

Galois cohomology and finite generalised imaginaries

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Amalgamation problems

Let $p_{12}(x, y), p_{23}(y, z), p_{13}(x, z)$ be three types over a set of parameters K . These three types amalgamate if there exists a type $p_{123}(x, y, z)$ such that whenever (a_1, a_2, a_3) realizes p_{123} , (a_1, a_2) realizes p_{12} , (a_2, a_3) realizes p_{23} and (a_1, a_3) realizes p_{13} . A theory has *3-existence* if any three types amalgamate. It is customary to assume that all tuples in question enumerate algebraically closed substructures of the monster model.

3-uniqueness (over $K = a_\emptyset$): the type p_{123} is unique, in other words, whenever $a_{ij} \models p_{ij}$ and $\sigma_{ij} \in \text{Aut}(a_{ij}/a_i a_j)$

$$\text{tp}(a_{12}a_{23}a_{13}/K) = \text{tp}(\sigma_{12}(a_{12})\sigma_{23}(a_{23})\sigma_{13}(a_{13})),$$

in other words, the restriction map

$$\text{Aut}(a_{123}/a_1 a_2 a_3) \rightarrow \text{Aut}(a_{12}/a_1 a_2) \times \text{Aut}(a_{23}/a_2 a_3) \times \text{Aut}(a_{13}/a_1 a_3)$$

is surjective, and 2-uniqueness amounts to the fact that for a 2-amalgamation problem

$$\text{Aut}(a_{12}/K) \rightarrow \text{Aut}(a_1/K) \times \text{Aut}(a_2/K)$$

is surjective.

How 3-uniqueness can break down

In stable theories, over an algebraically closed base set, 2-uniqueness (=stationarity over a.c. base) holds.

Therefore, if one considers $K = a_3$ as the base of amalgamation,

$$\text{Aut}(a_{123}/a_3) \rightarrow \text{Aut}(a_{13}/a_3) \times \text{Aut}(a_{23}/a_3)$$

is surjective. Therefore, 3-uniqueness amounts to the map $r : \text{Aut}(a_{12}/a_{23}a_{13}) \rightarrow \text{Aut}(a_{12}/a_1a_2)$ being surjective.

Remark (Hrushovski) If 3-uniqueness fails then image of $\text{Im } r$ in the Abelianisation $\text{Aut}(a_{12}/a_1a_2)^{ab}$ is a proper subgroup.

Proposition (Goodrick, Kolesnikov) A failure of 3-uniqueness is witnessed by existence of a definable over K groupoid that is not eliminable (definitions will come later).

It is a classic fact that group extensions with Abelian kernel are classified by second group cohomology. So it seems natural that a groupoid witnessing non-3-uniqueness ought to be related to it too.

Group cohomology

Let G be a group acting on an Abelian group A (it is then called a G -module). The group cohomology $H^n(G, A)$ is a collection of groups associated in a functorial way to A . One concrete way to define it is as follows:

A $(n-)$ cochain is a map $G^n \rightarrow A$. It is *cocycle* if it satisfies a certain condition which for small n is as follows:

$$\begin{aligned} \text{for } n = 1 & \quad h(\sigma\tau) = h(\sigma) + \sigma \cdot h(\tau) \\ \text{for } n = 2 & \quad h(\alpha\sigma, \tau) = h(\alpha, \sigma\tau) - h(\alpha, \sigma) + \alpha \cdot h(\sigma, \tau) \end{aligned}$$

It is a coboundary if

$$\begin{aligned} \text{for } n = 1 & \quad \text{there exists } g \in A \text{ such that } h(\sigma) = \sigma(g) - g \\ \text{for } n = 2 & \quad \text{there exists } g : G \rightarrow A \text{ such that } h(\sigma, \tau) = g(\sigma) - g(\sigma\tau) + \sigma \cdot g(\tau) \end{aligned}$$

The n -th cohomology group is the quotient of the group of n -cocycles by the group of n -coboundaries. The definition of H^1 can be stated for non-Abelian A , but it will no longer have structure of a group.

If G is a profinite group, $G = \varprojlim G/G_\alpha$, and the action of G on A is continuous, then one defines

$$H^n(G, A) = \varprojlim H^n(G/G_\alpha, A^{G_\alpha})$$

Galois cohomology and torsors

Let M be a model and let A be an Abelian definable group defined over a set of parameters K . Then $A(M)$ is naturally a $G = \text{Aut}(M/K)$ -module. If $M = \text{acl}(K)$ then G has a profinite structure and the action is continuous.

A *principal homogeneous space over A or torsor* is definable set X together with a free transitive action of A .

Proposition (Pillay) Suppose the theory we work in has elimination of imaginaries, and $A = \text{acl}(K)$. The set of isomorphism classes of torsors over A definable over K is in bijective correspondence with $H^1(G, A(M))$.

