid\; x > 0, y \geq 0 \,\}, \qquad W^- = \{\, (x,y) \in \mathbb{R}^2 \;\mid\; x > 0, y \leq 0 \,\},$$
$$R = W^+ \cap W^-,$$

and define the sheaf A^1 on $\mathbb{R}^2$ as follows: for any open U that does not intersect the ray R let $A^1(U)$ be the affine functions on U. For any open U such that $U \cap R \neq \emptyset$ let the sections of A^1 over U be piece-wise affine functions f that are affine on $U \cap W^+$ and $U \cap W^-$ and such that the function

$$f'(x,y) = \begin{cases} f(x+y,y) & (x,y) \in W^+ \\ f(x,y) & (x,y) \in W^- \end{cases}$$

is affine in the standard affine structure on $\mathbb{R}^2$.

Pick a point $i \in P$ and let U be a neighbourhood of i, then there exists a neighbourhood U of i and a continuous injective map $\iota : U \to \mathbb{R}^2$ such that $A^1_\Delta|_U = \iota^* A^1_{\mathbb{R}^2}$. We will slightly abuse notation denoting x, y the pull-backs of the standard coordinate functions to U.

Let $j \in \Delta$ be the point such that $x(j) = 0, y(j) = 1$. we may assume that there is no affine structure singularity at j, multiplying x and y by a scalar if necessary. Glue a 2-simplex to Δ so that one of its sides is ij and the third vertex e and two other sides, ie and je are not glued. Let π be a linear projection from $\tilde{\Delta} \to \Delta$ that e maps to the middle of the edge ij.

Let us now define the sheaf $A^1_{\tilde{\Delta}}$. Firstly, we put it to be isomorphic to A^1_Δ away from ije. For any neighbourhood U of i or j we put $A^1_{\tilde{\Delta}}(U) = \pi^* \iota^* A^1_\Delta(\iota(\pi(U)))$.

For any point p in the interior of ij and an open neighbourhood U of p that doesn't meet ie and je we model U on an open neighbourhood of the origin of a fan in $\mathbb{R}^3$ that is generated vectors

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (-1,-1,0), \quad e_4 = (0,0,1), \quad e_5 = (0,0,-1),$$

with the 2-dimensional cones being spans of pairs of vectors

$$\{ e_1, e_4 \}, \{ e_1, e_5 \}, \{ e_2, e_4 \}, \{ e_2, e_5 \}, \{ e_3, e_4 \}, \{ e_3, e_5 \}$$

(this fan is matroidal, see [JSS19, Definition 4.11]). For p in the interior of ie, resp. je, and an open neighbourhood U of p that is contained in the interior of ije and ie, resp. je, we let $A^1(U)$ to be the restrictions of affine functions f on ije such that $f(i) = f(j)$. Finally, let A^1 have no sections in a neighbourhood of e.

One can check that the projection $\pi : \tilde{\Delta} \to \Delta$ is a morphism of affine simplicial spaces, that is, $\pi^* A^1_\Delta \subset A^1_{\tilde{\Delta}}$.

Proposition 3.1.1. *The pairing*

$$H^q(\tilde{\Delta}, \Lambda^p) \otimes H^{2-q}(\tilde{\Delta}, \Lambda^{2-p}) \to \mathbb{R}, \qquad [\alpha] \otimes [\beta] \to \int \alpha \wedge \beta$$

is non-degenerate.

Proof. The pairing is well-defined by Stokes theorem.

The proof follows the strategy of [JSS19, Theorem 4.26]: it suffices to show that the pairing is non-degenerate on local charts. The only non-trivial case to consider is the neighbourhoods of points in the interior of intervals ij in the notation above, and the pairing is non-degenerate in this case by [JSS19, Proposition 4.19], since the sheaves of superforms are the same as ones defined on a matroidal fan. □

3.2. **Affine Harwey-Lawson criterion.** In complex geometry a Kähler current is a "singular" version of the notion of a Kähler form, and existence of a Kähler current characterizes manifolds that are bimeromorphic to Kähler manifoldds (Fujiki class $\mathscr{C}$ manifolds).

