A new proof of Zilber's relative trichotomy conjecture

Dmitry Sustretov
Ben Gurion University
sustreto@math.bgu.ac.il

April 23, 2014

Zilber's relative trichotomy: statement

Let M be an algebraic curve over an algebraically closed field k, and let $X \subset T \times M^2$ be a family of distinct curves on the surface M^2, $\dim T \geqslant 2$.

Conjecture. One can recover the field k starting from the data $M(k), X(k) \subset (T \times M^2)(k)$, moreover, one can do it in a definable way: the field is definable in the first-order structure (M, X).

Proved by Rabinovich (1993) for $M = \mathbb{P}^1$, X is allowed to be a family of constructible sets.

I will present the main ideas of the new proof (joint work with Assaf Hasson) which in particular has no restrictions on M.

In this talk I will assume that X_t is closed irreducible for t in an open dense subset of T (ensuring this is the first step of the proof). One can also without loss of generality assume that M is smooth.

Weil's birational group laws

Let G be an algebraic variety and let $m : G \times G \dashrightarrow G$ be a rational map such that

$(x, y) \mapsto (x, m(x, y))$ and $(x, y) \mapsto (m(x, y), y)$ are birational maps and
$m(x, m(y, z)) = m(m(x, y), z)$ (whenever it makes sense)

Then m is called a *birational group law*.

Theorem (*Weil*) Let $m : G \times G \to G$ be a birational group law. Then there exists an algebraic group G' such that G' is birationally equivalent to G and such that the group law on G' pulls back to m under an isomorphism of dense open subsets of G and G'.

The work of Artin generalises this result to group schemes.

Correspondences and compositions

Let X, Y be two varieties or, more generally, schemes. In this talk we will call a closed subscheme Z of $X \times Y$ a *correspondence* from X to Y if it projects surjectively on X and Y. Notation: $\alpha : X \vdash Y$, $\Gamma(\alpha) = Z$.

If $U \subset X(k)$ then $\alpha(U) = p_Y \circ p_X^{-1}(U)$. Similarly, if X is proper, and $\mathcal{L}$ is a coherent sheaf of $\mathcal{O}_X$-modules then $\alpha(\mathcal{L}) = p_{Y*}p_X^*(\mathcal{L})$ is a coherent sheaf of $\mathcal{O}_Y$-modules.

Given two correspondences $\alpha : X \vdash Y, \beta : Y \vdash Z$ one defines their composition $\beta \circ \alpha : X \vdash Z$

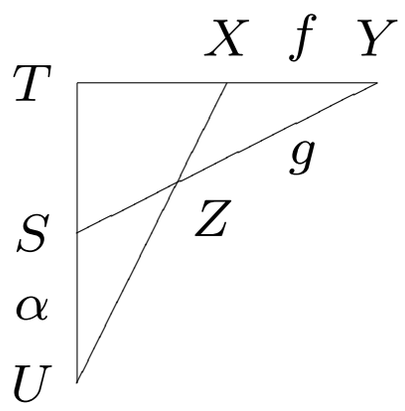
$$\Gamma(\beta \circ \alpha) = p_{XZ}(p_{XY}^{-1}(\Gamma(\alpha)) \cap p_{YZ}^{-1}(\Gamma(\beta))$$

Similarly, for correspondeneces between proper schemes one considers pull-backs and pushforwards of sheaves of ideals defining their graphs, and scheme theoretic intersection

$$\mathcal{I}_{\Gamma(\beta\circ\alpha)} = p_{XZ*}(p_{XY}^*(\mathcal{I}_{\Gamma(\alpha)}) \otimes_{\mathcal{O}_{XYZ}} p_{YZ}^*(\mathcal{I}_{\Gamma(\beta)}))$$

Hrushovski's group configuration

Hrushovski's theorem allows to recover a group law, in fact, even a group acting on a one-dimensional variety, from a collection of correspondences. One usually depicts the data as follows:



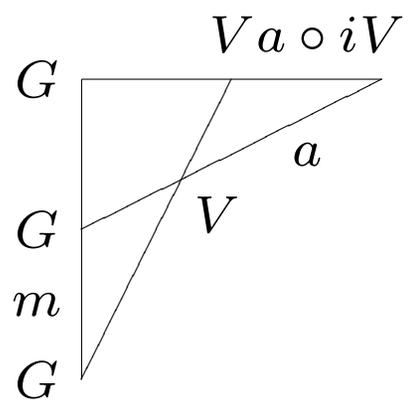
where $f : T \times X \vdash Y, g : S \times Z \vdash Y, \alpha : X \times Y \vdash Z$ are correspondences, finite-to-finite at generic points. The line $U - Z - X$ corresponds to the requirement that

$$\cup_{u \in \alpha(t,s)} \Gamma(g_s \circ f_t^{-1})$$

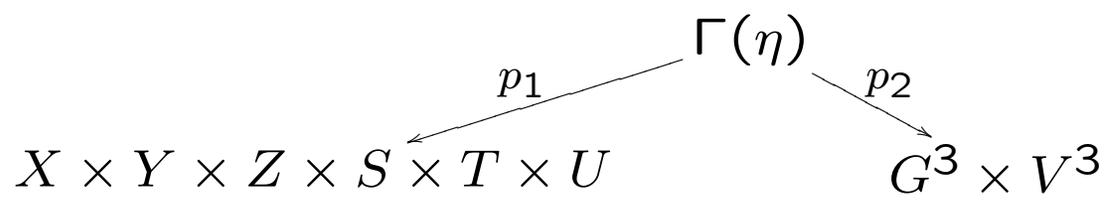
has an irreducible component that is a graph of a finite-to-finite correspondence h_u for generic $u \in U$.

Hrushovski's group configuration, continued

If G is a group with the group law $m : G \times G \to G$ and inverse $i : G \to G$, and $a : G \times V \to V$ is faithful group action then one has a naturally associated configuration



Theorem (*Hrushovski*). Given a group configuration as on the previous slide, $\dim S = \dim T = \dim U = 1$ there exists a (definable) group G, a definable set V, and a generically finite-to-finite correspondence η



such that $p_1^{-1}(\Gamma(\alpha)) = p_2^{-1}(\Gamma(\alpha'))$ share an irreducible component, for all respective correspondences α and α' in two configurations.

Endomorphisms of fat points

The scheme of the form $P_n = \mathrm{Spec}\, k[x]/(x^{n+1})$ is called a *fat point*. Recall that $\mathrm{Hom}(P_1, X) \cong TX(k)$.

One easily sees that $\mathrm{Aut}(P_1) = \mathbb{G}_m(k), \mathrm{Aut}(P_2) = \mathbb{G}_a \rtimes \mathbb{G}_m(k)$; in general, $\mathrm{Aut}(P_n)$ is some unipotent linear group.

If the graph of a correspondence $\alpha : X \vdash Y$ contains a point P, and the projection p_X is étale in a neighbourhood of P then by choosing closed embeddings $P_1 \to M$ such that $P_1 \times P_1 \to M^2$ maps the closed point to P and restricting $\Gamma(\alpha)$ to $P_1 \times P_1$ we get a graph of an endomorphism $\tau_\alpha : P_1 \to P_1$. We call τ_α an endomorphism of P_1 *associated to* α (at P).

Proposition. Let $\alpha, \beta : M \to M$ be two correspondences such that $P \in \Gamma(\alpha), \Gamma(\beta)$ for $P \in \Delta$, where Δ is the diagonal of M^2. Then

$$\tau_{\beta \circ \alpha} = \tau_\beta \circ \tau_\alpha$$

Finding a one-dimensional subfamily

There exists a point P such that curves passing through P induce infinitely many distinct associated endomorphisms of P_1.

Indeed, suppose the contrary. Then there exists a function $\varphi : M^2 \to \mathrm{End}(P_1)$ such that for any point Q every curve X_t that passes through Q (except finitely many) has associated endomorphism $\varphi(Q)$. Then each X_t expanded into formal series $y \in k[[x]]$ around some $Q \in M^2$, satisfies the equation

$$y' = f(x, y)$$

for some formal series $f \in k[[x, y]]$. But this ordinary differential equation has a unique solution with zero constant term (in char. 0), contradiction.

With some work one can actually find such a point on the diagonal $P \in \Delta \subset M^2$. We further denote as X^P the family of curves that pass through P, and it's parameter space as T^P.

Defining "tangency"

Let $X \to T, Y \to S$ be two families of curves in M^2 that both pass through a point P.

Fact. The length of the structure sheaf of the scheme-theoretic intersection of Y_s and X_t as an $\mathcal{O}_{M^2}$-module is constant for (t, s) in a dense open subset of $T \times S$. Same for the number of irreducible components of the module.

