Gromov-Hausdorff limits via model theory

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Gromov-Hausdorff distance

Let $(X, d_1), (Y, d_2)$ be two metric spaces. The Gromov-Hausdorff distance between X and Y is the infimum of numbers ε such that there exist isometric embeddings $X \hookrightarrow Z, Y \hookrightarrow Z$ such that X is in an ε-neighbourhood of Y and Y is in an ε-neighbourhood of X.

Properties:
1. The GH distance between a metric space and its completion is 0
2. GH distance satisfies the axioms of a metric on the "set" of isomorphism classes of complete compact metric spaces
3. Finite metric spaces are dense among compact metric spaces

Examples: points on a circle, annulus $\to$ circle.

Convergence criterion: let $\mathcal{C}$ be a class of metric spaces such that the diameter is uniformly bounded above and for every $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that any space $X \in \mathcal{C}$ admits an ε-dense subset of cardinality less than $N(\varepsilon)$. Then every sequence of elements of $\mathcal{C}$ has a GH-converging subsequence.

Gromov compactness theorem: the above is true for a class $\mathcal{C}$ of Riemannian manifolds with Ricci curvature uniformly bounded below and diameter uniformly bounded above.

structures, definable sets

Model theory studies definable sets in structures. Let M be a set, and let $R_1, R_2, \ldots$ be a collection of subsets of Cartesian powers of M — they are called *atomic relations*. A definable subset is an element of the collection of subsets of Cartesian powers of M closed under the following operations:

1. Boolean operations ($\cup$, $\cap$, $\setminus$)
2. Cartesian products
3. images under coordinate projections $M^{n+k} \to M^n$
4. taking fibres of the projections $M^{n+k} \to M^n$

A language is a choice of names and "arities" of atomic relations. A definable subset can be specified by a formula in a language (cf. solution sets of polynomials).

Examples:

1. $(\mathbb{C}, 0, 1, +, \times)$. Definable sets are constructible sets (Boolean combinations of closed).
2. $(\mathbb{R}, 0, 1, +, \times)$. Definable sets are semi-algebraic sets (Boolean combinations of sets defined by polynomial equations and inequalities).

ultraproducts

A *filter* F on a set X is a collection of subsets of X closed under intersections and taking supersets. An *ultrafilter* is a maximal filter. A *principal ultrafilter* is one of the form $\{ \ Y \subset X \ \mid \ x \in Y \ \}$.

Given a collection of structures $(M_i)_{i\in I}$ in a fixed language and an ultrafilter $\mathcal{U}$ on I one defines the *ultraproduct* of M_i-s as the quotient of $\prod_{i\in I} M_i$ by the relation

$$(x_i) \sim (y_i) \text{ iff } \{ \ i \in I \ \mid \ x_i = y_i \ \} \in \mathcal{U}$$

with atomic relations defined as follows

$$R_j(\overline{x}_i) \text{ iff } \{ \ i \in I \ \mid \ \overline{x}_i \in R_j^{M_i} \ \} \in \mathcal{U}$$

This carries over to formulas, i.e.

$$\varphi(\overline{x}_i) \text{ iff } \{ \ i \in I \ \mid \ \overline{x}_i \in \varphi^{M_i} \ \} \in \mathcal{U}$$

for a formula φ.

Example: $\prod_{\mathcal{U}} \mathbb{R}$ is a real closed field. The reals are naturally embedded into it (the image is called *standard reals*), there are inifintely big elements and infinitesimal elements. There exists a standard part map $st : \mathcal{O} \to \mathbb{R}$ from the convex hull of reals that kills the infinitesimals ($\mathcal{O}$ is acutally a value ring, and st is the residue map).

GH-limits as reductions

Let $X \to T$ be a family of complete metric spaces of bounded diameter definable (i.e. X is definable, and metric is uniformly definable) in a structure M that contains the reals, and suppose that the cardinality of ε-dense net is bounded uniformly for elements of the family.