When one considers a general affine simplicial space, it is not clear what the correct notion of a "top degree superform" is, and therefore it is not clear into which space of supercurrents the space of superforms of given bidegree naturally embeds. In particular, it is not clear what the dimension of a "Kähler supercurrent" should be. Instead of attempting to build a general theory, we propose here an *ad hoc* definition of Kähler supercurrents on spaces Δ and $\tilde{\Delta}$, assuming simply that Kähler supercurrents are continuous functionals on $(1,1)$-forms.

We fix a not necessarily closed symmetric positive $(1,1)$-superforms ψ_Δ and $\psi_{\tilde{\Delta}}$ on Δ and $\tilde{\Delta}$, which exist by Lemma 2.3.6.

Definition 3.2.1 (Kähler supercurrents)**.** Let ψ be a positive $(1,1)$-superform on Δ or $\tilde{\Delta}$. A *Kähler supercurrent* is a symmetric positive d''-closed (and hence d'-closed) $(1,1)$-supercurrent T such that $T-\psi$ is positive.

Proposition 3.2.2. *If $T \in \mathscr{D}'^{1,1}(\tilde{\Delta})$ is a Kähler supercurrent, then $\pi_* T$ is a Kähler supercurrent.*

Proof. Straightforward. □

Definition 3.2.3 (Weakly nef supercurrents)**.** A suppercurrent $T \in \mathscr{D}'^{1,1}$ is called *weakly nef* if it is a limit of positive $(n-1,n-1)$-forms.

Lemma 3.2.4. *Let $\theta \in \mathscr{A}^{1,1}(\Delta)$. If $(\theta,\omega) \geq 0$ for every positive $(1,1)$-superform ω such that $d'd''\omega = 0$. Then there exists a supercurrent $\chi \in \mathscr{D}'^{0,0}(\Delta_X)$ such that $\theta + d'd''\chi \geq 0$.*

Proof. Let

$$\begin{aligned} P &= \{\ \omega \in \mathscr{A}^{1,1}(\Delta_X)\ \mid\ \omega > 0\ \} \\ V &= \{\ \omega \in \mathscr{A}^{1,1}(\Delta_X)\ \mid\ d'd''\omega = 0\ \} \end{aligned}$$

By assumption θ defines a functional on V which is non-negative on the cone $P \cap V$, which is open in V.

If θ reaches 0 on a non-zero vector in P then since it is non-negative it must vanish on V. In this case θ defines a continuous functional on $d'd''\mathscr{A}^{1,1} \subset \mathscr{A}^{2,2}$ which can be extended to a continuous functional χ on $\mathscr{A}^{2,2}$. In other words there exists a supercurrent $\chi \in \mathscr{D}'^{0,0}(\Delta_X)$ such that $\theta = -d'd''\chi$, so $\theta + d'd''\chi = 0$.

If θ is strictly positive on $P \cap V$, the hyperplane $\theta^\perp \subset V$ is disjoint from the convex cone $P \cap V$ and therefore by Hahn-Banach theorem the functional defined by θ on $\mathscr{A}^{1,1}$ can be extended to a functional τ that vanishes on $\theta^\perp \cap V$ and is strictly positive on P.

Since $\theta^\perp$ is codimension 1 in V, θ and τ must be proportional on V. Therefore for some positive constant c the functional $\theta - c\tau$ vanishes on V, and by the same reasoning as above, $\theta - c\tau = -d'd''\chi$ for some supercurrent $\chi \in \mathscr{D}'^{0,0}(\Delta_X)$. It follows that $\theta + d'd''\chi = c\tau \geq 0$. □

Proposition 3.2.5. *The following conditions are equivalent:*

i) there exists a Kähler supercurrent on $\tilde{\Delta}$;
ii) every weakly nef d''-exact $(1,1)$-supercurrent is zero;
iii) there exists a symmetric $(1,1)$-superform α such that $(\alpha,\omega) \geq (\psi,\omega)$ for any positive symmetric $(1,1)$-superform ω such that $d'd''\omega = 0$.