In fact, the addition operation in H^1 can be defined geometrically.

Pillay has also worked out a definition of Galois cohomology in the setting where M is atomic over K where the above proposition is still true.

Group extensions

Let A, B be groups, A Abelian.

Proposition Consider a group extension

$$1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1$$

with A Abelian, and pick a section $\iota : B \rightarrow G$. Then $b \in B$ acts on A by conjugation by $\iota(b)$, the action being independent from ι . The set of isomorphism classes of group extensions with the given action of B on A is in bijective correspondence with elements of $H^2(B, A)$. Split extensions correspond to the trivial cohomology class.

The cohomology class is defined as follows: $h(\sigma, \tau) = \iota(\sigma)\iota(\tau)\iota(\sigma\tau)^{-1}$, which turns out to be a cocycle, and its cohomology class does not depend on ι .

If one is interested in profinite groups, H^2 only classifies extensions such that $G \rightarrow B$ has a continuous section.

Groupoids

A *groupoid* is a category such that all its morphisms are isomorphisms. If a groupoid is small, i.e. if its objects and its morphisms are sets, then it is defined by the following data: a tuple $X_\bullet = (X_0, X_1)$ of sets along with maps s, t, m, i, e , where s, t maps X_1 to X_0 (source and target objects), c maps $X_1 \times_{s, X_0, t} X_1$ to X_1 (composition of arrows), i maps X_1 to itself (inverse), $e : X_0 \rightarrow X_1$, satisfying the natural axioms.

A *definable groupoid* is a pair of definable sets X_0, X_1 along with the morphisms s, t, m, i, e satisfying the mentioned identities.

If $\text{Mor}(x, x)$ is isomorphic to a group A for all $x \in X_0$ then the groupoid X_\bullet is said to be *bounded* by A .

Example: G be a definable group, $\cdot : G \times X \rightarrow X$ be a group action. *action groupoid:* $G \times X \rightrightarrows X$ where $s(g, x) = x$ and $t(g, x) = g \cdot x$, and $(g, x) \cdot (h, gx) = (gh, x)$;

Groupoid torsors

Let X_\bullet be a groupoid. A *groupoid homogeneous space* for X_\bullet over Y is a morphism $p : P \rightarrow Y$ together with the *anchor map* $a : P \rightarrow X_0$ and *action map* $\cdot : P \times_{a, X_0, s} X_1 \rightarrow P$ which commutes with the projection to Y . A homogeneous space is called *principal* (or a *torsor*) if for any two $f, g \in P$ such that $p(f) = p(g)$ there exists a unique $m \in X_1$ such that $f \cdot m = g$.

A morphism of groupoid torsors P and Q is a map $\alpha : P \rightarrow Q$ that commutes with the action map: $\alpha(m \cdot f) = m \cdot \alpha(f)$ for any $a \in \text{Ob}(X_\bullet)$ and any $m \in \text{Mor}(a, s(f))$.

Let X_\bullet be a groupoid. Let E be the equivalence relation on X_0 which is the image of the map $(s, t) : X_1 \rightarrow X_0 \times X_0$. The quotient X_0/E is called the *groupoid quotient* and is denoted $[X_\bullet]$.

A groupoid X_\bullet is called *eliminable* if there exists a X_\bullet -groupoid torsor over $[X_\bullet]$. In the terminology introduced by Hrushovski groupoid torsors over $[X_\bullet]$ with all the relevant structure maps are *generalised imaginary sorts*.

Theorem (*Hrushovski*) In a stable theory with elimination of imaginaries, 3-uniqueness is equivalent to the fact that all groupoids with finitely many objects are eliminable.

Morita equivalence

A *Morita morphism* $f_{\bullet} : X_{\bullet} \rightarrow Y_{\bullet}$ is a pair of maps $f_0 : X_0 \rightarrow Y_0, f_1 : X_1 \rightarrow Y_1$ such that the diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & X_0 \times X_0 \\ \downarrow f_1 & & \downarrow f_0 \times f_0 \\ Y_1 & \longrightarrow & Y_0 \times Y_0 \end{array}$$

commutes, f_0 is surjective and for any $(x_1, x_2) \in X_0 \times X_0$ the map f_1 induces a bijection between $Mor(x, y)$ and $Mor(f_0(x), f_0(y))$. If one looks at groupoids as small categories, then the above conditions say precisely that Morita morphism defines a fully faithful functor which is surjective on objects.

Two groupoids X_{\bullet} and Y_{\bullet} are called *Morita equivalent* if there exists a third groupoid Z_{\bullet} together with two Morita morphisms $Z_{\bullet} \rightarrow X_{\bullet}$ and $Z_{\bullet} \rightarrow Y_{\bullet}$.

Proposition Morita equivalence preserves eliminability.

Proposition Generalised imaginary sorts corresponding to Morita equivalent groupoids are bi-interpretable.