Let ${}^*M$ be an ultrapower of M. Consider an equivalence relation on fibres X_t:

$$x \sim y \text{ iff } \operatorname{st} d(x, y) = 0$$

Let $t \in T$ be a non-standard point of T. Then $X_t/\sim$ is a Gromov-Hausdorff limit of a sequence of metric spaces (X_{t_i}, d_{t_i}).

Proof: for any ε there is an ε-dense finite subset $x_1, \ldots, x_n$ of X_t. The property of having an ε-dense set of given cardinality is expressible as a first-order formula depending on parameter t. By evaluating this formula in M we get X_{t_i} which is distance at most 2ε from the reduction $X_t/\sim$.

maximally degenerate families

Recall that a Hermitian manifold is a complex manifold X with a Hermitian metric h, which can be written down as $h = g - i\omega$, $g = 1/2(h + \overline{h})$, $\omega(x, y) = g(Ix, y)$, so g is a Riemannian metric and ω is an anti-symmetric form. If ω closed, X is called *Kähler*.

We will be considering families of triples $(X_t, \omega_t, \Omega_t)$ where parameter t belongs to the punctured disc. For every t, X_t is a Kähler manifolds, $\omega_t \in H^0(X_t, \mathcal{A}^{1,1})$ is the fundamental form of X_t, and Ω_t is a distinguished nowhere degenerate holomorphic n-form. The cohomology class of ω_t is assumed to be ample.

The volume of X_t can be computed as

$$\int_{X_t} \Omega_t \wedge \overline{\Omega}_t = C \log |t|^m \cdot |t|^{2l}(1 + o(1))$$

as $t \to 0$. If $m = n$ then one says that the family is *maximally degenerate*.

Example: Tate curve.

$X_t : \mathbb{C}/\langle 1, \frac{\log t}{2\pi i}\rangle$. $\Omega_t = dz, \omega_t = dz \wedge d\overline{z}$.

Conjectures of Konstevich and Soibelman

Let M be an n-dimensional manifold. An *integral affine structure* is given by an atlas $\{U_i, \varphi_i\}$ such that $\varphi_i \circ \varphi_j^{-1} \in \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{Z}_n)$.

Examples: torus. focus-focus singularity

K-S conjectures (“Homological mirror symmetry and torus actions”’2001): Let $(X_t, \omega_t, \Omega_t)$ be a maximally degenerate family of n-dimensional Calabi-Yau manifolds. Let d_t be the geodesic distance on X_t and D_t be the diameter of X_t wrt this distance. Then there exists a GH limit $(\overline{Y}, d)$ of a sequence $(X_{t_i}, d_{t_i}/D_{t_i})$ such that

1. $\dim \overline{Y} = n$ and there exists a dense subset $Y \subset \overline{Y}$ such that $\dim(\overline{Y} \setminus Y) = 2$
2. there is an integral affine structure on Y
3. the Riemannian metric on Y comes from a potenial $g_{ij} = \det \dfrac{\partial^2 K}{\partial x_i \partial x_j}$
4. the metric satisfies real Monge-Ampere equation: $\det(g_{ij}) = C$

Calabi-Yau manifolds

If one considers the volume of a thin solid cone of radius ε in the direction ξ then Ricci curvature is a symmetric 2-tensor having property that $Ric(\xi,\xi)$ measures how much bigger or smaller is the volume of this cone compared to the volume of a similar cone in the Euclidean space.

On a Kähler manifold Ricci curvature can be computed as follows: consider the Hermitian metric on the canonical bundle $\Omega^n_X = \Lambda T^* X$ given by $||\alpha|| = \dfrac{\alpha \wedge \overline{\alpha}}{\omega^n}$ then $Ric = \partial\overline{\partial}\ \log ||\alpha||$ where $\alpha \in H^0(X, \Omega^n_X)$.

Theorem (Yau). Let X be a Kähler manifold with fundamental form ω and trivial canonical bundle. Then there exists a unique up to rescaling Kähler metric with fundamental form cohomologous to ω which is Ricci-flat.