Proof. (i) ⇒ (ii) Let τ be the Kähler supercurrent, so $\tau = \theta + d'd''\chi \geq \psi$ for some θ such that $N\theta = 0, d''\theta = 0$. Let $T = d''S$, $NS = 0$, be a weak nef supercurrent such that $d'd''T = 0$, so $T = \lim \omega_n$ where $d'd''\omega_n = 0, \omega \geq 0$. We have

$$(\theta + d'd''\chi, \omega_n) = (\theta, \omega_n) \geq (\psi, \omega_n)$$

so $(\theta, T) \geq (\psi, T)$ in the limit. Furthermore,

$$(\theta, T) = (\theta, d''S) = 0$$

Therefore $(\psi, T) \leq 0$. Since $\psi > 0$, we have $T = 0$.

(ii) ⇒ (iii) Define

$$W = \{\ \omega \in V \cap P\ \mid\ (\omega, \psi) = 1\ \}$$

and let Y be the closure of W in $\mathscr{D}'^{1,1}(\tilde{\Delta})$. By assumption, no element of Y is d''-exact. Therefore, by Hahn-Banach theorem there exists $\eta \in \mathscr{A}^{1,1}(\Delta)$ such that $\eta|_Y > 0$, $d''\eta = 0$. Since Y is weakly compact, η regarded as a functional achieves a minimum on Y: $(\eta, T) \geq C, \forall T \in Y$, and since $\eta|_Y > 0$, $C > 0$. In particular,

$$\frac{(\eta, \omega)}{(\psi, \omega)} \geq C$$

since $\omega/(\omega, \psi) \in Y$. Taking $\alpha = \eta/C$, we obtain

$$(\alpha, \omega) \geq (\psi, \omega).$$

(iii) $\Rightarrow$ (i) If $\alpha \in \mathscr{A}^{1,1}(\Delta)$ is a d''-closed symmetric form such that $(\alpha, \psi) \geq (\omega, \psi)$ for any $d'd''$-closed superform ω, then $\alpha - \psi$ satisfies the conditions of Lemma 3.2.4, and therefore exists a supercurrent χ such that $\alpha + d'd''\chi \geq \psi$, so $\alpha + d'd''\chi$ is a Kähler supercurrent.

□

3.3. **Monodromy morphism.**

Lemma 3.3.1. *Let V be a vector space, then for all $p, q \geq 0$*

$$\begin{aligned} (N \circ \bar{N} \circ N)\alpha &= (q+1)N\alpha, \quad \textit{for all } \alpha \in \mathscr{A}^{1,q}(V), \\ (\bar{N} \circ N \circ \bar{N})\alpha &= (p+1)\bar{N}\alpha, \quad \textit{for all } \alpha \in \mathscr{A}^{p,1}(V). \end{aligned}$$

In particular, $\mathrm{id} - \frac{1}{q+1}\bar{N} \circ N$ *is a projection on* $\operatorname{Ker} N \subset \mathscr{A}^{1,q}$, *and* $\mathrm{id} - \frac{1}{p+1} N \circ \bar{N}$ *is respectively a projection on* $\operatorname{Ker} \bar{N} \subset \mathscr{A}^{p,1}$.

Proof. Let $x_1, \ldots, x_n$ be coordinate. Let α be a (p,q)-superform given by the expression

$$\alpha = \sum_{i, |J|=q} f_{i,J} d'x_I \wedge d''x_J.$$

We have then

$$N\alpha = \sum_{i=1}^{n} \sum_{|J|=q+1} f_{i,J}\ d''x_i \wedge d''x_J = \sum_{|J|=q+1} \left(\sum_{j \in J} \operatorname{sgn}(j, J) f_{j, J \setminus \{j\}} \right) d''x_J,$$

$$\bar{N}(N\alpha) = \sum_{i=1}^{n} \sum_{|J|=q} \operatorname{sgn}(i, J \cup \{i\}) \sum_{j \in J \cup \{i\}} \operatorname{sgn}(j, J \cup \{i\}) f_{j, J \cup \{i\} \setminus \{j\}}\ d'x_i \wedge d''x_J,$$

Applying the first formula to the second, we further get

$$N(\bar{N}(N\alpha)) = \sum_{|J|=q+1} \sum_{k \in J} \sum_{j \in J} \operatorname{sgn}(k, J) \operatorname{sgn}(k, J) \operatorname{sgn}(j, J) f_{j, J \setminus \{j\}} d''x_J,$$

which proves the first claim. The statement for $\bar{N}$ is proved analogously.