Groupoids and group cohomology

Notation: $G_K = \text{Aut}(\text{acl}(K)/\text{dcl}(K))$, $G_{L/K} = \text{Aut}(\text{dcl}(L)/\text{dcl}(K))$.

Theorem (S.) Suppose $M = \text{acl}(K)$. There exists a bijective correspondence

$$\left\{ \begin{array}{l} \text{Morita equivalence classes of} \\ \text{connected groupoids} \\ \text{definable over } K \\ \text{and bounded by a group } A \end{array} \right\} \Leftrightarrow \left\{ \text{cohomology classes in } H^2(G_K, A) \right\}$$

Eliminable groupoids correspond to the trivial cohomology class.

There is an operation (Baer sum) on groupoids that is mapped by the correspondence to addition in cohomology groups. One can check that the difference of two Morita equivalent groupoids is eliminable, and then it is left to verify the bijectivity.

Hochschild-Serre spectral sequence

Let $G_L \subset G_K$ be normal. Then the following sequence (the lower terms long exact sequence associated to Hochschild-Serre spectral sequence) is exact

$$\begin{aligned} \dots \rightarrow H^1(G_L, A)^{G_{L/K}} \xrightarrow{\text{tr}} H^2(G_{L/K}, A^{G_L}) \xrightarrow{\text{inf}} \\ \xrightarrow{\text{inf}} \text{Ker}(H^2(G_K, A) \rightarrow H^2(G_L, A)^{G_{L/K}}) \xrightarrow{\rho} H^1(G_{L/K}, H^1(G_L, A)) \end{aligned}$$

The inflation map inf on cohomology induced by pulling back a cochain along the quotient projection $G_K \rightarrow G_K/G_{L/K}$. Restriction map res is just the map induced on cohomology by restricting cochain to a subgroup.

The correspondence (\Rightarrow) Suppose that Q is a torsor for a groupoid X_\bullet , then let $P = \sqcup_{\sigma \in G_{L/K}} \sigma(Q)$, and let G be the group generated by action of $\text{Gal}(L/K)$ and A on P (one needs to trivialise the fibres of P in order for the action of A to be well-defined). Then X_\bullet is Morita equivalent to an action groupoid $Q \times G \rightrightarrows \sqcup_{\sigma \in G_{L/K}} \{\sigma(a)\}$, and this yields a class in $H^2(G_{L/K}, A)$.

Hochschild-Serre spectral sequence, contd

$$\dots \rightarrow H^1(G_L, A)^{G_{L/K}} \xrightarrow{\text{tr}} H^2(G_{L/K}, A^{G_L}) \xrightarrow{\text{inf}}$$

$$\xrightarrow{\text{inf}} \text{Ker}(H^2(G_K, A) \rightarrow H^2(G_L, A)^{G_{L/K}}) \xrightarrow{\rho} H^1(G_{L/K}, H^1(G_L, A))$$

The correspondence (\Leftarrow) $\text{Ker}(H^2(G_K, A) \rightarrow H^2(G_L, A)^{G_{L/K}})$ consists of classes of groupoids that are eliminable over L . We want to construct a Morita equivalence with a group action groupoid, we can only do it for L big enough, so that the class lives in the image of the inf map. One can achieve this by ensuring that the image of the class under ρ is trivial, using functoriality of the spectral sequence.

Remark The result above only seems to be vacuous for theories such as ACF that have 3-uniqueness over algebraically closed sets: see next slide.

The correspondence ought to generalise to the setting used by Pillay (and maybe beyond) for the right definition of H^2 where cocycles are represented by definable functions.

Non-standard Zariski geometries

Let X be an algebraic curve. Let H be a group acting on X freely and let $1 \rightarrow A \rightarrow G \rightarrow H \rightarrow 1$ where A is finite Abelian. Consider the structure M whose universe is formed by replacing every H -orbit (which is a copy of H) with a copy of G . There is natural action of G on M and a projection $M \rightarrow X$. Basic relations on M are pre-images of definable sets on X under the projection and graphs of the action of G on M . Incidentally, M also has a structure of a Zariski geometry (declare the mentioned basic relations and their positive Boolean combinations closed).

Proposition (Hrushovski) Let X be an elliptic curve, $H = \mathbb{Z}^2$, and G be the non-split extension of H by $\mathbb{Z}/2\mathbb{Z}$. Then M is not interpretable in an algebraically closed field.

Proposition (S.) If G, A are finite then M is interpretable in an algebraically closed field expanded with the generalised imaginary sort corresponding to the groupoid $G \times X \rightrightarrows X$. Therefore is interpretable in an algebraically closed field iff the group extension is split.

There are more intricate examples of Zariski geometries, constructed by Zilber (“A class of quantum Zariski geometries”, 2005) which are also interpretable in generalised imaginary sorts.