Classification of definable groupoids and Zariski geometries

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Motivation: Azumaya algebras

An *Azumaya algebra* is a generalisation of a central simple algebra for base rings that might be not fields. An Azumaya algebra A over a local ring R is an algebra such that $A \otimes S \cong M_n(S)$ where S is étale over R. An Azumaya algebra over a scheme X is a coherent sheaf of algebras such that its stalks are Azumaya algebras over local rings $\mathcal{O}_{X,x}$. So geometrically, Azumaya algebra is a sheaf of "twisted" endomorphisms of a vector bundle; in particular, End($\mathcal{E}$) for a locally free sheaf $\mathcal{E}$ is an Azumaya algebra.

While End($\mathcal{E}$) is arguably a geometric object — its sections act faithfully on the total space of the vector bundle corresponding to $\mathcal{E}$ — an arbitrary Azumaya algebra doesn't seem to act (faithfully) on anything.

One of the aims of this talk is to introduce objects of geometric nature on which an Azumaya algebra can act.

Model theory

Model theory is the study of definable sets.

Fix a language — a collection of symbols with arities. A structure in this language is a set M with a relation $R \subset M^n$ for each n-ary symbol of the language, 0-ary relations or constants by convention correspond to elements of M. The collection of definable sets is generated from the basic relations using Boolean operations (intersection, union, negation) and quantification ($\exists$).

Formulas use straightforward notational shortcuts: $\forall = \neg\exists\neg$, functions can be represented by their graphs, etc.

Examples: $(\mathbb{Z}, +, 0)$, $(V, +, 0, (a\cdot)_{a\in k})$, $(K, +, \cdot, 0, 1)$.

Zilber's Trichotomy Conjecture and Zariski geometries

Zilber has conjectured (ca. 1980) a certain classification statement for so-called strongly minimal structures. Although the stamenet has been disproved by Hrushovski (1993) it turned out to be true for a class of strongly minimal structures called Zarski geometries (Hrushovski & Zilber 1996).

A Zariski geometry is an axiomatisation of some of the properties of Zariski topology of algebraic varieties. It is a structure where every definable set is constructible in the sense of a topology.

Examples: let X be an algebraic variety over an algebraically closed field. Let M be a structure with the universe all k-points of X and the language consisting of all closed subsets of X^n intepreted naturally. Then M is a Zariski geometry.

Similarly, if X is a compact complex manifold, one considers the language consisting of all analytic subsets of X^n for all $n > 0$, this structure also is a Zariski geometry.

Zariski geometries, definition

Let M be a set and let $\tau_n(M), n \in \mathbb{N}$ be a collection of topologies on M^n. A *Zariski geometry* is a model-theoretic structure in the language containing an n-ary predicate for each closed subset of M^n for all n, subject to the following two sets of conditions.

Topological axioms

1. singletons and diagonal sets are closed;
2. coordinate projection maps are continuous;
3. permutation of coordinates maps are homeomorphisms;
4. quantifier elimination: any definable subset of M^n is constructible in the sense of $\tau_n(M)$;

Zariski geometries, contd

Dimension axioms

There exists a function dim from the collection of constructible subsets of M^n to non-negative integers

1. $\dim\{\ x\ \} = 0$, $\dim Z \cup W = \max(\dim Z, \dim W)$;

2. given a definable continuous surjective map of closed irreducibles $X \to Y$

$$\dim X = \dim Y + \min_y \dim X_y$$

3. given a definable continuous surjective map of closed irreducibles $X \to Y$ the function $y \mapsto \dim X_y$ is upper semi-continuous;

Pre-smoothness: for any closed definable sets $Z, W \subset X^n$

$$\dim Z \cap W = \dim Z + \dim W - n$$

Non local modularity: there exists a definable set $X \subset T \times M^2$ such that $\dim T = 2$ and X_t is closed irreducible in M^2.

Non-standard Zariski geometries

Theorem (H. and Z. 1996): Let M be a one-dimensional pre-smooth non locally modular Zariski geometry. Then there exists a definable equivalence relation on M with finite equivalence classes, and an open set $U \subset M$ such that $U/\sim$ and all its Cartesian powers are homeomorphic to Cartesian powers of an algebraic curve C.

Let E be an elliptic curve and let $\mathbb{Z}^2$ act on it freely. Consider a non-split central extension

$$1 \to \mathbb{Z}/n \to G \to \mathbb{Z}^2 \to 1$$

Replace every $\mathbb{Z}^2$-orbit with a copy of G getting a new structure $\tilde{E}$ with a natural map $\tilde{E} \to E$ induced by the map $p : G \to \mathbb{Z}^2$. In order to do it, one must non-canonically choose a representative in each orbit, but all the structures one thus obtains are isomorphic.

Declare sets of the form $p^{-1}(Z)$ closed for Z closed (and similarly for Cartesin powers), as well as graphs of actions by elements of the group, and consider the topology generated by such.

Theorem (H. & Z.) $\tilde{E}$ is a non locally modular Zariski geometry. The Cartesian powers $\tilde{E}^n$ are not homeomorphic to Zariski topologies of an algebraic curve, nor they are coarsenings of Zariski topologies of Cartesian powers of a curve.