Now $\operatorname{Im}(\mathrm{id} - \bar{N} \circ N) = \operatorname{Ker} N$ since by the claim just proved

$$N \circ (\mathrm{id} - \frac{1}{q+1} \bar{N} \circ N) = N - N = 0$$

and since $\mathrm{id} - \frac{1}{q+1}\bar{N} \circ N = \mathrm{id}$ on $\operatorname{Ker} N$. Moreover,

$$(\mathrm{id} - \frac{1}{q+1}\bar{N} \circ N) \circ (\mathrm{id} - \frac{1}{q+1}\bar{N} \circ N) = \mathrm{id} - \frac{1}{q+1}\bar{N} \circ N - \frac{1}{q+1}\bar{N} \circ N +$$
$$+ \frac{1}{(q+1)^2}\bar{N} \circ N \circ \bar{N} \circ N = \mathrm{id} - \frac{2}{q+1}\bar{N} \circ N \circ N + \frac{1}{q+1}\bar{N} \circ N = \mathrm{id} - \frac{1}{q+1}\bar{N} \circ N.$$

The proof that the operator $\mathrm{id} - \frac{1}{p+1}N \circ \bar{N}$ is a projection on $\operatorname{Ker} \bar{N} \subset \mathscr{A}^{p,1}$ is analogous. □

Lemma 3.3.2.

$$d' \circ N - N \circ d' = d'', \qquad d'' \circ \bar{N} - \bar{N} \circ d'' = d'.$$

Proof. Let $x_1, \ldots, x_n$ be coordinates let

$$\alpha = \sum_{I,J} f_{IJ} d'x_I \wedge d''x_J$$

be a (p, q)-superform. Then

$$d'\alpha = \sum_{|I|=p+1, |J|=q} \sum_{i \in I} \operatorname{sgn}(i, I) \frac{\partial f_{I \setminus \{i\}, J}}{\partial x_i} d'x_I \wedge d''x_J,$$

$$d''\alpha = \sum_{|I|=p, |J|=q+1} (-1)^{p-1} \sum_{j \in J} \operatorname{sgn}(j, J) \frac{\partial f_{I, J \setminus \{j\}}}{\partial x_i} d'x_I \wedge d''x_J,$$

$$Nd'\alpha = \sum_{|I|=p, |J|=q+1} \sum_{j \in J} (-1)^p \operatorname{sgn}(j, I \cup \{j\}) \operatorname{sgn}(j, J) \cdot$$
$$\cdot \sum_{i \in I \cup \{j\}} \operatorname{sgn}(i, I \cup \{j\}) \frac{\partial f_{I \cup \{j\} \setminus \{i\}, J \setminus \{j\}}}{\partial x_i} d'x_I \wedge d''x_J,$$

$$d'N\alpha = \sum_{|I|=p, |J|=q+1} \sum_{i \in I} \operatorname{sgn}(i, I) \sum_{j \in J} (-1)^{p-1} \operatorname{sgn}(j, I \cup \{j\} \setminus \{i\}) \operatorname{sgn}(j, J) \cdot$$
$$\cdot \frac{\partial f_{I \cup \{j\} \setminus \{i\}, J \setminus \{j\}}}{\partial x_i} d'x_I \wedge d''x_J,$$

$$d'N\alpha - Nd'\alpha = \sum_{|I|=p, |J|=q+1} \sum_{j \in J} \sum_{i \in I} \Big((-1)^{p-1} \operatorname{sgn}(i, I) \operatorname{sgn}(j, I \cup \{j\} \setminus \{i\}) \operatorname{sgn}(j, J) -$$
$$- (-1)^p \operatorname{sgn}(j, I \cup \{j\}) \operatorname{sgn}(i, I \cup \{j\} \operatorname{sgn}(j, J) \Big) \frac{\partial f_{I \cup \{j\} \setminus \{I\}, J \setminus \{j\}}}{\partial x_i} d'x_I \wedge d''x_J +$$
$$- \sum_{j \in J} (-1)^p \operatorname{sgn}(j, I \cup \{j\}) \operatorname{sgn}(j, I \cup \{j\}) \operatorname{sgn}(j, J) \frac{\partial f_{I \cup \{j\} \setminus \{j\}, J \setminus \{j\}}}{\partial x_j} d'x_I \wedge d''x_J.$$