Let N be the number of intersections $\#(X_t \cap X_s)$ (on the level of geometric points, without counting multiplicities) for (t, s) in a dense open subset of $T \times S$

Proposition. If $\tau_{X_t} = \tau_{Y_s}$ then $\#(X_t \cap Y_s) < N$, where $\tau_{X_t} = \tau_{Y_s}$ are the associated endomorphisms of P_1.

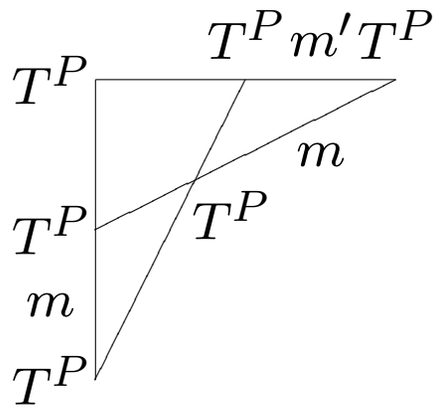
Notice that the opposite is not necessarily true. The relation $\#(X_t \cap X_s) < N$ is thus possibly coarser than the relation $\tau_{X_t} = \tau_{Y_s}$ and even if $T = S$, it might still be not transitive. However, it is definable in the structure (M, X) and it is enough to construct a group configuration.

Building the group configuration

Let N be the number of intersections $\#(X_t^P \circ X_s^P \cap X_u^P)$ for (t,s,u) in a dense open subset of T^3. Let $m : T^P \times T^P \vdash T^P$ be the correspondence defined as follows:

$$\begin{aligned} \Gamma(m) &= \{\ (t,s,u) \in (T^P)^3 \ \mid\ \#(X_t^P \circ X_s^P \cap X_u^P) < N\ \} \\ \Gamma(m') &= \{\ (t,s,u) \in (T^P)^3 \ \mid\ \#(X_t^P \circ X_u^P \cap X_s^P) < N\ \} \end{aligned}$$

Consider the configuration



One checks that

$$\{\ (u,z,x)\ \mid\ \tau_{X_x^P} = \tau_{X_z^P}^{-1} \circ \tau_{X_u^P}\ \} \subset \bigcup_{u \in \alpha(t,s)} \Gamma(m(s,-) \circ m(t,-))$$

Therefore, $\Gamma(m)$, which includes the lhs by the previous slide, intersects the rhs at a finite-to-finite correspondence, and the data satisfies the requirements of Hrushovski's theorem. There exists a definable one-dimensional group G.

Reduct of an algebraic group: getting a field

The one-dimensional group that we have defined is in one-to-one correspondence with T^P and hence with M. We can therefore consider the image of the family X in G^2 under the correspondence. There are other definable families of curves in G^2, and we can push them to G^2 as well. A standard argument implies that there exists a family that does not consist of cosets of subgroups of G^2.

So our new setting is this: a one-dimensional algebraic group G, a definable set $Z \subset G^2$ which is not a coset of a subgroup of G^2.

To complete the proof we need to define a field in the structure $(G, \cdot, Z)$. In fact, in suffices to define a two-dimensional group acting on a one-dimensional variety.

Theorem (*Cherlin, Hrushovski*) If G is a two-dimensional definable group (*in the setting of the conjecture*) acting on a definable one-dimensional set X, then G is definably isomorphic to $(\mathbb{G}_a \rtimes \mathbb{G}_m)(K)$ for a definable field K.

That G is isomorphic to $\mathbb{G}_a \rtimes \mathbb{G}_m$ can be directly observed for algebraic groups and varieties.

Reduct of an algebraic group: getting a field, continued

We follow the same strategy as before: find a one-dimensional family of curves through the fixed point $(0,0) \in G^2$ with infinitely many distinct associated endomorphisms of P_1 and then define a group configuration.

This time we use two operations: composition of endomorphisms and their addition using group law.

$$Z + W := \{ \ (x,y) \in M^2 \ \mid \ y = y_1 + y_2, y_1 \in Z, y_2 \in W \ \}$$

Lemma. Let G be an algebraic group over a field k. Then there exists an exact sequence

$$1 \to \mathbb{G}_a^{\dim G}(k) \to G(P_1) \to G(k) \to 1$$

so the group law on $T_0 G$ is that of a vector group.

