# Valuations, plurisubharmonic functions and multiplier ideals

learning seminar, winter semester 2020

Demailly and Kollár have asked [DK01, Remark 5.3] if the interval $\{c > 0 \mid \mathcal{I}(c \cdot \varphi) = \mathcal{O}_X\}$, where $\mathcal{I}(\varphi)$ is the multiplier ideal of a plurisubharmonic function $\varphi$ on a complex manifold $X$, is open ("openness cconjecture"). Favre and Jonsson [FJ05b] have answered this question [FJ05a] in dimension 2 using a valuative charaterization of multiplier ideals. Their approach consisted in the following: one considers a space of normalized valuations on $\mathcal{O}_{\mathbb{C}^2,0}$, (the "valuative tree" [FJ04]), and puts into correspondence to any germ at $0 \in \mathbb{C}^2$ of a psh function $\varphi$ a certain function on this space, called the tree transform $\hat{\varphi}$ of $\varphi$. Their proof of the conjecture is based on the fact that the multiplier ideal $\mathcal{I}(\varphi)$ can then be characterized in terms of $\hat{\varphi}$.

Later, this construction has been generalized in co-authorship with Boucksom [BFJ07] to germs of psh functions on $\mathbb{C}^n$. In the perspective [BFJ07] takes, one also considers the space $\mathcal{V}$ of normalized valuations of $\mathcal{O}_{\mathbb{C}^n,0}$ centered at 0, and to each psh $\varphi$ a valuative transform $\hat{\varphi}$ is associated. The key difference of the setting in dimension $n > 2$ and dimension 2 is that there is no natural notion of convexity for functions on $\mathcal{V}$, and in order to define the class of functions on $\mathcal{V}$ to which one can associate a multiplier ideal, one is lead to consider also the Riemann-Zariski space $\mathfrak{X} = \varprojlim X_\pi$, the projective limit of all proper modifications $X_\pi \to X$ above 0.

This formalism was further extended in [JM12] where a valuation transform of a graded sequence of ideals was defined, giving a unified valuative characterization of multiplier ideals associated to three kinds of objects: psh functions, ideals and graded sequences of ideals. This has allowed Jonsson and Mustaţă [JM14] to reduce the openness conjecture to a purely algerbaic statement about existence of a quasi-monomial valuation that computes the log-canonical threshhold of a graded system of ideals.

At the core of these activities lie valuative approximation techniques analogous to the Demailly approximation theorem in complex analysis; they enabled Boucksom, Favre and Jonsson to develop an approach to intersection theory for Weil divisors on the Riemann-Zariski space (also known as Shokurov $b$-divisors), as a side-product. This approach allowed them to introduce an algebraic construction for the "movable intersection product" initially defined in the analytic context in [Bou02a], [BDPP04]. It has been used in [BFJ06] to study the volume function on the big cone and give a formula for its derivative in the direction of an ample divisor, and in [BDFF$^+$12] to define the volume of an isolated normal singularity, a generalization of the Wahl's characteristic number for surface singularitiy.

The rough plan of the seminar is as follows:

- describe the structure of the valuative tree and various parameters of valuations (skewness, thinness, multiplicity) [FJ04], discuss monomialization [FJ05b]. Discuss basic properties of psh functions, Lelong and Kiselman numbers [Dem12] and Demailly approximation [Dem11, §13]. Define the tree transform of a psh function, tree potentials and their Laplacians, using valuative Demailly approximation show that the tree transform of a function with analytic singularities coincides with the tree transform of the corresponding ideal [FJ05b];

- define the multiplier ideal of a tree potential [FJ05a], prove subadditivity property and show that the multiplier ideal of a tree potential of an ideal coindcides with the usual definition of a multiplier ideal of an ideal. Sketch the proof of the fact that the multiplier ideal of a planar psh function coincides with the multiplier ideal of its tree transform, and deduce the openness conjecture in dimension 2.

- show that an integrally closed ideal is the multiplier ideal of a tree potential, and the tree potentials with the same multiplier ideal are equal. Give an overview of the proof of the Shokurov ACC conjecture in dimension 2;

- present the proof of the non-archimedean version of Demailly approximation theorem on a disc following Stevenson [Ste17], reviewing the necessary background on Berkovich spaces [Nic07], [Tem16], [MN12];

- give an overview of the result of Nicaise and Xu [NX$^+$16] that the weight function is constant on a face of a dual intersection complex only if it reaches its minimum (equal to the log canonical threshold);

- describe the structure of $\mathcal{V}$ and Zariski-Riemann space, Weil and Cartier divisors on it (Shokurov's $b$-divisors), define nef Weil divisors. Define formal psh functions and valuative multiplier ideals [BFJ07], discuss nef envelopes and valuative version of Demailly approximation (Theorems 3.9) and subadditivity of valuative multiplier ideals(Theorem 3.10 [BFJ07]). Define the formal psh function associated to a nef Weil divisor, and show that it coincides with the valuative transform of an ideal in case of a psh function with analytic singularities. Prove that the Weil divisor associated to a psh function is nef, and show that $\cup_{\epsilon\to 0}\mathcal{I}((1+\epsilon)u)$ for a psh function $\varphi$ is equal to the multiplier ideal of the formal psh function $\hat{u}$;

- give the valuative definition of the asymptotic mulitplier ideals and asymptotic jumping numbers of a graded system of ideals following [JM12]. Give an overview Theorem A: the infimum of the discrepancy function is reached on the space of valuations. Show that lct of a graded system of ideals is computed by a quasi-monomial valuation in dimension 2. Give an overview of the reduction of the analytic openness conjecture to the algebraic statement of that asymptotic lct is computed by a quasi-monomial valution [JM14].

There are further possible directions for the seminar:

- read [BDFF$^+$12] where the volume of an isolated normal singularity is defined as minus top self-intersection with respect to the movable intersection product of the dicrepancy $b$-divisor. We might want to start by first reading [BFJ06] where the definition of the movable intersection product from [BDPP04, Bou02b] is reformulated in the language of $b$-divisors, and comparing it to the construction in [BFJ07]. We might continue with the follow-up article [BdFFU13]) where valuative characterization of multiplier ideals on normal varieties is established.

- read the papers [BFJ11] and [BFJ15] that study non-archimedean analogues of formal psh functions "with non-trivial $c_1$" on proper varieties over a discretely valued field (singular semi-positive metrics) and their intersection theory, solving a non-Arcihmedean analgue of the complex Monge-Ampere equation. This work builds on [BFJ12] to prove a uniform Lipshitz bound for $\theta$-psh functions.

- read about the solution by Chenyang Xu [Xu19], of the conjecture of Jonsson and Mustaţă that lct of a graded system of ideals is computed on a quasi-monomial valuation;

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