The first sum in the expression for $Nd'\alpha + d'N\alpha$ vanishes and the second sum coincides with the expression for $d''\alpha$, proving the first claim. Indeed, as one observes easily

$$\operatorname{sgn}(i, I) \operatorname{sgn}(i, \{i, j\}) = \operatorname{sgn}(i, I \cup \{j\}),$$
$$\operatorname{sgn}(j, I \cup \{j\} \setminus \{i\}) \operatorname{sgn}(j, \{i, j\}) = \operatorname{sgn}(j, I \cup \{j\}),$$

and since $\operatorname{sgn}(i, \{i, j\}) = -\operatorname{sgn}(j, \{i, j\})$ by definition,

$$(-1)^p \operatorname{sgn}(j, I \cup \{j\}) \operatorname{sgn}(i, I \cup \{j\} \operatorname{sgn}(j, J) - (-1)^{p-1} \operatorname{sgn}(i, I) \operatorname{sgn}(j, I \cup \{j\} \setminus \{i\}) \operatorname{sgn}(j, J) =$$
$$= (-1)^p \operatorname{sgn}(j, I \cup \{j\}) \operatorname{sgn}(i, I \cup \{j\} \operatorname{sgn}(j, J) +$$
$$- (-1)^{p-1} (\operatorname{sgn}(i, \{i, j\}) \operatorname{sgn}(i, I \cup \{j\}))(\operatorname{sgn}(j, \{j, i\}) \operatorname{sgn}(j, I \cup \{j\})) \operatorname{sgn}(j, J) = 0$$

Now, $\bar{N} = J \circ N \circ J$, so

$$d''\bar{N}\alpha - \bar{N}d''\alpha = (J \circ (d' \circ N - N \circ d') \circ J)(\alpha) = (J \circ d'' \circ J)\alpha = d'\alpha.$$

□

3.4. **Resolution of A^1.** By Proposition 2.2.4(i) the sheaf A^1 is the kernel of the morphism $d'd'' : \mathscr{A}^{0,0} \to \mathscr{A}^{1,1}$. In fact, this morphism can be extended to a resolution, which will be crucially used in the arguments in the Section 4.

Proposition 3.4.1.
i) $\operatorname{Im}\{d'' : \mathscr{A}^{p,q} \to \mathscr{A}^{p,q+1}\} = \operatorname{Ker}\{d'' : \mathscr{A}^{p,q+1} \to \mathscr{A}^{p,q+2}\}$ *for all* $p \geq 0$*;*
ii) $d'd''(\operatorname{Ker}\{N : \mathscr{A}^{p,0} \to \mathscr{A}^{p-1,1}\}) = \operatorname{Ker} d'' \cap \operatorname{Ker}\{N : \mathscr{A}^{p,1} \to \mathscr{A}^{p-1,2}\}$.
iii) $d''(\operatorname{Ker}\{N : \mathscr{A}^{1,q} \to \mathscr{A}^{0,q+1}\}) = \operatorname{Ker} d'' \cap \operatorname{Ker}\{N : \mathscr{A}^{1,q+1} \to \mathscr{A}^{0,q+2}\}$ *for all* $q \geq 1$*;*

Proof. It follows from Definition 2.2.3 that all four statements reduce to correspnoding statements about superforms on a vector space. In particular, the first statement reduces to [Lag12, Lemma 1.10] or [JSS19, Theorem 2.16].

For the second statement, the inclusion from left to right follows from Proposition 2.2.4(ii) and from right to left from [Lag12, Lemma 1.13].

In the last statement, the inclusion from left to right follows from Lemma 2.1.2. Let $\alpha \in \mathscr{A}^{1,q}$, $d''\alpha = N\alpha = 0$ for some $q \geq 2$. Then by Proposition 2.2.4(iv) there exists $\beta \in \mathscr{A}^{1,q-1}$ such that $d''\beta = \alpha$. By Lemma 2.1.2 again, $d''N\beta = 0$, so by Proposition 2.2.4(iv) there exists $\gamma \in \mathscr{A}^{0,q-1}$ such that $d''\gamma = \beta$. Consider $\beta + d''\bar{N}\gamma$: we have