In model-theoretic parlance, we say that $\tilde{E}$ is *not interpretable* in the theory of algebraically closed fields.

Zilber's quantum Zariski geometries

Structures of two sorts $M = ((V, +), (k, +, \cdot, 0, 1), p : V \to X(k), \cdot : k \times V \to V)$, k algebraically closed field, X an affine variety defined over k. For every $x \in X$, the fibre $V_x = p^{-1}(x)$ has the structure of a vector space over k.

Example (Zilber 2006): a quantum Zariski geometry M which is acted upon definably by the quantum torus algebra

$$A = k\langle \mathbf{u}, \mathbf{v}, \mathbf{u}^{-1}, \mathbf{v}^{-1} \mid \mathbf{uv} = q\mathbf{vu}, \mathbf{uu}^{-1} = \mathbf{u}^{-1}\mathbf{u} = 1, \mathbf{vv}^{-1} = \mathbf{v}^{-1}\mathbf{v} = 1\rangle$$

where $q^n = 1$. A is an Azumaya algebra over its center $k[x, x^{-1}, y, y^{-1}] = k[X]$ where $X = \mathbb{G}_m \times \mathbb{G}_m$.

In a fibre $V_{(a,b)}$ we pick a basis and choose n-th roots of a, b — $\mu^n = a, \nu^n = b$ — and define the action of $\mathbf{u}, \mathbf{v}$ by matrices

$$\mathbf{u} = \begin{pmatrix} \mu & 0 & 0 & \dots & 0 \\ 0 & q\mu & 0 & \dots & 0 \\ 0 & 0 & q^2\mu & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & & & \dots & q^{n-1}\mu \end{pmatrix} \qquad \mathbf{v} = \begin{pmatrix} 0 & 0 & \dots & 0 & \nu \\ \nu & 0 & \dots & 0 & 0 \\ 0 & \nu & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & & \dots & \nu & 0 \end{pmatrix}$$

Quantum Zariski geometries, contd

If we replace μ, ν with conjugates, the matrices $\mathbf{u}(\mu, \nu), \mathbf{v}(\mu, \nu)$ will change to $\mathbf{u}(q^l \mu, q^m \nu), \mathbf{v}(q^l \mu, q^m \nu)$. We regard this as a choice of a point in a fibre of a cover $Y \to X$ where $Y = \mathbb{G}_m \times \mathbb{G}_m$ and the map is raising both coordinates to the n-th power.

In order for the structure to be well-defined the matrices for $\mathbf{u}, \mathbf{v}$ ought to be matrices of the same transformations in a different basis. More precisely, there ought to be a function $g : \mathrm{Gal}(Y/X) \to \mathrm{PGL}_n(\mathcal{O}_X)$ such that

$$\begin{array}{rcl} \mathbf{u}(\gamma \cdot y_0) & = & g(\gamma)\, \mathbf{u}(y_0)\, g^{-1}(\gamma) \\ \mathbf{v}(\gamma \cdot y_0) & = & g(\gamma)\, \mathbf{v}(y_0)\, g^{-1}(\gamma) \\ g(\alpha\beta) & = & g(\alpha)\alpha^{-1} g(\beta)\alpha \end{array}$$

In case of our structure

$$g(q,0) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \qquad g(0,q) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & q^{n-1} \end{pmatrix}$$

Proposition (Zilber) Structure M is a Zariski geometry, its isomorphism type does not depend on the choice of μ, ν in each fibre $V_{(a,b)}$.

Definable groupoids

A *groupoid* is a category such that all its morphisms are isomorphisms. If a groupoid is small, i.e. if its objects and its morphisms are sets, then it is defined by the following data: a tuple $X_\bullet = (X_0, X_1)$ of sets along with maps s, t, m, i, e, where s, t maps X_1 to X_0 (source and target objects), c maps $X_1 \times_{s,X_0,t} X_1$ to X_1 (composition of arrows), i maps X_1 to itself (inverse), $e : X_0 \to X_1$, satisfying the natural axioms.

A *definable groupoid* is a pair of definable sets X_0, X_1 along with the morphisms s, t, m, i, e satisfying the mentioned identities.

If $\mathrm{Mor}(x, x)$ is isomorphic to a group A for all $x \in X_0$ then the groupoid $X_\bullet$ is said to be *bounded* by A.

Example: G be a definable group, $\cdot : G \times X \to X$ be a group action. *action groupoid*: $G \times X \rightrightarrows X$ where $s(g, x) = x$ and $t(g, x) = g \cdot x$, and $(g, x) \cdot (h, gx) = (gh, x)$;

Groupoid torsors

Let $X_\bullet$ be a groupoid. A *groupoid homogeneous space for* $X_\bullet$ *over* Y is a morphism $p : P \to Y$ together with the *anchor map* $a : P \to X_0$ and *action map* $\cdot : P \times_{a,X_0,s} X_1 \to P$ which commutes with the projection to Y. A homogeneous space is called *principal* (or a *torsor*) if for any two $f, g \in P$ such that $p(f) = p(g)$ there exists a unique $m \in X_1$ such that $f \cdot m = g$.