Kähler manifolds with Ricci-flat metric are called *Calabi-Yau manifolds*.

o-minimal structures

o-minimal structure is a structure M that has a total order relation on its universe M such that the only definable subsets of M are unions of generalised open intervals (ends can be $+\infty, -\infty$) and points.

Examples:
1. $(\mathbb{R}, +, \times, 0, 1)$ (Tarski)
2. $\mathbb{R}_{an}$: add graphs of analytic functions restricted to the unit cube $[0,1]^n$ (van den Dries)
3. $\mathbb{R}_{Pfaff}$: add solutions to Pfaffian systems of ODEs — more on it later (van den Dries, Wilkie, Speisseger)

Some properties:
1. A definable function has finitely many isolated zeroes
2. A definable function $R \to R$ is differentiable away from finitely many points, infinitely differentiable away from countably many points
3. A definable set is a union of cells

Robinson's asymptotic fields

Let $^*\mathbb{R}$ be an ultrapower of $\mathbb{R}$. Let α be an infinitely big element of $^*R$. Define

$$O(\alpha) := \cup[-\alpha^n, \alpha^n] \qquad \mathfrak{m}(\alpha) := \cap[-\alpha^{-n}, \alpha^{-n}] \qquad K := O(\alpha)/\mathfrak{m}(\alpha)$$

One checks that $O(\alpha)$ is a convex value ring with the maximal ideal $\mathfrak{m}(\alpha)$. Let K be the residue field. One can use a similar construction for an o-minimal expansion of reals R, closing $O(\alpha)$ under definable functions.

Theorem (van den Dries). K is an ultrapower of R. In particular, there is a natural interpretation of atomic relations of R on K.

On K one can define a real-valued valuation

$$v(x) = -st \log_\alpha |x|$$

Indeed, one checks that $st \log_\alpha |x + y| \leqslant \max\{st \log_\alpha |x|, st \log_\alpha |y|\}$.

An o-minimal structure R which expands the real numbers is called *polynomially bounded* if for any definable function $f : R \to R$ there exists a polynomial $p(x)$ such that $f(x) \leqslant p(x)$ for sufficiently big x.

Theorem (van den Dries, Loewenberg). The induced structure on the value group of K is generated by F-linear maps where F is the "field of exponents" of R (for $\mathbb{R}_{an}$ it is $\mathbb{Q}$, for example).

Tate curve in detail

Consider the family of elliptic curves $X_t : \mathbb{C}^\times/q_t^{\mathbb{Z}}$ where $|q_t| \to 0$ as $|t| \to 0$. Let $\Omega_t = dz$ and $\omega_t = \frac{1}{|z|} dz \wedge d\overline{z}$.

The Riemannian metric is $g = \frac{1}{|z|} dz \otimes d\overline{z}$.

Distance function on radial geodesics is

$$d(z_1, z_2) = \left| \int_{|z_1|}^{|z_2|} \frac{dx}{|x|} \right| = |\log|z_2| - \log|z_1||$$

The diameter of X_t is thus $\log|q_t|$, and supposing for example $q_t = t$ the normalized distance is

$$d(z_1, z_2) = |\log_{|t|}|z_2| - \log_{|t|}|z_1||$$

Hopes

(credit: Ehud Hrushovski)

As examplified by the Tate curve: Gromov-Hausdorff limit is the quotient of a set definable in the Robinson field K by the relation $x \sim y$ iff $st \log_{|t|} \frac{|y|}{|x|} = 0$. The quotient is naturally identified with $[0,1] \subset \mathbb{R}$, with ends glued.

One hopes to have a situation similar to Tate curve example in general. Start with an o-minimal expansion of reals so that X_t, ω_t is definable, then GH limit is of the form $X/\sim$, where $\sim$ is definable in the Robinson field associated to a polynomially bounded expansion of the real field, and the quotient can be embedded into Γ^n. As a result we have a natural affine structure on the Gromov-Hausdorff limit.