$$d''(\beta - \frac{1}{q-1} d''\bar{N}\gamma) = d''\beta = \alpha,$$

so it is left to prove that this superform belongs to $\operatorname{Ker} N$. Indeed, applying Lemmas 3.3.1 and 3.3.2 we get

$$N(\beta - \frac{1}{q-1} d''\bar{N}\gamma) = N\beta - \frac{1}{q-1}(Nd'\gamma + N\bar{N}d''\gamma) =$$
$$= N\beta - \frac{1}{q-1}(d'N\gamma - d''\gamma + N\bar{N}N\beta) = N\beta - \frac{1}{q-1}(-N\beta + q \cdot N\beta) = N\beta - N\beta = 0.$$

□

It follows that the complex of sheaves

$$0 \to \mathscr{A}^{0,0} \xrightarrow{d'd''} \mathscr{A}^{1,1}_{KerN} \xrightarrow{d''} \mathscr{A}^{1,2}_{\operatorname{Ker} N} \xrightarrow{d''} \mathscr{A}^{1,3}_{\operatorname{Ker} N} \to \dots \tag{3.1}$$

is the resolution of A^1 and one observes easily that its natural inclusion into $\operatorname{Cone}(N)$

$$0 \to \mathscr{A}^{0,0} \oplus \mathscr{A}^{0,1} \xrightarrow{(d'', N + d'')} \mathscr{A}^{0,1} \oplus \mathscr{A}^{1,1} \xrightarrow{(d'', N + d'')} \mathscr{A}^{0,2} \oplus \mathscr{A}^{1,2} \to \dots$$

is a quasi-isomorphism. In particular, we have a long exact sequence of cohomology associated to the short exact sequence of sheaves $0 \to \Lambda^0 \cong \mathbb{R} \to A^1 \to \Lambda^1 \to 0$:

$$\dots \to H^{0,1}(\tilde{\Delta}_X) \xrightarrow{N} H^{1,0}(\tilde{\Delta}_X) \to H^{1,1}_{BC}(\tilde{\Delta}) \xrightarrow{i} H^{1,1}(\tilde{\Delta}_X) \to H^{0,2}(\tilde{\Delta}_X) \to \dots \tag{3.2}$$

where

$$H^{1,1}_{BC}(\tilde{\Delta}) = \frac{\{ \alpha \in (\mathscr{D}^{1,1}_{\mathrm{Ker}\,N})'(\tilde{\Delta}) \mid d''\alpha = 0 \}}{d'd''\,\mathscr{D}'^{0,0}(\tilde{\Delta})},$$

and the morphism i on the level of cocycles is induced by the simple inclusion of the space of d''-closed $(1,1)$-superforms in $\mathrm{Ker}\,N$ into the the space of d''-closed $(1,1)$-superforms.

By [Lag11, Lemma 1.10] the resolution (3.2) can be replaced with a similar one, where supercurrents are used:

$$0 \to \mathscr{D}'^{0,0} \xrightarrow{d'd''} \mathscr{D}'^{1,1}_{\mathrm{Ker}\,N} \xrightarrow{d''} \mathscr{D}'^{1,2}_{\mathrm{Ker}\,N} \xrightarrow{d''} \mathscr{D}'^{1,3}_{\mathrm{Ker}\,N} \xrightarrow{d''} \dots \tag{3.3}$$

and so the group $H^{1,1}_{BC}(\tilde{\Delta})$ can be computed with supercurrents.

By Serre duality for superforms (Proposition 3.1.1) the sequence

$$\dots \to (H^{0,2}(\tilde{\Delta}))^* \xrightarrow{N} (H^{1,1}(\tilde{\Delta}))^* \xrightarrow{p} (H^1_{BC}(\tilde{\Delta}))^* \to (H^{1,2}(\tilde{\Delta}))^* \to \dots \tag{3.4}$$

is exact, where

$$(H^{1,1}_{BC}(\tilde{\Delta}))^* = \frac{\{ \alpha \in \mathscr{A}^{1,1}_{\mathrm{Ker}\,N}(\tilde{\Delta}) \mid d'd''\alpha = 0 \}}{\{ \alpha \in \mathscr{A}^{1,1}_{\mathrm{Ker}\,N}(\tilde{\Delta}) \mid \exists \beta \in \mathscr{A}^{1,0}_{\mathrm{Ker}\,N}(\tilde{\Delta}), d''\beta = \alpha \}}$$

4. Cohomology of sheaves Λ^p on $\tilde{\Delta}$

4.1. Map Q.

Lemma 4.1.1. *i) The map $j_2 : H^{2,0}(\tilde{\Delta}) \to H^{0,2}(\tilde{\Delta})$ induced by J is an isomorphism;*
ii) the map $N : H^{1,0}(\tilde{\Delta}) \to H^{0,1}(\tilde{\Delta})$ is injective.