Therefore, End(P_1) has a structure of a ring (actually a field $\cong k$), composition of curves inducing multiplication on End(P_1) and group law of G inducing addition.

Reduct of an algebraic group: finding a one-dimensional family

We consider the one-dimensional family of shifts of the definable curve $Z \subset G^2$

$$X_t := Z - t, t \in T^0 = Z$$

One ensures that the associated endomorphisms of P_1 are distinct.

$G = \mathbb{G}_a$ **and** $G = \mathbb{G}_m$. Passing to formal power series the condition that associated endomorphisms are constant in t amounts to a differential equation

$$y' = a \qquad \text{(for } \mathbb{G}_a\text{)} \qquad\qquad \frac{y'}{y} = \frac{a}{x} \qquad \text{(for } \mathbb{G}_m\text{)}$$

for some $a \in k$. Solving these equations we get that Z is a coset.

G — **elliptic curve**. Suppose to the contrary that the T_tZ is constant as a subspace of T_0G^2. Then TZ is trivial, hence Z is an elliptic curve, hence a coset of a subgroup of $G \times G$.

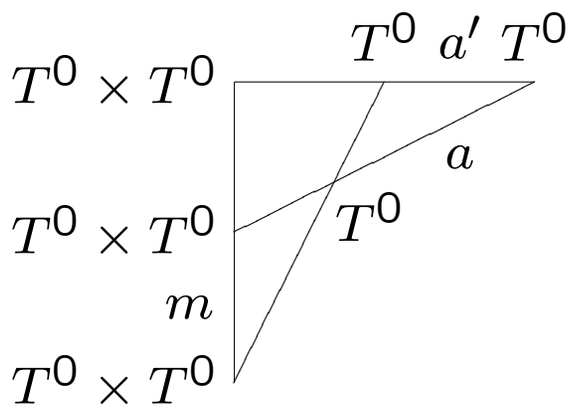
Reduct of an algebraic group: two-dimensional group configuration

Define

$$\begin{aligned}
\Gamma(m) &= \{(t_1,t_2),(s_1,s_2),(u_1,u_2)) \in (T^0\times T^0)^3 \mid \\
&\quad \#(X_{t_1}\circ X_{s_1}\cap X_{u_1}) < N_1 \text{ and} \\
&\quad \#(X_{t_1}\circ X_{s_2}+X_{t_2}\cap X_{u_2}) < N_2\} \\
\Gamma(a) &= \{\ ((t_1,t_2),x,y)\in (T^0)^3 \ \mid\ \#(X_{t_1}\circ X_x + X_{t_2}\cap X_y) < K\ \} \\
\Gamma(a') &= \{\ ((t_1,t_2),x,y)\in (T^0)^3 \ \mid\ \#(X_{t_1}\circ X_y + X_{t_2}\cap X_x) < K\ \}
\end{aligned}$$

where N_1, N_2, K are numbers of intersections of generic elements of respective families of curves.

Consider the configuration



There are certain additional requirements for two-dimensional group configuration which are satisfied but I wish not to mention them here.

Remarks on characteristic p

Caveat: ODEs don't have unique solutions any more. Given

$$y' = f(x, y)$$

the solutions with zero constant term are a torsor under $k[[x^p]]$.

Proposition. There exists a point P and a one-dimensional family of curves $X^P \to T^P$ such that one of the following holds

1. there are infinitely many endomorphisms of P_1 associated to curves X_t^P;

2. expansion of X_t^P into power series lies in $k[[x]]^{p^n}$ for almost all $t \in T^P$;

3. expansion of X_t^P into power series is of the form $ax + (f_t)^{p^n}$ for almost all $t \in T^P$.

Remarks on characteristic p, continued

In the first case we can proceed as in char. 0.

In the second case, we consider the family of curves of the form $X_{s_0}^{-1} \circ X_t$, and there are infinitely many associated endomorphisms of P_1.

In the third case we consider the family of curves of the form $X_{s_0}^{-1} \circ X_t$, and there are infinitely many associated endomorphisms of P_{p^n}.

One encounters similar problems with the second group configuration. Due to non-uniqueness of solutions of respective ODEs it might be the case that we have a non-coset Z such that its shifts $Z - t$ have constant associated endomorphism of P_1. However, using group operation and composition, it is always possible to construct a definable set that does not have this property.