A morphism of groupoid torsors P and Q is a map $\alpha : P \to Q$ that commutes with the action map: $\alpha(m \cdot f) = m \cdot \alpha(f)$ for any $a \in \mathrm{Ob}(X_\bullet)$ and any $m \in \mathrm{Mor}(a, s(f))$.

Let $X_\bullet$ be a groupoid. Let E be the equivalence relation on X_0 which is the image of the map $(s,t) : X_1 \to X_0 \times X_0$. The quotient X_0/E is called the *groupoid quotient* and is denoted $[X_\bullet]$.

A groupoid $X_\bullet$ is called *eliminable* if there exists a $X_\bullet$-groupoid torsor over $[X_\bullet]$.

Morita equivalence

A *Morita morphism* $f_\bullet : X_\bullet \to Y_\bullet$ is a pair of maps $f_0 : X_0 \to Y_0, f_1 : X_1 \to Y_1$ such that the diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \times X_0 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0 \times f_0} \\ Y_1 & \longrightarrow & Y_0 \times Y_0 \end{array}$$

commutes, f_0 is surjective and for any $(x_1, x_2) \in X_0 \times X_0$ the map f_1 induces a bijection between $Mor(x, y)$ and $Mor(f_0(x), f_0(y))$. If one looks at groupoids as small categories, then the above conditions say precisely that Morita morphism defines a fully faithful functor which is surjective on objects.

Two groupoids $X_\bullet$ and $Y_\bullet$ are called *Morita equivalent* if there exists a third groupoid $Z_\bullet$ together with two Morita morphisms $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$.

Proposition Morita equivalence preserves eliminability.

Brauer group

Two Azumaya algebras A, A' over X are called *Morita equivalent* if there exists a vector bundle $V \to X$ such that $A \otimes \mathsf{End}(V) \cong A' \otimes \mathsf{End}(V)$ where $\mathsf{End}(V)$ is the $k[X]$-algebra of endomorphisms of the vector bundle V. The Morita equivalence classes of Azumaya algebras over X form a group under the tensor operation called the *Brauer group of* X.

An Azumaya algebra A over X is a twisted form of a sheaf of endomorphism of a locally free sheaf over X. The descent data for A gives a Čech cocycle in $\check{H}^1(X_{\text{ét}}, \mathsf{PGL}_n)$. The image of this cocycle in the cohomology long exact sequence

$$\to H^1(X_{\text{ét}}, \mathsf{GL}_n) \to H^1(X_{\text{ét}}, \mathsf{PGL}_n) \xrightarrow{\delta} H^2(X_{\text{ét}}, \mathbb{G}_m) \to \ldots$$

is a torsion cocycle in $H^2(X_{\text{ét}}, \mathbb{G}_m)$.

Theorem (Gabber). The Brauer group of an affine scheme X is isomorphic to the torsion part of $H^2(X_{\text{ét}}, \mathbb{G}_m)$.

Splitting groupoid

Let us get back to the quantum torus example (keeping notation).

The *splitting groupoid* $S_\bullet$ has the objects set $S_0 = Y$ and the morphisms set S_1 is $(Y \times_X Y) \times k^\times$ with the obvious source and target maps. The composition is defined as follows:

$$(y_0, \alpha \cdot y_0, a) \circ (\alpha \cdot y_0, \beta\alpha \cdot y_0, b) = (y_0, \beta\alpha y_0, g(\alpha)\alpha^{-1} g(\beta)\alpha(g(\alpha\beta))^{-1} ab)$$

One can think of objects — points of S_0 — as isomorphisms of fibres of the Azumaya algebra with $M_v(k(y))$ (splittings) and morphisms — points of S_1 — as isomorphisms between them.

One can show that the set of bases in which operators $\mathbf{u}, \mathbf{v}$ are represented by matrices from before for *some* choice of roots μ, ν is a torsor under $S_\bullet$. This groupoid torsor "lives" in V.

Theorem (S.) There is a bijective correspondence between Morita equivalence classes of finite definable groupoids defined over K, and bounded by a group G and $H^2(\mathrm{Gal}(K^{\mathrm{alg}}/K^{\mathrm{perf}}), G(K^{\mathrm{alg}}))$ and eliminable groupoids correspond to the trivial class.

This is a specialisation (to the theory of algebraically closed fields) of a theorem that classifies definable groupoids in a general model-theoretic setting.

Interpretability and generalised imaginaries

An Azumaya algebra is called *split* if it is of the form $\mathrm{End}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$.

The class in $H^2(\mathrm{Gal}(k(X)^{\mathrm{sep}}/k(X)), \mathbb{G}_m)$ corresponding to the splitting groupoid associated to a quantum Zariski geometry is the class of the restriction of the Azumaya algebra corresponding to it to the generic point of X.

Theorem (S.) Let A be an Azumaya algebra over a variety X, and suppose that the restriction of A to a generic point of one of positive-dimensional subvarieties of X is not split. Then the quantum Zariski structure corresponding to A is not interpetable in an algebraically closed field.