Proof. Let $\omega \in \mathscr{A}^{2,0}(\tilde{\Delta}_X), d''\omega = 0$. Then $d'\omega = 0$ for dimension reasons and $d''J\omega = 0$. If there exists a superform $\alpha \in \mathscr{A}^{0,1}, d''\alpha = J\omega$ then

$$d''(\omega \wedge \alpha) = d''\omega \wedge \alpha + \omega \wedge d''\alpha = \omega \wedge J\omega$$

By Stokes theorem $\int_{\tilde{\Delta}_X} d''(\omega \wedge \alpha) = 0$, but since $\omega \wedge J\omega$ is a positive superform, $\omega = 0$. Therefore the map j_2 is injective. Reasoning in a symmetric way, we get that the inverse of the map j_2 is injective too.

Let $\alpha \in \mathscr{A}^{1,0}(\tilde{\Delta}), d''\alpha = 0$, so α represents a class in $H^{1,0}(\tilde{\Delta})$. If $N\alpha = d''f$ then $\alpha = d'f$ and $d'd''f = 0$. But since $\tilde{\Delta}$ is compact, f mult be 0, so $\alpha = 0$. ☐

For the rest of this section we assume that the map $N : H^0(\tilde{\Delta}, \Lambda^1) \to H^1(\tilde{\Delta}, \mathbb{R})$ is an isomorphism then by Poincaré duality the map

$$H^{1,2}(\tilde{\Delta}_X) \to H^{2,1}(\tilde{\Delta}_X)$$

induced by J is an isomorphism, and the map

$$H^1(\tilde{\Delta}, A^1) \to H^1(\tilde{\Delta}, \Lambda^1)$$

is an injection. In view of Lemma 4.1.1 this is true in particular when $H^1(\tilde{\Delta}, \mathbb{R}) = 0$.

Denote

$$\begin{aligned} Z^{1,1}_{d'd''} &= \{ \omega \in \mathscr{A}^{1,1}_{\mathrm{Ker}\,N}(\tilde{\Delta}) \mid d'd''\omega = 0 \}, \\ Z^{1,1}_{d''} &= \{ \omega \in \mathscr{A}^{1,1}(\tilde{\Delta}) \mid d''\omega = 0 \}. \end{aligned}$$

Proposition 4.1.2. *Assume that $N : H^{1,0}(\tilde{\Delta}) \to H^{0,1}(\tilde{\Delta})$ is an isomorphism. There exists a linear map $Q : Z^{1,1}_{d'd''} \to Z^{1,1}_{d''}$ such that*

i) $Q(\omega) = \omega + Nd'\alpha$ *for some* $\alpha \in \mathscr{A}^{1,0}(\tilde{\Delta})$*;*
ii) $Q(d''\gamma) = d''\gamma$ *for all* $\gamma \in \mathscr{A}^{1,0}(\tilde{\Delta})$*.*

Proof. We will define $Q(\omega) = \omega + Nd'\alpha$ for the unique form $\alpha \in \mathscr{A}^{1,0}(\tilde{\Delta})$ such that $\omega + Nd'\alpha$ is d''-closed. This immediately implies that $Q(d''\gamma) = d''\gamma$. This map is also linear since if $Q(\omega_i) = \omega_i + Nd'\alpha_i, i = 1, 2$ then clearly $\omega_1 + \omega_2 + Nd'(\alpha_1 + \alpha_2)$ and $c \cdot \omega_1 + Nd'(c \cdot \alpha_1)$ are d''-closed for any constant $c \in \mathbb{R}$.

First let us show that such form α exists for any $d'd''$-closed $\omega \in \operatorname{Ker} N$. Consider $d'\omega \in \mathscr{A}^{2,1}$, then since it is d''-closed by assumption, it defines a class in $H^{2,1}(\tilde{\Delta})$. The form $Jd'\omega$ is a d''-coboundary in $\mathscr{A}^{1,2}(\tilde{\Delta})$. Since by by assumption of the lemma J induces an isomorphism $H^{1,0}(\tilde{\Delta}) \to H^{0,1}(\tilde{\Delta})$, so by Fact 3.1.1, J induces isomorphism $H^{2,1}(\tilde{\Delta}) \to H^{1,2}(\tilde{\Delta})$, $d'\omega$ is a d''-coboundary in $\mathscr{A}^{2,1}(\tilde{\Delta})$ and therefore there exists $\beta \in \mathscr{A}^{2,0}$ such that $d''\beta = d'\omega$. Then by Lemma 3.3.2

$$d''N\beta = Nd''\beta = Nd'\omega = d'N\omega - d''\omega = -d''\omega$$

The form $J\beta$ is d''-closed for degree reasons and therefore defines a class in $H^{0,2}(\tilde{\Delta})$. The image of this class under J is represented by a form $\tilde{\beta} \in \mathscr{A}^{2,0}, d''\tilde{\beta} = 0$. Since J induces an isomorphism $H^{0,2} \to H^{2,0}(\tilde{\Delta})$ by Lemma 4.1.1, $J\beta$ is cohomologous to $J\tilde{\beta}$, and so there exists $\alpha \in \mathscr{A}^{1,0}(\tilde{\Delta})$, $J\beta = J\tilde{\beta} + d''J\alpha$, so $\beta = \tilde{\beta} + d'\alpha$, and since $d''\beta = d''d'\alpha$,

$$d''(\omega + Nd'\alpha) = d''\omega - d''\omega = 0.$$

Let us prove that such form $\alpha \in \mathscr{A}^{1,0}(\tilde{\Delta})$ is unique. Suppose that there two forms α_1, α_2 such that both $\omega + Nd'\alpha_i$, $i = 1, 2$ are d''-closed. Then

$$d''Nd'(\alpha_1 - \alpha_2) = Nd''d'(\alpha_1 - \alpha_2) = 0$$

Since $N : \mathscr{A}^{2,1}(\tilde{\Delta}) \to \mathscr{A}^{1,2}(\tilde{\Delta})$ is injective, $d'(\alpha_1 - \alpha_2)$ represents a class in $H^{2,0}$ which is mapped to a trivial class in $H^{0,2}$. By Lemma 4.1.1, the class of $d'(\alpha_1 - \alpha_2)$ is trivial in $H^{2,0}$, so is represented by 0, concluding the proof of uniqueness.

□

4.2. **Proof of Theorem A.**

Proposition 4.2.1. *There exists a Kähler supercurrent on* Δ*.*

Proof. By Propositions 3.2.5 and 3.2.2 it suffices to show that every weakly nef d''-exact supercurrent $T \in \operatorname{Ker} N \subset \mathscr{D}'^{1,1}$ is zero.

Clearly, such current T is d''-closed and therefore represents a class in $H^{1,1}_{BC}(\tilde{\Delta})$. By Proposition 4.1.2, the composition of natural morphisms

$$H^{1,1}_{BC}(\tilde{\Delta}) \to H^1(\tilde{\Delta}, \Lambda^1_{\tilde{\Delta}}) \to H^1(\tilde{\Delta}, \Lambda^1_{\tilde{\Delta}})^* \to H^{1,1}(\tilde{\Delta}, \Lambda^1_{\tilde{\Delta}})$$

defines an isomorphism. The image of $[T]$ in $H^{1,1}_{BC}(\tilde{\Delta})$ under this morphism vanishes, since T is $d'd''$-closed. Therefore $[T] = 0 \in H^{1,1}_{BC}(\tilde{\Delta})$. Let $\pi : \tilde{\Delta} \to \Delta$ be the natural projection, then π_*T is still $d'd''$-exact and positive, in particular, $\pi_*T = d'd''f$ for some global convex function $f \in \mathscr{A}^{0,0}(\Delta)$. But the only convex functions on Δ are the constant functions, therefore, $\pi_*T = 0$ and $T = 0$. □

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